S.Novikov. Which PDE Systems are Hamiltonian?

Why is it useful to know?

S.Novikov. Which PDE Systems are Hamiltonian?

Why is it useful to know?

XX Century Theoretical Physics: Fundamental Laws should be expressed by Lagrangian and Hamiltonian

S.Novikov. Which PDE Systems are Hamiltonian?

Why is it useful to know?

XX Century Theoretical Physics: Fundamental Laws should be expressed by Lagrangian and Hamiltonian (i.e. "**Conservative**") Systems. Otherwise:

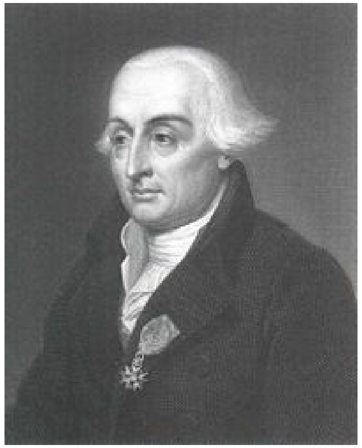
1. No rigorous notion of energy-momentum can be defined;

S.Novikov. Which PDE Systems are Hamiltonian?

Why is it useful to know?

XX Century Theoretical Physics: Fundamental Laws should be expressed by Lagrangian and Hamiltonian (i.e. "**Conservative**") Systems. Otherwise:

1. No rigorous notion of energy-momentum can be defined;
2. Only conservative systems can be quantized.



Lagrange

Hamilton

Poincaré

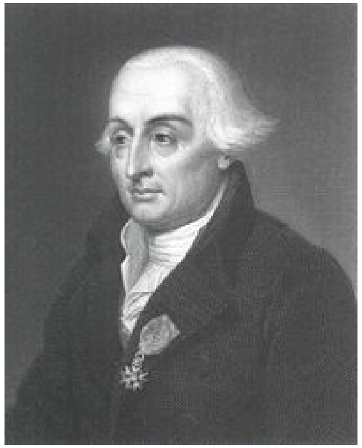
Bohr

S.Novikov. Which PDE Systems are Hamiltonian?

Why is it useful to know?

XX Century Theoretical Physics: Fundamental Laws should be expressed by Lagrangian and Hamiltonian (i.e. "**Conservative**") Systems. Otherwise:

1. No rigorous notion of energy-momentum can be defined;
2. Only conservative systems can be quantized.



Lagrange

Hamilton

Poincaré

Bohr

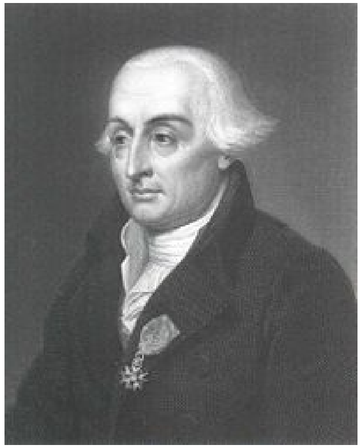
Any system is a subsystem of a Hamiltonian one.

S.Novikov. Which PDE Systems are Hamiltonian?

Why is it useful to know?

XX Century Theoretical Physics: Fundamental Laws should be expressed by Lagrangian and Hamiltonian (i.e. "**Conservative**") Systems. Otherwise:

1. No rigorous notion of energy-momentum can be defined;
2. Only conservative systems can be quantized.



Lagrange *Hamilton* *Poincaré* *Bohr*

Any system is a subsystem of a Hamiltonian one.

We ask: is the whole system Hamiltonian or not.

Part I. Classical Approach:

Part I. Classical Approach: In the Canonical Coordinates $p_i, , q_j$ every Hamiltonian $H(p,q)$
defines time evolution: $dq/dt = \partial H/\partial p; dp/dt = -\partial H/\partial q$

Part I. Classical Approach: In the Canonical Coordinates $p_i, , q_j$ every Hamiltonian $H(p,q)$
defines time evolution: $dq/dt = \partial H/\partial p; dp/dt = -\partial H/\partial q$
Symplectic Geometry works in the arbitrary coordinates in terms of differential forms.

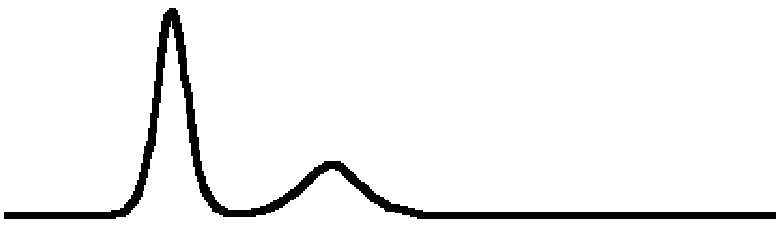
Part I. Classical Approach: In the Canonical Coordinates $p_i, , q_j$ every Hamiltonian $H(p, q)$
defines time evolution: $dq/dt = \partial H/\partial p; dp/dt = -\partial H/\partial q$
Symplectic Geometry works in the arbitrary coordinates in terms of differential forms.
For PDE partial derivatives should be replaces by "variational derivatives". The indices i, j became continuous like $(i, x), (j, y)$: $\partial p_i(x)/\partial t = \delta H/\delta q_i(x)$,
$\partial q_j(x)/\partial t = -\delta H/\delta p_j(x)$

Part I. Classical Approach: In the Canonical Coordinates $p_i, , q_j$ every Hamiltonian $H(p, q)$
defines time evolution: $dq/dt = \partial H/\partial p; dp/dt = -\partial H/\partial q$

Symplectic Geometry works in the arbitrary coordinates in terms of differential forms.

For PDE partial derivatives should be replaces by "variational derivatives". The indices i, j became continuous like $(i, x), (j, y)$: $\partial p_i(x)/\partial t = \delta H/\delta q_i(x)$,
$\partial q_j(x)/\partial t = -\delta H/\delta p_j(x)$

Phenomenon of Complete Integrability by the Inverse Scattering Transform (IST) was discovered by M.Kruscal et al for KdV (1965-1968)



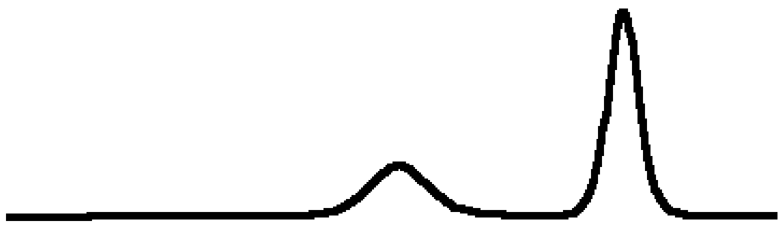


Fig 1 Solitons before interaction Fig 2 Solitons after interaction

The Modern Approach

The Modern Approach
is a by-product of the Theory of Solitons (IST).
It is based on the **Field-Theoretical Poisson Brackets**.

