Diffusion in Hamiltonian systems

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The study is reported of a diffusion in a model of degenerate Hamiltonian systems. The Hamiltonian under consideration is the sum of a linear function of action variables and a periodic function of angle variables. Under certain choices of these functions the diffusion of action variables exists. In the case of two degrees of freedom during the process of diffusion, the vector of the action variables returns many times near its initial value. In the case of three degrees of freedom the choice of Hamiltonian allows one to obtain a diffusion rate faster than any prescribed one. © *1998 American Institute of Physics.* [S1054-1500(98)01701-7]

The phenomenon of diffusion, originally discovered by Arnol'd, plays a significant role in the study of mechanisms underlying the onset of chaos in multidimensional Hamiltonian systems. Diffusion is the slow drift of action variables and random migration in gaps between Kolmogorov tori. Analytic investigations of diffusion in nondegenerate systems have been greatly impeded by the fact that the diffusion rate is exponentially small (Nekhoroshev's theorem). In this paper, we investigate diffusion in highly degenerate Hamiltonian systems to which the results of Kolmogorov–Arnold'd–Moser (KAM) theory and the Nekhoroshev theorem do not apply. In such systems, the diffusion of action variables can grow in accordance with a power-type law.

I. INTRODUCTION

According to Poincaré, the basic task of dynamics is to investigate the Hamilton equations

$$\frac{dx_k}{dt}=\frac{\partial H}{\partial y_k},\quad \frac{dy_k}{dt}=-\frac{\partial H}{\partial x_k};\quad k=1,\dots,n,$$
$$H=H_0(y)+\epsilon H_1(x,y), \tag{1}$$

where $x=(x_1,\dots,x_n) \bmod 2\pi$ are the canonical angular coordinates, $y=(y_1,\dots,y_n)$ are the canonical momenta, and ϵ is a small parameter. The Hamiltonian H is 2π-periodic in each angular coordinate $x_1,\dots,x_n$. When $\epsilon=0$, we have a completely integrable system and the variables y, x are the *action-angle variables* of this system.

In accordance with KAM theory, when a system with nondegenerate Hamiltonian

$$\det\left\|\frac{\partial^2 H_0}{\partial y^2}\right\|\neq 0 \tag{2}$$

is perturbed, most (in the sense of Lebesque measure) invariant tori

$$x \bmod 2\pi, y=y_0$$

do not vanish and are only slightly deformed. These are the so-called *Kolmogorov tori*. When $n=2$, the Kolmogorov tori divide the three-dimensional energy surface $H=\text{const}$ into invariant regions, so that the action variables y do not evolve for small values of ϵ. Arnol'd[1] has used a model example to show that for $n>2$ the system can drift in the gap between Kolmogorov tori. The action variables y can then change by a finite amount. This phenomenon is called *Arnol'd diffusion.* The mechanisms underlying this diffusion in multidimensional systems is not clear yet.

Nekhoroshev[2] has found an estimate for the rate of Arnol'd diffusion in the case of analytic systems with a *steep* Hamiltonian [the steepness condition strengthens the inequality given by Eq. (2)]: there exist positive constants ϵ_0, a, b such that for all

$$0\leq t\leq \frac{1}{\epsilon}\exp\frac{1}{\epsilon^a},\quad 0<\epsilon<\epsilon_0 \tag{3}$$

the inequality

$$|y(t)-y(0)|<\epsilon^b \tag{4}$$

is satisfied. The constants a and b depend only on the unperturbed Hamiltonian H_0.

In this paper, we present examples of Hamiltonian systems for which the rate of diffusion is significantly greater than the exponentially small rate. However, in the examples considered below, the systems will be highly degenerate [i.e., Eq. (2) will not be satisfied].

For a better insight into Eqs. (3) and (4), let us consider the case (it will be encountered below) in which the increase $y(t)-y_0(t)$ in the action variables is equal to $\epsilon \log t$. For example, if $b=1/2$ in Eq. (4), we find that

$$|y(t)-y(0)|<\sqrt{\epsilon},$$

which is valid on the exponentially long interval

$$0\leq t\leq \exp\frac{1}{\sqrt{\epsilon}}.$$

On the other hand, when $b=0$, we find that the increase in the action variables is 1 on the somewhat longer interval

$$0\leq t<\exp\frac{1}{\epsilon}.$$

Moreover, we note that estimates of the form given by Eqs. (3)–(4) were first obtained by Littlewood[3,4] in the restricted circular three body problem. Indeed, in this case, $n=2$ and the KAM theorem gives a stronger result on the secular stability of action variables.

