Operators on Graphs and Lattices: Their Factorizability

New Discretisation of Complex Analysis: Equilateral Triangle Lattices in Euclidean and Hyperbolic Planes

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items 136,137,140 148,159,163,

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Part I: Discrete Symmetries and Completely Integrable Systems.

The famous completely integrable systems like KdV and many others are associated with linear operators. Following "strong and weak factorization" properties of 1D and 2D second order operators play fundamental role here:

$$(1D) L = -\partial_x^2 + u(x) = QQ^+ + c$$

$$L\psi = \lambda\psi, Q = \partial_x + a(x)$$

$$a_x + a^2 = u - c.$$

(2D,a),"hyperbolic": $L = -Q_1 Q_2 + V$

$$= -(\partial_x + A)(\partial_y + B)) + V, L\psi = 0$$

(2D,b),"elliptic": $L = QQ^+ + V =$

$$-(\partial + A)(\bar{\partial} + B) + V, L\psi = 0$$

"Darboux Transformation" (Euler, 1742) for $D = 1$ follows from the strong factorization,

$$L \to \tilde{L} = Q^{+}Q + c, \psi \to Q^{+}\psi = \tilde{\psi}$$

preserves all solutions for all λ (except may be one such that $Q^{+}\psi_0 = 0$.)The isospectral deformations $dL/dt = [A, L]$ lead to KdV and other famous systems.

The Laplace transformation for $D = 2$ follows from the weak factorization,

$$2D, a : \psi \to Q_2\psi, L \to VQ_2V^{-1}Q_1 + V$$

and similar for (2D,b), replacing

$$Q_1 \to Q, Q_2 \to Q^+ = -\bar{\partial} + B$$

. The whole chain L_n of Laplace transformations $L_n \to L_{n+1}$ with potentials $V_n =$ exp$\{f_{n+1} - f_n\}$ is equivalent to the 2D Toda Lattice $f_{n,xy} =$ or $f_{n.z\bar{z}} =$

$$= \text{exp}\{f_{n+1} - f_n\} - \text{exp}\{f_n - f_{n-1}\}$$

. One-level analog of iso-spectral deformations appears here $dL/dt = [A, L]+ BL$ leading to the 2D analogs of KdV like the so-called NV(Novikov-Veselov) Hierarchy with better properties than KP.

Part II: Discretization of Linear Operators.

What is the best discretization? We are looking for the discretizations of linear operators preserving the discrete symmetries described above. For 1D case we have a shift operator $T : n \to n+1$ and take such class of operators

$$L = c_n T + T^{-1} c_n + v_n$$

that factorization $L = QQ^{+} + c$ is always possible, $Q = a_n T + b_n, Q^{+} = T^{-1} a_n + b_n$. Iso-spectral deformations

$dL/dt = [A, L]$ appear here like "Toda Lattice" or "discrete KdV" for the sub-family $v_n = 0$.

For the 2D case there are two different discretizations. Take the shifts $T_1(m, n) = (m + 1, n), T_2(m, n) = (m, n + 1)$:

Hyperbolic (see Fig 1): We take a square lattice and equation $L\psi = 0$

$L = a_{m,n} + b_{m,n}T_1 + c_{m,n}T_2 + d_{m,n}T_1T_2$

They admit gauge transformations:

$$L \to fLg, \psi \to g^{-1}\psi$$

where the functions f, g are nonzero everywhere,so L depends on two gauge invariant functions. We always can present L in the form $L =$

$$f((1+uT_1)(1+vT_2)+w) = f(Q_1Q_2+w)$$

implying the same gauge-invariant Laplace transformation as above.

Fig 1

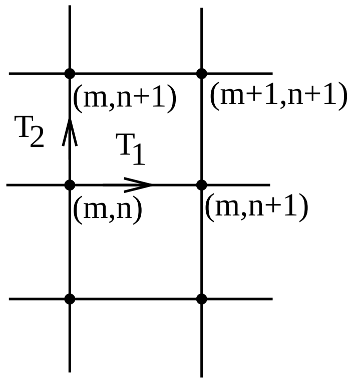
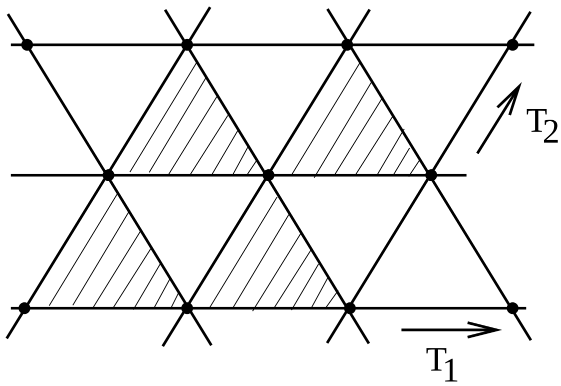


Fig 2



Elliptic, real self-adjoint (see Fig 2): We take an equilateral triangle lattice and operators of the form $L=$

$$a+bT_1+cT_2+dT_1^{-1}T_2+T_1^{-1}b+T_2^{-1}c+T_2^{-1}T_1d$$

We always can factorize them

$$L=QQ^{+}+V, Q=u+vT_1+wT_2$$

so the Laplace transformations are well-defined.