The Modern Approach
is a by-product of the Theory of Solitons (IST).
It is based on the **Field-Theoretical Poisson Brackets**.
They should satisfy to the Jacoby and "Leibnitz Identity":
$\{fg, h\} = f\{g, h\} + g\{f, h\}$.

The Modern Approach

is a by-product of the Theory of Solitons (IST).

It is based on the **Field-Theoretical Poisson Brackets**.

They should satisfy to the Jacoby and "Leibnitz Identity":

$\{fg,h\} = f\{g,h\}+g\{f,h\}$. This approach allows degeneracy:

Function f is called **"Casimir" or "Annihilator"**

if $\{f,g\} = 0$ for all g.

The Modern Approach

is a by-product of the Theory of Solitons (IST).

It is based on the **Field-Theoretical Poisson Brackets**.

They should satisfy to the Jacoby and "Leibnitz Identity":

$\{fg, h\} = f\{g, h\} + g\{f, h\}$. This approach allows degeneracy:

Function f is called **"Casimir" or "Annihilator"**

if $\{f, g\} = 0$ for all g.

For ODE the relations

$\{p_i, p_j\} = \{q_i, q_j\} = 0$, $\{p_i, q_j\} = \delta_{ij}$ define

the canonical coordinates,

The Modern Approach
is a by-product of the Theory of Solitons (IST).
It is based on the **Field-Theoretical Poisson Brackets**.
They should satisfy to the Jacoby and "Leibnitz Identity":
$\{fg, h\} = f\{g, h\}+g\{f, h\}$. This approach allows degeneracy:
Function f is called **"Casimir" or "Annihilator"**
if $\{f, g\} = 0$ for all g.
For ODE the relations
$\{p_i, p_j\} = \{q_i, q_j\} = 0$, $\{p_i, q_j\} = \delta_{ij}$ define
the canonical coordinates,
for PDE we have $\{p_i(x), q_j(y)\} = \delta_{ij}\delta(x - y)$
In the Modern Approach Poisson Brackets are presented
in the **Noncanonical Coordinates**

The Liouville form:

The Liouville form:

Hamilton equation

$df/dt = \{f, H\}$

The Liouville form:

Hamilton equation

$df/dt = \{f, H\}$ can be easily quantized replacing f by quantum operators $\hat{f}$,

The Liouville form:

Hamilton equation
$df/dt = \{f, H\}$ can be easily quantized replacing f by quantum operators $\hat{f}$, and Poisson Bracket by commutator
$\{f, g\} \to ih[\hat{f}, \hat{g}] + O(h^2)$.

The Liouville form:

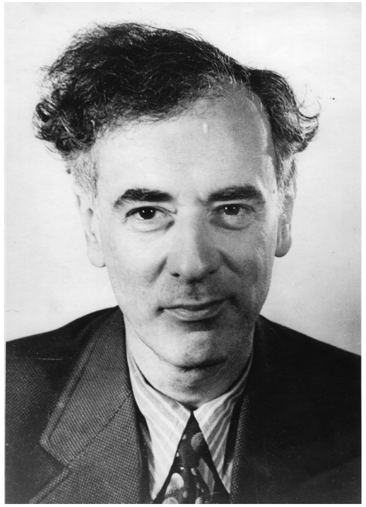
Hamilton equation
$df/dt = \{f, H\}$ can be easily quantized replacing f by quantum operators $\hat{f}$, and Poisson Bracket by commutator
$\{f, g\} \rightarrow ih[\hat{f}, \hat{g}] + O(h^2)$.
It became also a base of Kinetic Theory for multiparticle systems.

The Liouville form:

Hamilton equation $df/dt = \{f, H\}$ can be easily quantized replacing f by quantum operators $\hat{f}$, and Poisson Bracket by commutator $\{f, g\} \to ih[\hat{f}, \hat{g}] + O(h^2)$.

It became also a base of Kinetic Theory for multiparticle systems.

Heisenberg



Landau

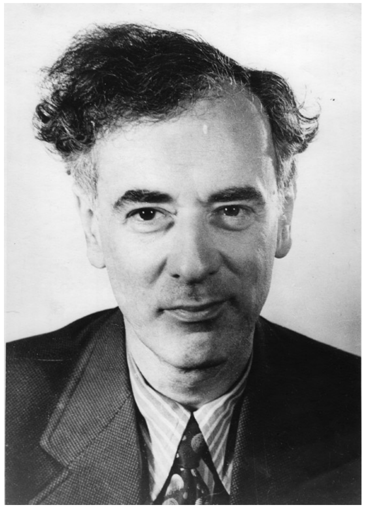
The Liouville form:

Hamilton equation

$df/dt = \{f, H\}$ can be easily quantized replacing f by quantum operators $\hat{f}$, and Poisson Bracket by commutator

$\{f, g\} \to ih[\hat{f}, \hat{g}] + O(h^2)$.

It became also a base of Kinetic Theory for multiparticle systems.

Heisenberg *Landau*

The Poisson Brackets for Hydrodynamics were written by L.Landau in 1940 as "quantum commutators" for superfluid He^4.

It was not realized till the late 1970s that they immediately imply Hamiltonian structure for the Euler equations.

It was not realized till the late 1970s that they immediately imply Hamiltonian structure for the Euler equations.
For Lie Algebras and Representations Hamiltonian ideas were invented in 1960s in Symplectic terms not knowing XIX Century results of S.Lie about linear Poisson Brackets.

Part II. The most important types of P.Brackets:

Part II. The most important types of P.Brackets:
1. The Canonical Brackets

$$\{p(x), q(y)\} = \delta(x - y)$$

Part II. The most important types of P.Brackets:

1. The Canonical Brackets

$$\{p(x), q(y)\} = \delta(x - y)$$

2.The Ultralocal Brackets

$$\{u_i(x), u_j(y)\} = A_{ij}(u(x))\delta(x - y)$$

Here matrix $A = A_{ij}(u)$ is smooth on the target space.

