# Modern Completely Integrable Systems

and

# New Discretisation of Complex Analysis

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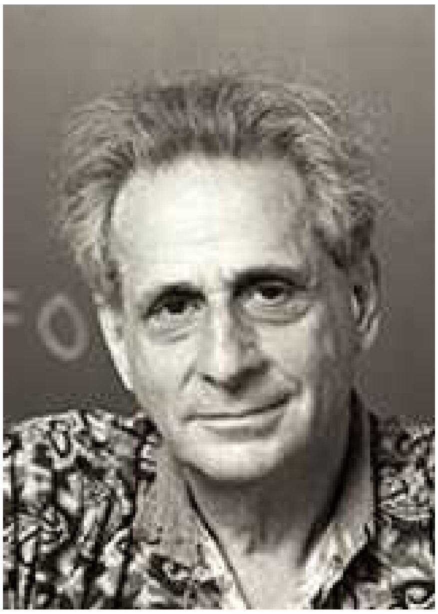
Homepage www.mi.ras.ru/~*snovikov* (click publications), items 137,140 148,159,163, collaborators: A.Veselov, I.Krichever, I.Dynnikov

# The Inverse Scattering Transform (IST): Discovery.



Martin Kruskal et al

(1965-67)

**Peter Lax (1968)**

**(1968)**

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**Numerics, Integrals: Kruskal and Zabuski, Soliton Interaction, 1965**

# The IST Solution: Gardner, Green, Kruskal, Miura, 1967

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Generalizations: Higher analogs of KdV

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**Hamiltonian Treatment of These Systems, 1971-...**

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Analog of IST for Periodic Boundary Conditions. Riemann Surfaces, Finite-Gap Operators and KdV Solutions, 1974-...

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2D Schrodinger $L = -\partial_x^2 - \partial_y^2 + A\partial_x + B\partial_y + W$ Parabolic $L = \sigma\partial_t + \partial_x + W$ Dirac L= ...

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## Example 1 (Factorization):

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Solve equation $a_x + a^2 + C = u(x)$
Isospectral Map: $QQ^* \to Q^*Q$
(Euler-Darboux-Backlund)

Example 2 (Weak Factorization):

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Conclusion: 2D Complex Analysis is similar to the Completely Integrable Systems. Is it possible to preserve this property after Discretization?

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## Discretization of Most Fundamental Linear Operators: How to preserve Spectral Symmetry?

For 1D Case we take:

$T(n) = n+1, L = c_n T + T^{-1} c_n + v_n$

The factorization $L = QQ^* + C$ is always possible,

$Q = a_n T + b_n, Q^* = T^{-1} a_n + b_n$.

Iso-spectral deformations $dL/dt = [A, L]$ appear ( "Toda Lattice" and "Volterra=Discrete KdV=..." for the subfamily $v_n = 0$).

2D Case and Quadrilateral (Square) Lattice: shifts $T_1(m,n) = (m+1,n), T_2(m,n) = (m,n+1)$: Take equation $L\psi = 0$:

$$L = a_{m,n} + b_{m,n}T_1 + c_{m,n}T_2 + d_{m,n}T_1T_2$$

The "Weakly Factorized" form is $f^{-1}L =$

$$(1+uT_1)(1+vT_2)+w = Q_1Q_2+w$$

and gauge group acts

$$L \sim f^{-1}Lg, \psi \sim g^{-1}\psi$$

Fig 1

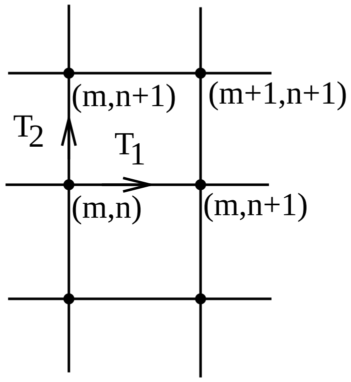
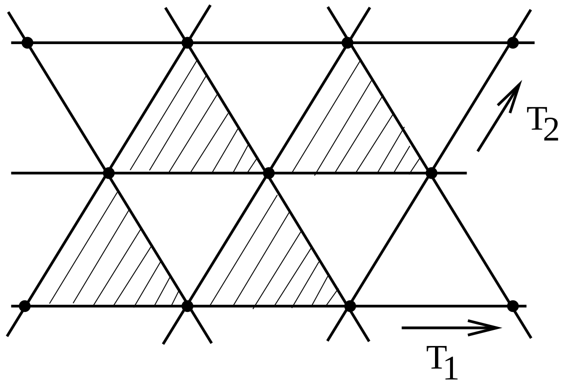