II. HIGHLY DEGENERATE SYSTEMS

Consider the model Hamiltonian

$$H=H_0(y)+\epsilon H_1(x),\quad H_0=\sum_{k=1}^{n}\omega_k y_k, \tag{5}$$

where $\omega_1,\dots,\omega_n$ are rationally incommensurate real numbers and H_1 is an analytic function on the n-dimensional torus $T^n=\{x \bmod \pi\}$. The phase space is the direct product of the torus $T^n=\{x \bmod 2\pi\}$ and $R^n=\{y\}$. This system has a complete set of multivalued commuting integrals

$$\omega_n x_1-\omega_1 x_n,\omega_{n-1}x_1-\omega_1 x_{n-1},\dots,$$
$$\omega_2 x_1-\omega_1 x_2,H$$

and is therefore explicitly integrable. However, as we shall see later, the question of the evolution of action variables rests on the complex problem of small denominators.

The canonical equations with the Hamiltonian given by Eq. (5) have the following explicit form:

$$\frac{dx_k}{dt}=\omega_k,\quad \frac{dy_k}{dt}=\epsilon f_k(x),\quad k=1,\dots,n,$$
$$f_k=-\frac{\partial H_1}{\partial x_k}. \tag{6}$$

The first group of equations determines the conditionally periodic motion on the n-dimensional torus

$$x_k=\omega_k t+x_k(0),\quad 1\leqslant k\leqslant n.$$

The numbers $\omega_1,\dots,\omega_n$ are the incommensurate frequencies of the conditionally periodic motion and $x_1(0),\dots,x_n(0)$ are the initial phases. The action variables are the integrals of the conditionally periodic functions

$$y_k(t)=y_k(0)+\epsilon\int_0^t f_k(\omega\tau+x(0))d\tau.$$

Since the phase averages of f_k over the torus T^n are zero, we find (according to Weyl's theorem) that $y_k(t)-y_k(0)=O(t)$, $t\to\infty$. However, this estimate of the rate of diffusion is a weak result because of its great generality. (Weyl's theorem is valid for continuous and even Riemann-integrable functions). We note that for smooth functions f_k, the Weyl estimate is uniform in the initial phases $x(0)$.

Systems with the Hamiltonian given by Eq. (5) were considered by Kozlov[5] from the standpoint of the conditions for the existence of single-valued integrals. It was shown there that for $n=2$ it is possible to find an irrational number ω_1/ω_2 and analytic function $H_1:T^2\to R$ such that Eqs. (6) have an H-independent integral from the class C^d without there being an additional integral from the class C^{d+1}. Detailed proofs are given by Moshchevitin.[6] An analogous result for arbitrary $n\geqslant 2$ has also been discussed by Moshchevitin.[7]

III. SYSTEMS WITH TWO DEGREES OF FREEDOM

Before we present our results, we note that the difference $y(t)-y(0)$ with fixed time t depends only on the initial phases $x_k(0)$, $k=1,\dots,n$. We shall adopt the following definition:

$$Y_k(T)=\max_{l\leqslant t\leqslant T}\max_{x(0)}|y_k(t)-y_k(0)|,$$

$$Y(T)=\max_{k=l,\dots,n}Y_k(T).$$

THEOREM 1:
Suppose that $n=2$, ω_1/ω_2 is irrational, and H_1 is analytic. Then:

(i) *There exists a sequence $t_\nu\uparrow\infty$ such that the separation between points $(x(t_\nu),y(t_\nu))$ and $(x(0),y(0))$ (in the standard metric on $T^2\times R^2$) tends to zero uniformly in the initial data as $\nu\to\infty$.*

(ii) *There exists a constant $c>0$ and sequence $T_\nu\uparrow\infty$ such that*

$$Y(T_\nu)<\epsilon c\log T_\nu,\quad\forall\nu.$$

Conclusion (i) was actually proved in Ref. 8. It establishes the uniform reversion of phase trajectories of system (5). Conclusion (ii) is proved by considering cases A and B below. Since the function f is analytic, we have

$$f_k(x)=\sum_{m\in Z^2\{0\}}f_k(m)\exp(i\langle m,x\rangle),$$

$$|f_k(m)|\leqslant\gamma_1\exp(-\gamma_2|m|),\quad |m|=|m_1|+|m_2|.$$

Case A. If for all $m\neq 0$ we have $|\langle m,\omega\rangle|>\exp(-\lambda'|m|)$ with a certain $\gamma'\in(0;\gamma_2)$, then $Y(T)=O(\epsilon)$ for $T\to\infty$, which completes the proof.