Part II. The most important types of P.Brackets:

1. **The Canonical Brackets**

$\{p(x), q(y)\} = \delta(x - y)$

2.**The Ultralocal Brackets**

$\{u_i(x), u_j(y)\} = A_{ij}(u(x))\delta(x - y)$

Here matrix $A = A_{ij}(u)$ is smooth on the target space.

3.**The Local Brackets**

$\{u_i(x), u_j(y)\} = A_{ij}(u(x))[\delta(x - y)]$,

Part II. The most important types of P.Brackets:

1. The Canonical Brackets

$\{p(x), q(y)\} = \delta(x - y)$

2.The Ultralocal Brackets

$\{u_i(x), u_j(y)\} = A_{ij}(u(x))\delta(x - y)$

Here matrix $A = A_{ij}(u)$ is smooth on the target space.

3.The Local Brackets

$\{u_i(x), u_j(y)\} = A_{ij}(u(x))[\delta(x - y)]$,

with differential Poisson operator $A = A_{ij}$ on target space::

$A = \sum_{k=0}^{k=m} A_{ij}^k(u(x), u'(x), ..., u^k(x))\partial_x^k$

Part II. The most important types of P.Brackets:

1. The Canonical Brackets

$\{p(x), q(y)\} = \delta(x - y)$

2.The Ultralocal Brackets

$\{u_i(x), u_j(y)\} = A_{ij}(u(x))\delta(x - y)$

Here matrix $A = A_{ij}(u)$ is smooth on the target space.

3.The Local Brackets

$\{u_i(x), u_j(y)\} = A_{ij}(u(x))[\delta(x - y)]$,

with differential Poisson operator $A = A_{ij}$ on target space::

$A = \sum_{k=0}^{k=m} A_{ij}^k(u(x), u'(x), ..., u^k(x))\partial_x^k$

The Hamilton Equation is PDE for local Hamiltonian

$u_t = A[\delta H/\delta u(x)]$

Part II. The most important types of P.Brackets:

1. **The Canonical Brackets**

$\{p(x), q(y)\} = \delta(x - y)$

2.**The Ultralocal Brackets**

$\{u_i(x), u_j(y)\} = A_{ij}(u(x))\delta(x - y)$

Here matrix $A = A_{ij}(u)$ is smooth on the target space.

3.**The Local Brackets**

$\{u_i(x), u_j(y)\} = A_{ij}(u(x))[\delta(x - y)]$,

with differential Poisson operator $A = A_{ij}$ on target space::

$A = \sum_{k=0}^{k=m} A_{ij}^k(u(x), u'(x), ..., u^k(x))\partial_x^k$

The Hamilton Equation is PDE for local Hamiltonian **(and Jacoby Identity Should be carefully checked!)**

$\{J, H\} = \int \delta J/\delta u(x) A[\delta H/\delta u(x)]dx$

$u_t = A[\delta H/\delta u(x)]$

An Important Example:

An Important Example: For KdV ($u_t = 6uu_x + u_{xxx}$) we have Hamiltonian Systems

An Important Example: For KdV ($u_t = 6uu_x + u_{xxx}$) we have Hamiltonian Systems in the form

$u_t = \partial_x[\delta H / \delta u(x)]$

An Important Example: For KdV ($u_t = 6uu_x + u_{xxx}$) we have Hamiltonian Systems in the form
$u_t = \partial_x[\delta H/\delta u(x)]$ where
$A_0 = \partial_x$, the GZF Bracket,, $H = H_0 = \int (u_x^2/2 + u^3)dx$ (energy), $P = \int u^2 dx$ (momentum) and $C = \int u dx$ (Casimir)

An Important Example: For KdV ($u_t = 6uu_x + u_{xxx}$) we have Hamiltonian Systems in the form
$u_t = \partial_x[\delta H/\delta u(x)]$ where
$A_0 = \partial_x$, the GZF Bracket,, $H = H_0 = \int (u_x^2/2 + u^3)dx$ (energy), $P = \int u^2 dx$ (momentum) and $C = \int u dx$ (Casimir)
Another local P.B. for KdV is the LM Bracket:

An Important Example: For KdV ($u_t = 6uu_x + u_{xxx}$) we have Hamiltonian Systems in the form
$u_t = \partial_x[\delta H/\delta u(x)]$ where
$A_0 = \partial_x$, the GZF Bracket,, $H = H_0 = \int (u_x^2/2 + u^3)dx$ (energy), $P = \int u^2 dx$ (momentum) and $C = \int u dx$ (Casimir)
Another local P.B. for KdV is the LM Bracket:
$A_1 = \partial_x^3 + 3/2(u(x)\partial_x + \partial_x u(x))$ and
$H_1 = \int u^2 dx, P_1 = \int u dx$, The Casimir C_1 is nonlocal.

An Important Example: For KdV ($u_t = 6uu_x + u_{xxx}$) we have Hamiltonian Systems in the form
$u_t = \partial_x[\delta H/\delta u(x)]$ where
$A_0 = \partial_x$, the GZF Bracket,, $H = H_0 = \int(u_x^2/2 + u^3)dx$ (energy), $P = \int u^2 dx$ (momentum) and $C = \int u dx$ (Casimir)
Another local P.B. for KdV is the LM Bracket:
$A_1 = \partial_x^3 + 3/2(u(x)\partial_x + \partial_x u(x))$ and
$H_1 = \int u^2 dx, P_1 = \int u dx$, The Casimir C_1 is nonlocal.
The KdV equation is **Bihamiltonian**, i.e.
Hamiltonian corr to all Poisson operators in the "pencil" $\lambda A_0 + \mu A_1$.

An Important Example: For KdV ($u_t = 6uu_x + u_{xxx}$) we have Hamiltonian Systems in the form

$u_t = \partial_x[\delta H/\delta u(x)]$ where

$A_0 = \partial_x$, the GZF Bracket,, $H = H_0 = \int (u_x^2/2 + u^3)dx$ (energy), $P = \int u^2 dx$ (momentum) and $C = \int u dx$ (Casimir)

Another local P.B. for KdV is the LM Bracket:

$A_1 = \partial_x^3 + 3/2(u(x)\partial_x + \partial_x u(x))$ and

$H_1 = \int u^2 dx, P_1 = \int u dx$, The Casimir C_1 is nonlocal.

The KdV equation is **Bihamiltonian**, i.e.

Hamiltonian corr to all Poisson operators in the "pencil" $\lambda A_0 + \mu A_1$.