Fig 2



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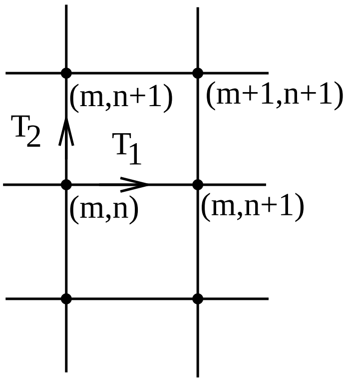
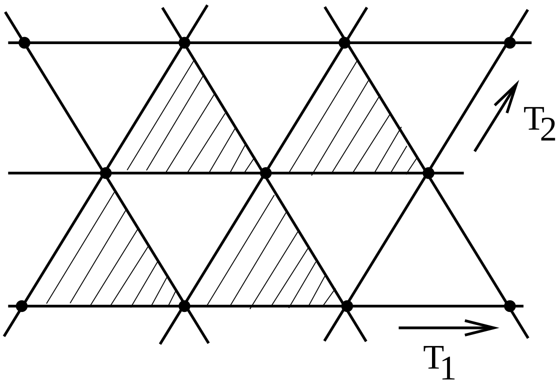



Fig 2




For 2D Case and Equilateral Triangle Lattice $L = a + bT_1 + cT_2 +$

$$+ dT_1^{-1}T_2 + T_1^{-1}b + T_2^{-1}c + T_2^{-1}T_1 d$$

The Weakly Factorized Form is:

$$\pm L = Q^{b*}Q^b + V, Q^b = u + vT_1 + wT_2$$

Example: Laplace -Beltrami Operator on The Equilateral Triangle Lattice:

$$-\Delta = Q^{b*}Q^{b} - 9$$

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so we have:

$$a = 6, b = c = d = -1$$

$$u = v = w = 1, V = -9$$

**Definition.** We call $Q^b = u + vT_1 + wT_2$ "Black Triangle Operator", and the adjoint operator $Q^{b*}$ "White Triangle Operator" $Q^w$ on the Equilateral Triangle Lattice.

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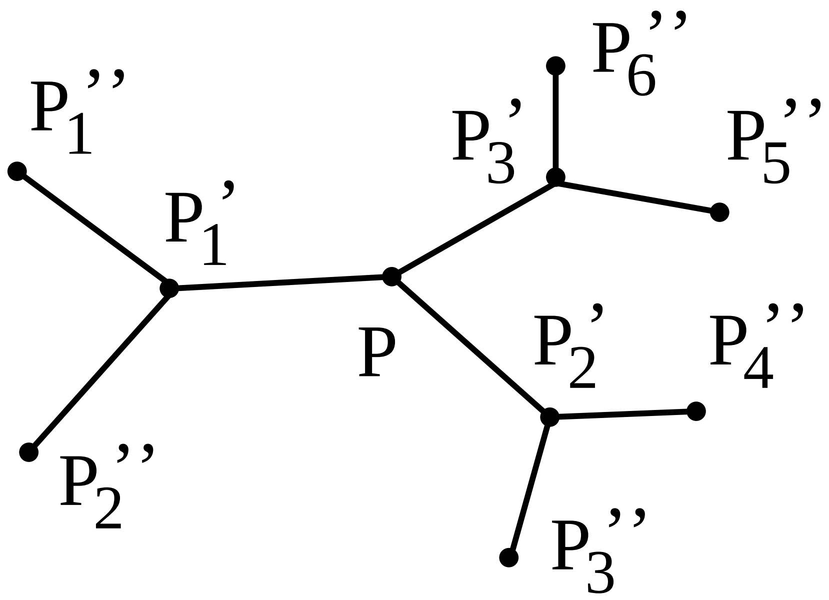
We call $Q^b\psi = 0$ "Black Triangle Equation" and $Q^w\psi = 0$ "White Triangle Equation".