Case B. If we have infinitely many integer number vectors $m_\nu=(p_\nu,q_\nu)\in Z^2$, $q_\nu>0$ such that

$$|\langle m_\nu,\omega\rangle|=|\omega_i p_\nu+\omega_2 q_\nu|<\exp(-\gamma'(|p_\nu|+q_\nu));$$

then for a certain $\gamma''>0$

$$y_1(q_\nu/\omega_2)-y_1(0)=\epsilon\int_0^{q_\nu/\omega_2}f_1(\omega\tau+x(0))d\tau$$
$$=O(\epsilon\exp(-\gamma''q_\nu)),$$

and since

$$y_1(\lambda q_\nu/\omega_2)-y_1(0)=\epsilon\sum_{l=0}^{\lambda-1}\int_0^{q_\nu/\omega_2}f_1(\omega\tau+\xi_l)d\tau$$
$$=O(\epsilon\lambda\exp(-\gamma''q_\nu))+O(\epsilon)$$

we have

$$y_1(\lambda q_\nu)=O(\epsilon\lambda\exp(-\gamma''q_\nu))+O(\epsilon),\quad\forall\lambda$$

and, consequently,

$$y_1(t)=O(\epsilon\lambda\exp(-\gamma''q_\nu))+O(\epsilon q_\nu),\quad\forall t\in(0;\lambda q_\nu).$$

Taking $\lambda_\nu = \exp(-\gamma' q_\nu)$ and $T_\nu = \lambda_\nu q_\nu$, we have

$$Y_1(T_\nu) = O(\epsilon \log T_\nu).$$

It remains to recall the existence of the energy integral $H = w_1 y_1 + w_2 y_2 + \epsilon H_1(x) = \text{const}$ and conclusion (ii) in theorem 1 can be regarded as proved.

Before we formulate theorems 2 and 3 we recall once again that the difference $y_k(t) - y(0)$ depends (for given t) on the initial phases $x(0)$ only.

THEOREM 2:

For $n=2$ and any positive-valued monotonically increasing function $g(t)$ given in advance and such that $g(t) = \bar{\bar{o}}(t)$, $t \to \infty$, we can find an irrational number ω_1/ω_2 and analytic function $H_1 : T^2 \to R$ such that:

(i) *For each value h system (6) is transitive on the energy level $H=h$.*

(ii) *There is a constant $c>0$ and sequence $t_\nu \uparrow \infty$ such that*

$$\left(\int_{T^2} |y_k(t_\nu) - y(0)|^2 dx(0) \right)^{1/2} \geqslant \epsilon c g(t_\nu).$$

Transitivity means the existence of trajectories that densely fill the whole (i) three-dimensional surface $H=h$. This result was established in Ref. 8. We thus have here the unbounded variation of action variables, and we can speak of diffusion in the Hamiltonian system (6), except that property (ii) does not ensure fast diffusion. Because of reversion, this type of diffusion can be referred to as *flickering*. Both conclusions of theorem 2 can be proved in standard fashion by using simple properties of cylindric cascades generated by the Hamiltonian system [see Eq. (6)].

IV. SYSTEMS WITH THREE DEGREES OF FREEDOM

The case $n \geqslant 3$ is significantly different from the case $n=2$.

THEOREM 3:

Let $n=3$ and let $g(t)$ be an arbitrary positive function that increases as $t \to \infty$ and is such that $g(t) = \bar{\bar{o}}(t)$. It is then possible to find incommensurate frequencies, ω_1, ω_2, ω_3 and analytic function $H_1 : T^3 \to R$ such that, for any t,

$$\left(\int_{T^3} |y_1(t) - y_1(0)|^2 dx(0) \right)^{1/2} > \epsilon g(t). \tag{7}$$

Thus for $k=3$ we no longer have the property of uniform reversion although, as was shown in Ref. 9, individual reversion of each $y_k(t)$ separately does occur. Theorem 3 is a consequence of a general result on integrals of conditional-periodic functions.[10] Evidently, we can find examples of Hamiltonian systems with three degrees of freedom satisfying theorem 3 and transitive on energy levels $H = \text{const}$, but this involves some subtle constructions in the theory of diophantine approximations.

As an example, let us put $g(t) = t^\alpha$, $0<\alpha<1$. From Eq. (7) we obtain the estimate

$$\max_{x(0)} |y_1(t) - y_1(0)| > 1, \tag{8}$$

which is valid for any

$$t \asymp \epsilon^{-1/\alpha}. \tag{9}$$

The estimates given by Eqs. (8) and (9) show that the action variables change by a finite amount in a time interval of the order of $\epsilon^{-\beta}$, $\beta>1$. It follows that the rate of diffusion will no longer be exponentially small, and that it increases in accordance with a power law. Nekhoroshev[2] has given examples of systems with nonsteep Hamiltonian H_0 and action variable increasing in accordance with a power law. However, Ref. 2 was concerned with individual trajectories whereas estimate (7) is generally valid.

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