The Recurrence Operator"

is defined by this pencil $R = A_1 A_0^{-1} = \partial_x^2 + 3/2u(x) + 3/2\partial_x \times u(x) \times \partial_x^{-1}$

4. Nonlocal P.B.

4. **Nonlocal P.B.**

The Poisson operator A is nonlocal.

4. **Nonlocal P.B.**

The Poisson operator A is nonlocal. Specific case of **Weekly Nonlocal P.B.** plays important role for one-dimensional x:

4. **Nonlocal P.B.**

The Poisson operator A is nonlocal. Specific case of **Weekly Nonlocal P.B.** plays important role for one-dimensional x: $A = A_{local} + \sum_s f_s \times \partial_x^{-1} \times g_s$ where $f_s(x), g_s(y)$ are "local functions".

4. **Nonlocal P.B.**

The Poisson operator A is nonlocal. Specific case of **Weekly Nonlocal P.B.** plays important role for one-dimensional x: $A = A_{local} + \sum_s f_s \times \partial_x^{-1} \times g_s$ where $f_s(x), g_s(y)$ are "local functions".

Example: For KdV all Poisson Operators $A_p = R^p A_1$ for $p = 2, 3, 4, ...$ are nonlocal but only **Weekly Nonlocal** because of remarkable cancelations.

4. **Nonlocal P.B.**

The Poisson operator A is nonlocal. Specific case of **Weekly Nonlocal P.B.** plays important role for one-dimensional x: $A = A_{local} + \sum_s f_s \times \partial_x^{-1} \times g_s$ where $f_s(x), g_s(y)$ are "local functions".

Example: For KdV all Poisson Operators $A_p = R^p A_1$ for $p = 2, 3, 4, \dots$ are nonlocal but only **Weekly Nonlocal** because of remarkable cancelations.

The simplest weekly nonlocal P.B. is

$\{u(x), u(y)\} = u \times \partial_x^{-1} \times u$.

4. **Nonlocal P.B.**

The Poisson operator A is nonlocal. Specific case of **Weekly Nonlocal P.B.** plays important role for one-dimensional x: $A = A_{local} + \sum_s f_s \times \partial_x^{-1} \times g_s$ where $f_s(x), g_s(y)$ are "local functions".

Example: For KdV all Poisson Operators $A_p = R^p A_1$ for $p = 2, 3, 4, \ldots$ are nonlocal but only **Weekly Nonlocal** because of remarkable cancelations.

The simplest weekly nonlocal P.B. is $\{u(x), u(y)\} = u \times \partial_x^{-1} \times u$. **Theorem.** It defines local (PDE) Hamiltonian equation with every translational invariant Hamiltonian functional

$$u_t = u(x)(\partial_x^{-1})u(x)[\delta H/\delta(u(x))]$$

4. **Nonlocal P.B.**

The Poisson operator A is nonlocal. Specific case of **Weekly Nonlocal P.B.** plays important role for one-dimensional x: $A = A_{local} + \sum_s f_s \times \partial_x^{-1} \times g_s$ where $f_s(x), g_s(y)$ are "local functions".

Example: For KdV all Poisson Operators $A_p = R^p A_1$ for $p = 2, 3, 4, ...$ are nonlocal but only **Weekly Nonlocal** because of remarkable cancelations.

The simplest weekly nonlocal P.B. is $\{u(x), u(y)\} = u \times \partial_x^{-1} \times u$. **Theorem.** It defines local (PDE) Hamiltonian equation with every translational invariant Hamiltonian functional

$$u_t = u(x)(\partial_x^{-1})u(x)[\delta H/\delta(u(x))]$$

5.The Local Symplectic Structures

5.**The Local Symplectic Structures** define the equations

$A'[\partial u/\partial t] = \delta H/\delta u(x)$

5.**The Local Symplectic Structures** define the equations $A'[\partial u/\partial t] = \delta H/\delta u(x)$ where A' is a local operator as above.

5.**The Local Symplectic Structures** define the equations $A'[\partial u/\partial t] = \delta H/\delta u(x)$ where A' is a local operator as above. The Poisson Bracket is nonlocal in this case.

5.**The Local Symplectic Structures** define the equations $A'[\partial u/\partial t] = \delta H/\delta u(x)$ where A' is a local operator as above. The Poisson Bracket is nonlocal in this case.
In the same way we define **Weekly Nonlocal Symplectic Structures** where the operator $A' = A^{-1}$ is weekly nonlocal.

5.**The Local Symplectic Structures** define the equations $A'[\partial u/\partial t] = \delta H/\delta u(x)$ where A' is a local operator as above. The Poisson Bracket is nonlocal in this case.
In the same way we define **Weekly Nonlocal Symplectic Structures** where the operator $A' = A^{-1}$ is weekly nonlocal. For KdV the recurrence operator R defines a family of nonlocal Symplectic Structures with operators $A'_p = R^{-p}A_0, p > 0$. It was proved that they are Weekly Nonlocal.

5.**The Local Symplectic Structures** define the equations $A'[\partial u/\partial t] = \delta H/\delta u(x)$ where A' is a local operator as above.
The Poisson Bracket is nonlocal in this case.
In the same way we define **Weekly Nonlocal Symplectic Structures** where the operator $A' = A^{-1}$ is weekly nonlocal.
For KdV the recurrence operator R defines a family of nonlocal Symplectic Structures with operators $A'_p = R^{-p}A_0, p > 0$. It was proved that they are Weekly Nonlocal.
The theory of Weekly Nonlocal Structures was developed few years ago but many examples appeared much earlier.

5.**The Local Symplectic Structures** define the equations $A'[\partial u/\partial t] = \delta H/\delta u(x)$ where A' is a local operator as above. The Poisson Bracket is nonlocal in this case.

In the same way we define **Weekly Nonlocal Symplectic Structures** where the operator $A' = A^{-1}$ is weekly nonlocal. For KdV the recurrence operator R defines a family of nonlocal Symplectic Structures with operators $A'_p = R^{-p}A_0, p > 0$. It was proved that they are Weekly Nonlocal.

The theory of Weekly Nonlocal Structures was developed few years ago but many examples appeared much earlier. Some other important classes of Poisson Brackets were developed in the Theory of Completely Integrable Systems, f.e. the "Yang-Baxter Brackets" which will not be discussed here.