Exotic Example:(S.N.-I.Krichever, 1999): For trivalent tree (see Fig 3) Every Self-adjoint Real 4th order operator is Weakly Factorizable: $L\psi(P) = \Sigma_i b_{PP_i''}\psi(P'')+$

$$+\sum_j b_{PP_j'}\psi(P_j') + V(P)\psi(P) =$$

$$= (QQ^{+} + v)\psi(P)$$

Fig 3




2nd order ball $B_2(P)$

Completely integrable systems appear on this graph. Nothing like that exists for the second order operators on this graph.

## $GL_n$ Connections=Overdetermined Systems of Linear Equations:

The Triangle Operators, Data:

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The Triangle Operators, Data:

1. Triangulated surface with selected family of triangles $X$; 2. Coefficients $b_{T:P} \neq 0$ for every Triangle $T \in X$ and vertex $P \in T$.

$$Q^X \psi(T) = \sum_{P \in T} b_{T:P} \psi(P)$$

acts on the functions of vertices.

B/W surfaces (The Discrete Conformal Structure): all triangles are colored into black and white colors. We have operators $Q^b$ and $Q^w$ where family $X = b$ consists of all black triangles, and $X = w$ of all white triangles.

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Another Example: $X = b \cup w$ is the set of all triangles $T$. We call corresponding triangle equation $Q\psi = 0$ "Discrete $GL_n$ Connection". What is Curvature?

Let all $b_{T:P} = 1$. We call solutions to the equation $Q^b\psi = 0$ on B/W Surfaces "Discrete (d) Holomorphic Functions".

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For Equilateral Triangle Lattice: We have

$$Q^b = 1 + T_1 + T_2$$

$$Q^w = 1 + T_1^{-1} + T_2^{-1}$$

Following picture explains how nontrivial curvature appears for such "connections" (see Fig 5). For every vertex $P$ we start from the vertex $P_1$ in its star. Knowing $\psi(P)$ and $\psi(P_1)$ we calculate all $\psi(P_i)$ "along the circle" for $n = 2$ in the star. Contradiction might appear after returning to the original point $P_1$ as a triangle matrix $C_P$. We call $C_P$ "curvature operator". Holonomy is defined for the Thick Paths. Important Case: $b_{T:P} = 1$ and $n = 2$. " The Zero Curvature" property $C_P = 1$ simply means that even number of triangles enter $P$. For the case $b_{T:P} = 1$ and $C_P = 1$ holonomy belongs to the permutation group $S_n$.

Fig 4

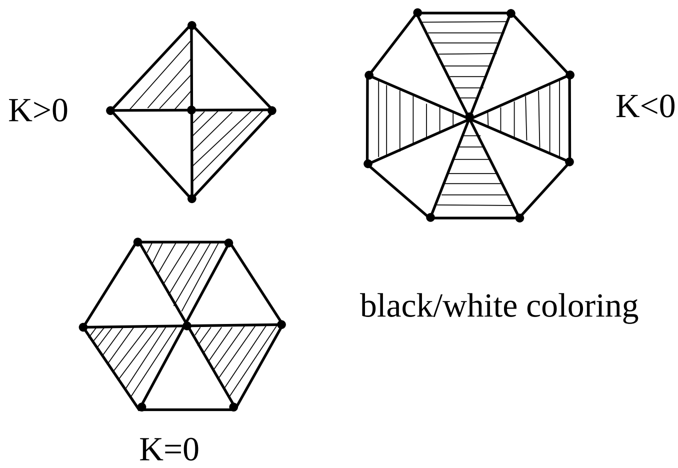
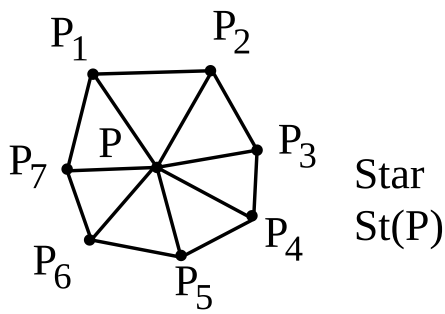



Fig 5




Theory of curvature was developed recently.

## New Discretization of Complex Analysis.

Classical discrete complex analysis is based on the quadrilateral lattice (Lelong-Ferrand, 1940). Weak Points: 1.Discrete Analog of Cauchy-Riemann Operator $\bar{\partial}$ is in fact a second-order difference operator. 2.Factorization Property is missing here.

Our Discretization is based on the properties of EquilateraL Triangle Lattice with Factorization

$$\Delta = -Q^{b*}Q^{b}+9, Q^{b} = 1+T_1+T_2$$

In both approaches d-holomorphic functions Do Not Form a Ring

For every 2-manifold with B/W triangulation and $b_{T:P} = 1$ we define d.(i.e. discrete) holomorphic functions as real functions satisfying to the equation: $Q^b\psi = 0$ and d.anti-holomorphic functions $Q^w\psi = 0$

''The Covariant Constants'' are such functions that $Q\psi = 0$ i.e.:

$$Q^b\psi = 0, Q^w\psi = 0$$

$$-2\Delta + 3m_P = Q^*Q =$$

$$= 2Q^{b*}Q^b = 2Q^{w*}Q^w$$

Here $m_P$ is equal to the number or triangles entering $P$ where $6 - m_P$ is a "Scalar Curvature" of the Triangulated Surface. For $m_P = const$ the zero modes of $Q^*Q$ coincide with maximal modes of Laplace-Beltrami Operator $\Delta$.

Let us remind "The Instanton Trick": For factorizable operators 0-minima of functional $(L\psi, \psi)$

satisfy to "smaller (self-duality) equation": $Q^*Q\psi = 0$ implies $(Q\psi, Q\psi) = 0$ implies $Q\psi = 0$. Therefore d-holomorphic function on compact surface is covariant constant:

$Q^b\psi = 0$ implies $Q^{b*}Q^b\psi = 0$ implies $Q^*Q\psi = 0$ implies $(Q\psi, Q\psi) = 0$ implies $Q^b\psi = 0, Q^w\psi = 0$.

So discrete analog of Liouville Principle is true here.

We assume now that the space of covariant constants is $R^2$.

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Continuous Limit: Take covariant constant $f_0$ whose values are $1, \zeta, \zeta^2$ where $\zeta^3 = 1$. Use Gauge $L \to f_0^{-1} L f_0, \psi \to f_0^{-1}\psi$ such that one covariant constant became ordinary constant. Extend field to $C$. In the continuous limit one half of our theory converges to the ordinary complex analysis, second half is divergent for the small scales.

Maximum Principle is also true: Consider finite domain $D$ consisting of black triangles $T$. The Evaluation Map $E_\psi(T)$ treats d-holomorphic functions as $R^2$-valued functions of black triangles: it assigns to black triangle $T$ with vertices $P, P', P''$ unique covariant constant $R^2$ defined by the triple $\psi(P), \psi(P'), \psi(P'')$ on $T$ Theorem. The image $E_\psi(D))$ coincides with the convex hull of the image of boundary triangles.

Previous results are true for all B/W surfaces.

**D-Holomorphic Polynomials and Taylor Series** We work now with equilateral triangle lattice in the plane with shifts $T_1, T_2$ (see Fig 6). Our operators $Q^b, Q^w$ map here the space of functions of vertices into itself: $Q^b = 1 + T_1 + T_2, Q^w = (Q^b)^* = 1 + T_1^{-1} + T_2^{-1}$

How to define polynomials without multiplication?