Part III. The Hydrodynamic Type Systems:

Part III. The Hydrodynamic Type Systems:

The first order homogeneous quasilinear systems

Part III. The Hydrodynamic Type Systems:

The first order homogeneous quasilinear systems

similar to the Euler Equations or Gas Dynamics (let $n = 1$):

$u_t^i = \sum_j a_j^i(u) u_x^j (*)$

Part III. The Hydrodynamic Type Systems:

The first order homogeneous quasilinear systems similar to the Euler Equations or Gas Dynamics (let $n=1$):

$u^i_t = \sum_j a^i_j(u) u^j_x (*)$

The matrices $a^i_j = A$ are 2-tensors in the "target space". They are diagonal if coordinates u are **the Riemann Invariants**.

Part III. The Hydrodynamic Type Systems:

The first order homogeneous quasilinear systems
similar to the Euler Equations or Gas Dynamics (let $n = 1$):
$u_t^i = \sum_j a_j^i(u) u_x^j (*)$
The matrices $a_j^i = A$ are 2-tensors in the "target space". They are diagonal if coordinates u are **the Riemann Invariants**.
Examples: Gas Dynamics and Whitham Modulation Equations

Part III. The Hydrodynamic Type Systems:

The first order homogeneous quasilinear systems
similar to the Euler Equations or Gas Dynamics (let $n = 1$):

$u^i_t = \sum_j a^i_j(u) u^j_x (*)$

The matrices $a^i_j = A$ are 2-tensors in the "target space". They are diagonal if coordinates u are **the Riemann Invariants**.

Examples: Gas Dynamics and Whitham Modulation Equations

For gas dynamics we have $(u^1 = p, u^2 = \rho, u^3 = s)$, densities of momentum, mass and entropy for $n = 1$.

Part III. The Hydrodynamic Type Systems:

The first order homogeneous quasilinear systems

similar to the Euler Equations or Gas Dynamics (let $n = 1$):

$u_t^i = \sum_j a_j^i(u) u_x^j (*)$

The matrices $a_j^i = A$ are 2-tensors in the "target space". They are diagonal if coordinates u are **the Riemann Invariants**.

Examples: Gas Dynamics and Whitham Modulation Equations

For gas dynamics we have $(u^1 = p, u^2 = \rho, u^3 = s)$, densities of momentum, mass and entropy for $n = 1$.

In the Theory of Solitons such systems appear in the asymptotic studies of "Rapidly Oscillating Solutions"

Part III. The Hydrodynamic Type Systems:

The first order homogeneous quasilinear systems

similar to the Euler Equations or Gas Dynamics (let $n = 1$):

$u^i_t = \sum_j a^i_j(u) u^j_x (*)$

The matrices $a^i_j = A$ are 2-tensors in the "target space". They are diagonal if coordinates u are **the Riemann Invariants**.

Examples: Gas Dynamics and Whitham Modulation Equations

For gas dynamics we have $(u^1 = p, u^2 = \rho, u^3 = s)$, densities of momentum, mass and entropy for $n = 1$.

In the Theory of Solitons such systems appear in the asymptotic studies of "Rapidly Oscillating Solutions" like Nonlinear Analog of WKB approximation or "Averaging Procedure" against the family of quasiperiodic N-gap solutions found through Riemann Surfaces and finite-pap potentials

(after Whitham, 1960s for $N = 1$),derived by Flashka-Mclaughlin, Lax-Levermore-Venakides in 1980s on the base of multisoliton and quasiperiodic finite-gap solutions.

(after Whitham, 1960s for $N = 1$),derived by Flashka-Mclaughlin, Lax-Levermore-Venakides in 1980s on the base of multisoliton and quasiperiodic finite-gap solutions.

Dubrovin and myself developed the Hamiltonian Theory in 1980s which I am going to discuss here

(after Whitham, 1960s for $N = 1$),derived by Flashka-Mclaughlin, Lax-Levermore-Venakides in 1980s on the base of multisoliton and quasiperiodic finite-gap solutions.
Dubrovin and myself developed the Hamiltonian Theory in 1980s which I am going to discuss here (including many results obtained by other people later, in particular by Mokhov, Tsarev, Ferapontov, Pavlov, Maltsev in the Hamiltonian theory ,and by Avilov, Krichever, Potemin, Tian , Grava in the study of special asymptotic solutions like "dispersive shock wave").

(after Whitham, 1960s for $N = 1$),derived by Flashka-Mclaughlin, Lax-Levermore-Venakides in 1980s on the base of multisoliton and quasiperiodic finite-gap solutions.
Dubrovin and myself developed the Hamiltonian Theory in 1980s which I am going to discuss here (including many results obtained by other people later, in particular by Mokhov, Tsarev, Ferapontov, Pavlov, Maltsev in the Hamiltonian theory ,and by Avilov, Krichever, Potemin, Tian , Grava in the study of special asymptotic solutions like "dispersive shock wave").

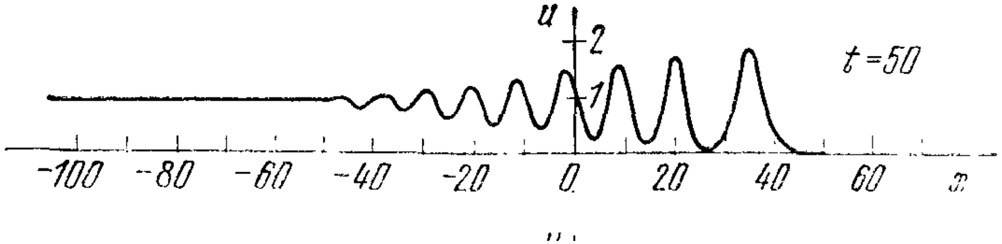


Fig 3 Oscillating Zone

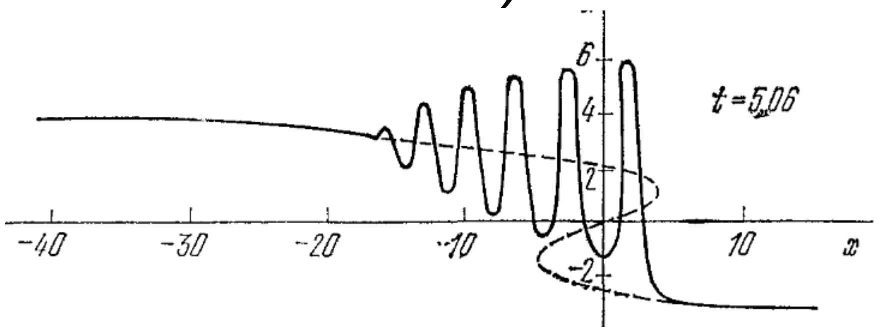


Fig 4 Dispersive Shock Wave

Definitions:

Definitions:

The Hydrodynamic Type Functional is $H = \int h(u(x)dx$.