We call d.holomorphic function Polynomial of degree $k$ if

$$(Q^w)^{k+1}\psi = 0$$

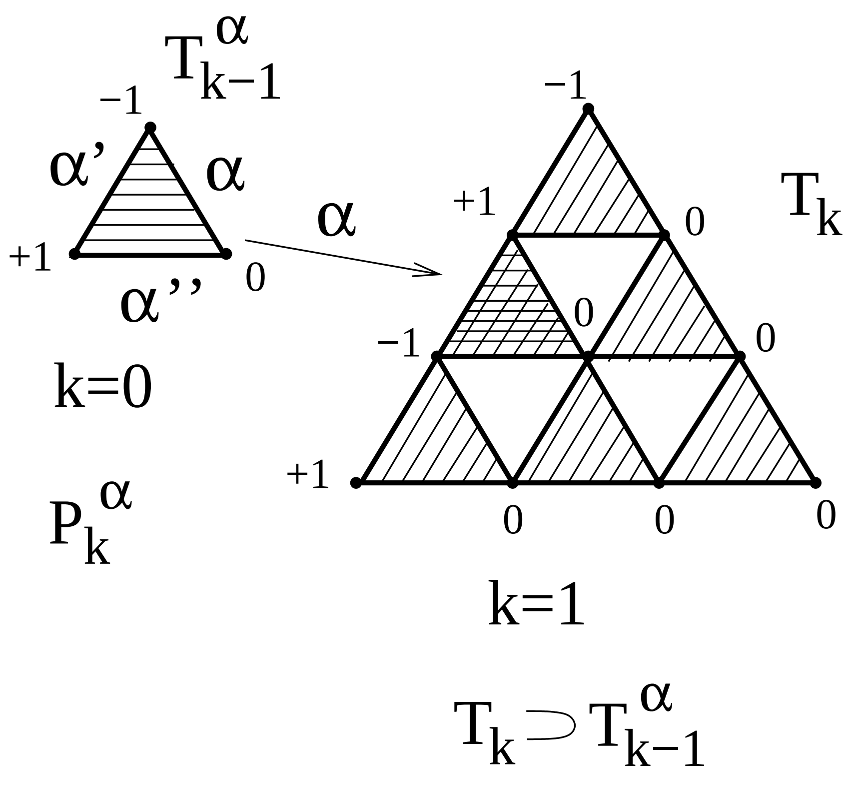
d-analog of Ball here is any big equilateral triangle $T_k$ whose edges are black from inside and contain exactly $2k+2$ vertices (see Fig 6).

Theorem (The Taylor Approximation).

For every d.holomorphic function $\psi$ and big triangle $T_k$ there exists exactly one holomorphic polynomial $P_k$ of degree $k$ such that $\psi - P_k = 0$ in the triangle $T_k$.

The space $H_k$ of holomorphic polynomials has dimension $2k+2$ over $R$. Its basis can be chosen using "Balls", see Fig 6.

Fig 6



How to define d-analog of Cauchy Kernel $1/z$?

Cauchy Formula.

Let $\psi$ be d.holomorphic in the bounded domain $D$ in the equilateral triangle lattice. We can easy construct fundamental solution $G(x-y)$ such that

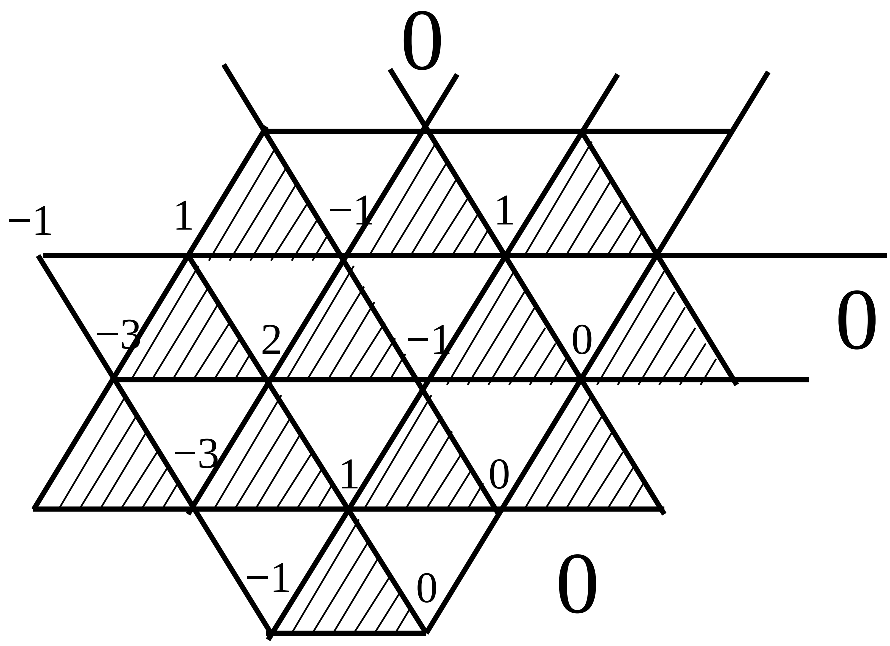
$$Q^b G(x-y) = \delta(x-y)$$

where $x = (m,n)$ and $\delta(x) = 1, 0 = x$, and zero otherwise.

One such function is given in Fig 7. It is equal to zero for all $x = (m, n)$ where $m > 0$ or $n > 0$. Its values at the boundary are $(-1)^m$ in the points $(-m, 0)$ and $(-1)^n$ in the points $(0, -n)$ and $G = (-1)^{m+n}\frac{(m+n)!}{m!n!}$ for $m < 0, n < 0$ (The Pascal Triangle).

Is it right analog of $1/z$? It is OK for Cauchy Formula but has exponential growth in some directions.

Fig 7




"Pascal Triangle" G(x) x=(m,n)

Let $\psi$ is d-holomorphic in finite domain $D$. Take function $\tilde{\psi} = \psi$ in $D$ and zero outside. The function $Q^b\tilde{\psi}$ is concentrated along the boundary $\partial D$ which is a "strip".

Theorem. Following Cauchy Formula is valid for $x \in D$:

$$\sum_y (Q^b\tilde{\psi}(y))G(x-y) = \psi(x)$$

Any Green function can be used here. Our function looks more hyperbolic than elliptic. Recently Grinevich and R.Novikov found "really elliptic" function $G(x-y)$ decreasing for $|x-y| \to \infty$. Such Green function (The Cauchy Kernel) is unique. It can be simply found by the Fourier Transform. They obtained a number of results using it. So all rational functions are naturally defined in our theory

## Hyperbolic (Lobatchevslki) Plane.

Recently we started to develop d-complex analysis for the equilateral lattices on hyperbolic plane. Neither analogs of Taylor Polynomials nor Grinevich-R.Novikov type Green function are known here. We have negative curvature if number of edges entering every vertex is $m_P > 6$. In our case it should be even number. For the homogeneous triangulations with $m_P = 8, 10, 12, \ldots$ we have a big group preserving triangulation. Let us concentrate on the minimal case $m_P = 8$.

Problem: How to describe boundary of $r$-ball for every integer $r$?

A picture is presented below for $r = 0, 1, 2$.

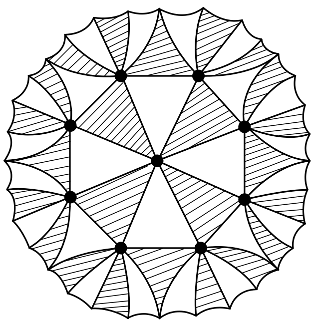


Fig 8 r=0,1,2

We define a class of the Right-Convex oriented simplicial paths–see Fig 9a,b,c,d. Their local picture from the right side is following by definition

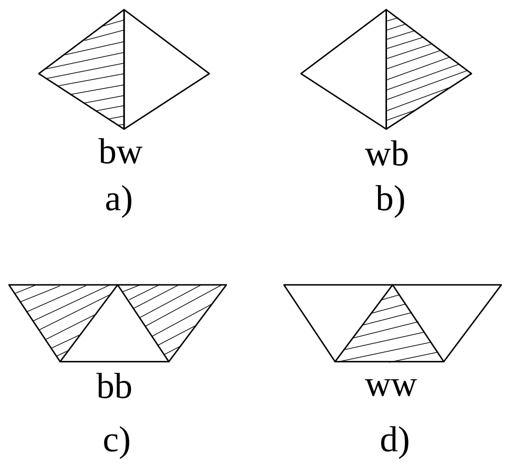



Fig 9

We are coding right-convex oriented paths by the words in 2 symbols $b, w$ assigning $bw$ to fig 9a, $wb$ to fig 9b, $bb$ to fig 9c and $ww$ to fig 9d.