Definitions:

The Hydrodynamic Type Functional is $H = \int h(u(x)dx$.

The Hydrodynamic Type Poisson Bracket is

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_l \Gamma_l^{ij}(u)u_x^l \delta(x-y)$$

Definitions:

The Hydrodynamic Type Functional is $H = \int h(u(x)dx$.

The Hydrodynamic Type Poisson Bracket is

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_l \Gamma_l^{ij}(u)u_x^l\delta(x-y)$$

The Geometrization Theorem:

Definitions:

The Hydrodynamic Type Functional is $H = \int h(u(x)dx$.

The Hydrodynamic Type Poisson Bracket is

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_l \Gamma_l^{ij}(u)u_x^l \delta(x-y)$$

The Geometrization Theorem: g^{ij} transforms as a metric tensor on the target u-manifold,

Definitions:

The Hydrodynamic Type Functional is $H = \int h(u(x)dx$.

The Hydrodynamic Type Poisson Bracket is

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_l \Gamma^{ij}_l(u)u^l_x\delta(x-y)$$

The Geometrization Theorem: g^{ij} transforms as a metric tensor on the target u-manifold, $\Gamma^{ij}_l = g^{is}\Gamma^j_{sl}(u)$ transforms as Cristoffel Symbols.

Definitions:

The Hydrodynamic Type Functional is $H = \int h(u(x)dx$.

The Hydrodynamic Type Poisson Bracket is

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_l \Gamma^{ij}_l(u)u^l_x\delta(x-y)$$

The Geometrization Theorem: g^{ij} transforms as a metric tensor on the target u-manifold, $\Gamma^{ij}_l = g^{is}\Gamma^j_{sl}(u)$ transforms as Cristoffel Symbols. Jacoby Identity implies that this metric is flat and Cristoffel Symbols are canonical.

Definitions:
The Hydrodynamic Type Functional is $H = \int h(u(x)dx$.
The Hydrodynamic Type Poisson Bracket is

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_l \Gamma_l^{ij}(u)u_x^l \delta(x-y)$$

The Geometrization Theorem: g^{ij} transforms as a metric tensor on the target u-manifold, $\Gamma_l^{ij} = g^{is}\Gamma_{sl}^j(u)$ transforms as Cristoffel Symbols. Jacoby Identity implies that this metric is flat and Cristoffel Symbols are canonical. For $n = 1$ and nondegenerate case $\det g^{ij} \neq 0$

Definitions:

The Hydrodynamic Type Functional is $H = \int h(u(x)dx$.

The Hydrodynamic Type Poisson Bracket is

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_l \Gamma^{ij}_l(u)u^l_x\delta(x-y)$$

The Geometrization Theorem: g^{ij} transforms as a metric tensor on the target u-manifold, $\Gamma^{ij}_l = g^{is}\Gamma^j_{sl}(u)$ transforms as Cristoffel Symbols. Jacoby Identity implies that this metric is flat and Cristoffel Symbols are canonical. For $n = 1$ and nondegenerate case $\det g^{ij} \neq 0$
we have full local **Classification**: Poisson Bracket is completely determined by the Signature of Metric g^{ij}

In all applications this metric has Signature which is either Ultrahyperbolic (N, N) or "Almost Ultrahyperbolic" $(N, N+1)$

In all applications this metric has Signature which is either Ultrahyperbolic (N, N) or "Almost Ultrahyperbolic" $(N, N+1)$

Examples: For Gas Dynamics $n = 1$ in the variables $(p, \rho, s) = (u^1, u^2, u^3)$ this **Metric is Degenerate**

In all applications this metric has Signature which is either Ultrahyperbolic (N, N) or "Almost Ultrahyperbolic" $(N, N+1)$
Examples: For Gas Dynamics $n = 1$ in the variables $(p, \rho, s) = (u^1, u^2, u^3)$ this **Metric is Degenerate**

$$g^{ij} = \gamma^{ij} + \gamma^{ji}, \gamma^{1i} = (p, \rho, s), \gamma^{ji} = 0, j \neq 1$$

In all applications this metric has Signature which is either Ultrahyperbolic (N, N) or "Almost Ultrahyperbolic" $(N, N+1)$
Examples: For Gas Dynamics $n = 1$ in the variables $(p, \rho, s) = (u^1, u^2, u^3)$ this **Metric is Degenerate**

$$g^{ij} = \gamma^{ij} + \gamma^{ji}, \gamma^{1i} = (p, \rho, s), \gamma^{ji} = 0, j \neq 1$$

$$\Gamma_l^{ij} = \partial\gamma^{ij}/\partial u^l$$

In all applications this metric has Signature which is either Ultrahyperbolic (N, N) or "Almost Ultrahyperbolic" $(N, N+1)$

Examples: For Gas Dynamics $n = 1$ in the variables $(p, \rho, s) = (u^1, u^2, u^3)$ this **Metric is Degenerate**

$$g^{ij} = \gamma^{ij} + \gamma^{ji}, \gamma^{1i} = (p, \rho, s), \gamma^{ji} = 0, j \neq 1$$

$$\Gamma_l^{ij} = \partial\gamma^{ij}/\partial u^l$$

What do we know about Whitham Equations?

In all applications this metric has Signature which is either Ultrahyperbolic (N, N) or "Almost Ultrahyperbolic" $(N, N+1)$ **Examples**: For Gas Dynamics $n = 1$ in the variables $(p, \rho, s) = (u^1, u^2, u^3)$ this **Metric is Degenerate**

$$g^{ij} = \gamma^{ij} + \gamma^{ji}, \gamma^{1i} = (p, \rho, s), \gamma^{ji} = 0, j \neq 1$$

$$\Gamma^{ij}_l = \partial\gamma^{ij}/\partial u^l$$

What do we know about Whitham Equations?

1.They admit Riemann Invariants, Whitham for $N = 1$, Flashka-McLaughlin for all N, identifying Riemann Invariants with branching points of Spectral Riemann Surfaces of finite-gap Schrodinger Operators.

2. They are Hamiltonian with P. B. of Hydrodynamic Type (S.N.-Dubr.) so Riemannian Metric Appears. It is Diagonal in the Riemann Invariants.