Let us introduce Structural Transformation $T$ on the space of infinite periodic right-convex oriented paths by the formulas $T: bw \longrightarrow bwbw$, $wb \longrightarrow wbwb$, $bb \longrightarrow bwb$,$ww \longrightarrow wbw$ We apply $T$ to every pair of neighboring letters in the word and after that delete old letters.

For the word $R_1 = ...bwbwbwbw...$ which is a boundary of 1-ball,we have

$T(R_1) =$

$= ...bwbwwbwbbwbwwbwbbwbwwbwbbwbwwbwb... = R_2$

Lemma 1. The image of right-convex path exactly coincides with the right-convex path which is a closest neighbor from the left side. In particular,

$$T^r(R_1) = R_r = \partial D^r, r \geq 1$$

for $r$-balls $D_r$

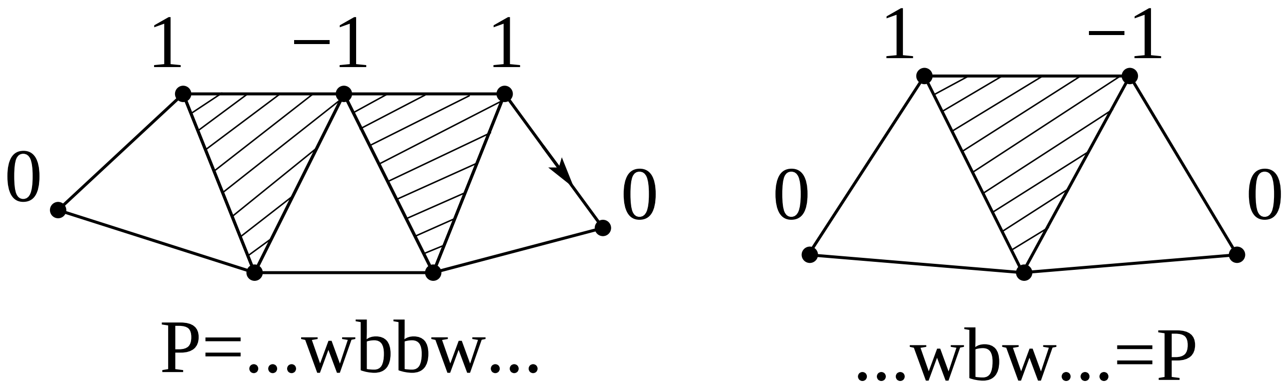
This type of maps are standard for people working in symbolic dynamics. Mike Boyle from the University of Maryland helped me:

Lemma 2. For every word $A$ we have: $|T(A)|/|A|$ asymptotically equal to $2 + \sqrt{3}, |A| \to \infty$ . This asymptotic almost exact for $r \geq 4, A = R_r$

We have $|R_1| = 8, |R_2| = 32, |R_3| = 120, |R_4| = 448, |R_5| = 1672, R_6 = 6230, \ldots$

Construct basis of d-holomorphic functions $z_P^r(x)$ such that $z_P^r = 0$ for all points $x$ in $R_k, k < r$ and for all points in the path $R_r$ except of the selected place $P \subset R_r$ where $P = wbbw$ or $P = wbw$ (see Fig 10 for the values of these functions in $P$)

Fig 10



Conjecture: There exists basis of d-holomorphic functions $z_P^r$ which are globally bounded in the Hyperbolic Plane. Their linear combinations are similar to polynomials $\sum_{k=0}^n a_k z^k$ in the unit disc (Poincare' model in the continuous case).

Theorem. Dimension of the space of d-holomorphic functions restricted to the boundary $\partial D_r = R_r$, is equal to $1 + |R_r|/2$

It is quite similar to the continuous case. On the boundary $R_r$ linear span of these spaces is exactly all space of functions, and their intersection is exactly covariant constants.