2.They are Hamiltonian with P. B. of Hydrodynamic Type (S.N.-Dubr.) so Riemannian Metric Appears. It is Diagonal in the Riemann Invariants.

3.Conclusion (Tsarev Theorem): Every Hamiltonian H.T. System with Riemann Invariants is Completely Integrable. The integration is based on **Diagonal Riemannian Metric.**

2.They are Hamiltonian with P. B. of Hydrodynamic Type (S.N.-Dubr.) so Riemannian Metric Appears. It is Diagonal in the Riemann Invariants.

3.Conclusion (Tsarev Theorem): Every Hamiltonian H.T. System with Riemann Invariants is Completely Integrable. The integration is based on **Diagonal Riemannian Metric.**

The Hamiltonian property (i.e. Riemannian Metric) plays fundamental role in the solution of this problem.

2.They are Hamiltonian with P. B. of Hydrodynamic Type (S.N.-Dubr.) so Riemannian Metric Appears. It is Diagonal in the Riemann Invariants.

3.Conclusion (Tsarev Theorem): Every Hamiltonian H.T. System with Riemann Invariants is Completely Integrable. The integration is based on **Diagonal Riemannian Metric.**

The Hamiltonian property (i.e. Riemannian Metric) plays fundamental role in the solution of this problem. Hydrodynamic Type systems were studied since times of Riemann but no Riemannian Geometry appeared here until Hamiltonian Theory was developed.

2.They are Hamiltonian with P. B. of Hydrodynamic Type (S.N.-Dubr.) so Riemannian Metric Appears. It is Diagonal in the Riemann Invariants.
3.Conclusion (Tsarev Theorem): Every Hamiltonian H.T. System with Riemann Invariants is Completely Integrable. The integration is based on **Diagonal Riemannian Metric.**
The Hamiltonian property (i.e. Riemannian Metric) plays fundamental role in the solution of this problem. Hydrodynamic Type systems were studied since times of Riemann but no Riemannian Geometry appeared here until Hamiltonian Theory was developed.
Actually, Differential Geometry is not enough: The important solutions were found using combination of Numerical methods and Riemann Surfaces.

Three types of coordinates were used for Whitham Equations:

Three types of coordinates were used for Whitham Equations:
1.Physical Cooordinates,

Three types of coordinates were used for Whitham Equations:
1.Physical Cooordinates, used for the averaging procedure: For $N = 1$ the averaged densities of energy, momentum and Casimir for the GZF Bracket are: $\bar{\epsilon}$ where $\epsilon = u_x/2 + u^3, H = \int \epsilon dx, \bar{p} = \bar{u^2}, c = \bar{u}$.

Three types of coordinates were used for Whitham Equations: 1.Physical Cooordinates, used for the averaging procedure: For $N = 1$ the averaged densities of energy, momentum and Casimir for the GZF Bracket are: $\bar{\epsilon}$ where $\epsilon = u_x/2 + u^3$, $H = \int \epsilon dx$, $\bar{p} = \bar{u^2}$, $c = \bar{u}$. Poisson Brackets are calculated in these coordinates.

Three types of coordinates were used for Whitham Equations:

1.Physical Cooordinates, used for the averaging procedure: For $N = 1$ the averaged densities of energy, momentum and Casimir for the GZF Bracket are: $\bar{\epsilon}$ where $\epsilon = u_x/2 + u^3, H = \int \epsilon dx, \bar{p} = \bar{u^2}, c = \bar{u}$. Poisson Brackets are calculated in these coordinates.

2. The **flat coordinates** s.t. Riemannian Metric is constant.

Three types of coordinates were used for Whitham Equations:
1.Physical Cooordinates, used for the averaging procedure: For $N = 1$ the averaged densities of energy, momentum and Casimir for the GZF Bracket are: $\bar{\epsilon}$ where $\epsilon = u_x/2 + u^3, H = \int \epsilon dx, \bar{p} = \overline{u^2}, c = \bar{u}$. Poisson Brackets are calculated in these coordinates.
2. The **flat coordinates** s.t. Riemannian Metric is constant. They exist by the Geometrization Theorem and give densities of Casimirs for the "Averaged" Hydrodynamic Type P.Bracket.

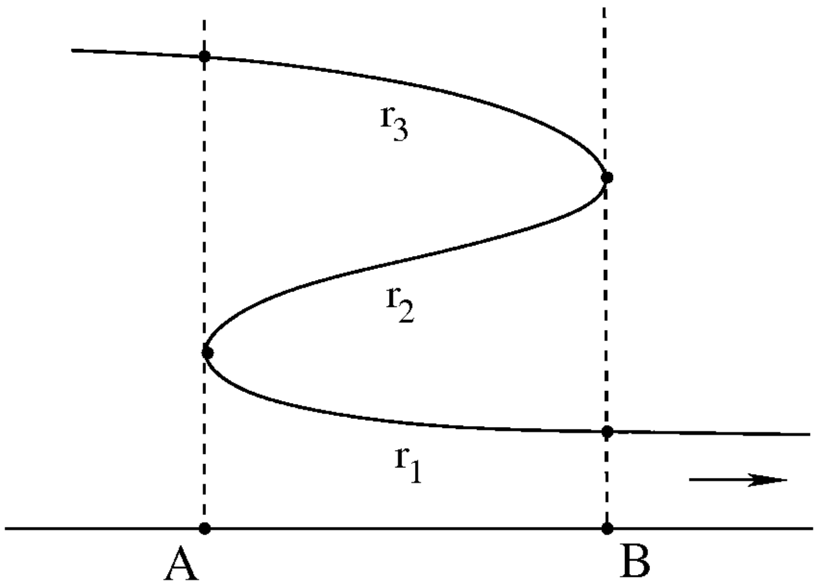
Three types of coordinates were used for Whitham Equations:
1.Physical Cooordinates, used for the averaging procedure: For $N = 1$ the averaged densities of energy, momentum and Casimir for the GZF Bracket are: $\bar{\epsilon}$ where $\epsilon = u_x/2 + u^3, H = \int \epsilon dx, \bar{p} = \bar{u^2}, c = \bar{u}$. Poisson Brackets are calculated in these coordinates.
2. The **flat coordinates** s.t. Riemannian Metric is constant. They exist by the Geometrization Theorem and give densities of Casimirs for the "Averaged" Hydrodynamic Type P.Bracket.
3.The Riemann Invariants used for the integration procedure.

Three types of coordinates were used for Whitham Equations:
1.Physical Cooordinates, used for the averaging procedure: For $N = 1$ the averaged densities of energy, momentum and Casimir for the GZF Bracket are: $\bar{\epsilon}$ where $\epsilon = u_x/2 + u^3, H = \int \epsilon dx, \bar{p} = \bar{u^2}, c = \bar{u}$. Poisson Brackets are calculated in these coordinates.
2. The **flat coordinates** s.t. Riemannian Metric is constant. They exist by the Geometrization Theorem and give densities of Casimirs for the "Averaged" Hydrodynamic Type P.Bracket.
3.The Riemann Invariants used for the integration procedure. They are are coming from the Periodic version of the Inverse Scattering Transform as branching points of the Spectral Riemann Surfaces for the finite-gap periodic/quasiperiodic Schrodinger operators.

The Dispersive Shock wave

The Dispersive Shock wave (defined and studied by physicists Gurevich and Pitaevski in 1970s) is a very special selfsimilar solution to the Whitham equation $(N = 1)$ joined at the boundary with cubical solution for $N = 0$

The Dispersive Shock wave (defined and studied by physicists Gurevich and Pitaevski in 1970s) is a very special selfsimilar solution to the Whitham equation $(N = 1)$ joined at the boundary with cubical solution for $N = 0$



KdV:

$$u_t + uu_x + u_{xxx} = 0$$

Whitham:

$$\dot{r_1} = -\left[\frac{r_1+r_2+r_3}{3} - \frac{2}{3}(r_2 - r_1)\frac{K(s)}{K(s)-E(s)}\right] r_{1x}$$

$$\dot{r_2} = -\left[\frac{r_1+r_2+r_3}{3} - \frac{2}{3}(r_2 - r_1)\frac{(1-s^2)K(s)}{E(s)-(1-s^2)K(s)}\right] r_{2x}$$

$$\dot{r_3} = -\left[\frac{r_1+r_2+r_3}{3} - \frac{2}{3}(r_3 - r_1)\frac{(1-s^2)K(s)}{E(s)}\right] r_{3x}$$

$$s^2 = \frac{r_2-r_1}{r_3-r_1}$$

Fig 5, *Dispersive Shock wave: at the endpoints* A, B *it is singular*

As a corollary from the complete integrability we proved that it is C^1-smooth (which is a strongly overdetermined property), As a corollary from the complete integrability we proved that it is C^1-smooth (which is a strongly overdetermined property), and calculated it by the exact formulas. Quite nonstandard singular boundary conditions are needed here at the ends joining Whitham equations for different values of N. A number of other problems (including the KdV-Burgers averaged system with weak viscosity) were investigated later.

Nonlinear Schrodinger System $NLS_{\pm}$:

Nonlinear Schrodinger System $NLS_{\pm}$: For the "stable" repulsive case $(+)$ it was investigated by D.Levermore et al. For the unstable case $(-)$ this problem remains unsolved.

Nonlinear Schrodinger System $NLS_{\pm}$: For the "stable" repulsive case $(+)$ it was investigated by D.Levermore et al. For the unstable case $(-)$ this problem remains unsolved.

Sine-Gordon System::

Nonlinear Schrodinger System $NLS_{\pm}$: For the "stable" repulsive case $(+)$ it was investigated by D.Levermore et al. For the unstable case $(-)$ this problem remains unsolved.
Sine-Gordon System:: The averaging procedure might lead to Nonhyperbolic Whitham Equation depending on the type of "finite-gap background".

Nonlinear Schrodinger System $NLS_{\pm}$: For the "stable" repulsive case $(+)$ it was investigated by D.Levermore et al. For the unstable case $(-)$ this problem remains unsolved.

Sine-Gordon System:: The averaging procedure might lead to Nonhyperbolic Whitham Equation depending on the type of "finite-gap background". If it is Hyperbolic, it can be completely investigated now because very effective description of finite-gap real solutions was obtained recently by P.Grinevich and myself.

Nonlinear Schrodinger System $NLS_{\pm}$: For the "stable" repulsive case $(+)$ it was investigated by D.Levermore et al. For the unstable case $(-)$ this problem remains unsolved.
Sine-Gordon System:: The averaging procedure might lead to Nonhyperbolic Whitham Equation depending on the type of "finite-gap background". If it is Hyperbolic, it can be completely investigated now because very effective description of finite-gap real solutions was obtained recently by P.Grinevich and myself. Such description was missing many years, so long period people could not use exact periodic solutions to the SG System.

Nonlinear Schrodinger System $NLS_{\pm}$: For the "stable" repulsive case $(+)$ it was investigated by D.Levermore et al. For the unstable case $(-)$ this problem remains unsolved.
Sine-Gordon System:: The averaging procedure might lead to Nonhyperbolic Whitham Equation depending on the type of "finite-gap background". If it is Hyperbolic, it can be completely investigated now because very effective description of finite-gap real solutions was obtained recently by P.Grinevich and myself. Such description was missing many years, so long period people could not use exact periodic solutions to the SG System.
Remark. In the latest works of several people (Dubrovin Krichever, Zakharov) geometric calculus borrowed from the Hamiltonian Theory of Hydrodynamic Type Poisson Brackets was used

Nonlinear Schrodinger System $NLS_{\pm}$: For the "stable" repulsive case $(+)$ it was investigated by D.Levermore et al. For the unstable case $(-)$ this problem remains unsolved.

Sine-Gordon System:: The averaging procedure might lead to Nonhyperbolic Whitham Equation depending on the type of "finite-gap background". If it is Hyperbolic, it can be completely investigated now because very effective description of finite-gap real solutions was obtained recently by P.Grinevich and myself. Such description was missing many years, so long period people could not use exact periodic solutions to the SG System.

Remark. In the latest works of several people (Dubrovin Krichever, Zakharov) geometric calculus borrowed from the Hamiltonian Theory of Hydrodynamic Type Poisson Brackets was used for the study of 2D Topological Quantum Field Theory and for the solution of the classical problem of Differential Geometry:

How to classify all orthogonal coordinates in Euclidean Space?

My homepage is www.mi.ras.ru/∼snovikov, click publications

All highlighted items can be taken directly. The items **nn 80,93-95, 107,109, 116, 121, 127,128, 130, 154, 155, 156** were used in this talk. I already mentioned the names of people who participated in these investigations.