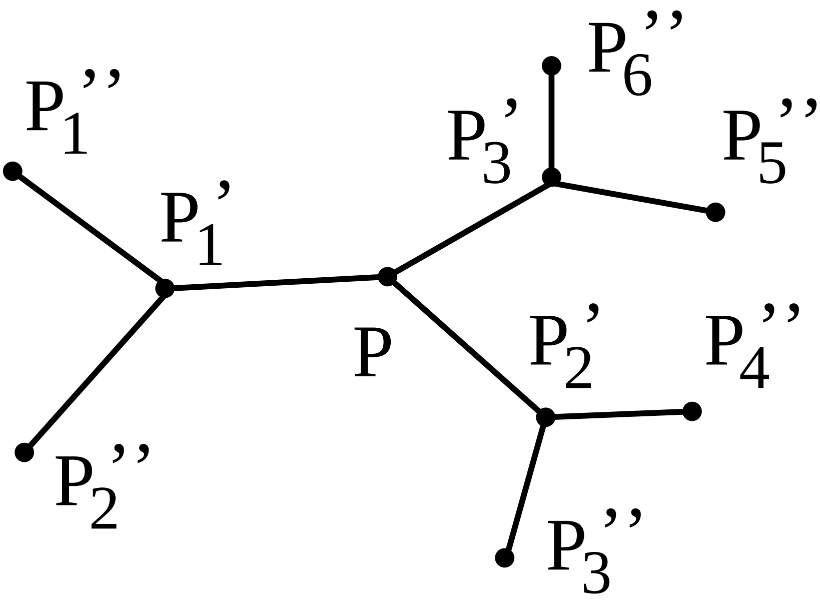
Definition. We call $Q=u+vT_1+wT_2$ "Black Triangle Operator" and Q^{+} "White Triangle Operator" on the Equilateral Triangle Lattice. We call

$Q\psi = 0$ "Black Triangle Equation" and $Q^{+}\psi = 0$ "White Triangle Equation".

Another interesting class is following(S.N.-I.Krichever, 1999): Consider a trivalent tree (see Fig 3) and any real self-adjoint real 4th order operator $L\psi(P) =$

$$\sum_i b_{PP_i''}\psi(P'') + \sum_j b_{PP_j'}\psi(P_j') + V(P)\psi(P)$$

Fig 3



2nd order ball $B_2(P)$

Such operators can be factorized $L = QQ^+ + v$ through the second order operators Q, and completely integrable systems appear on this graph. Nothing like that exists for the second order operators L on this graph.

Part III: Discrete GL_n Connections and Triangle Equation.

Let K be a simplicial complex (n-manifold) with fixed family of n-simplices X, and set of coefficients is fixed $b_{T:P} \neq 0$ for every n-simplex $T \in X$ and its vertex $P \in T$.Following Triangle Operator

$$Q^X \psi(T) = \sum_{P \in T} b_{T:P} \psi(P)$$

is defined on the functions of vertices.

Three families $X-$black, white, all$-$ will be especially considered: Let all n simplices of K are colored into black and white colors. We have operators Q^b and Q^w where X is the set of black (or white) simplices. Another example is the case where X is simply set of all n-simplices $T \in K$. We call corresponding triangle equation $Q\psi = 0$ "Discrete GL_n Connection". We call solutions to the equation $Q^b\psi = 0$ for $n = 2, b_{T:P} = 1$ "Discrete Holomorphic Functions", and for $Q^w\psi = 0$ we call them "Discrete Anti-Holomorphic Functions". (see Fig 4). Following picture explains how nontrivial curvature appears for such "connections" (see Fig 5). For every vertex P we start from the vertex P_1 in its star. Knowing $\psi(P)$ and $\psi(P_1)$ we calculate all $\psi(P_i)$ "along the circle" for $n = 2$ in the star. However, contradiction might appear after returning to the original point P_1 in the form of non-unit triangle matrix C_P. We call it "curvature operator". For $b_{T:P} = 1$ and $n = 2$ " the zero curvature" property $C_P = 1$ simply means that even number of edges (triangles) enter P. Holonomy is defined here along the "thick paths". For the case $b_{T:P} = 1$ and $C_P = 1$ holonomy belongs to the permutation group S_n.

Fig 4

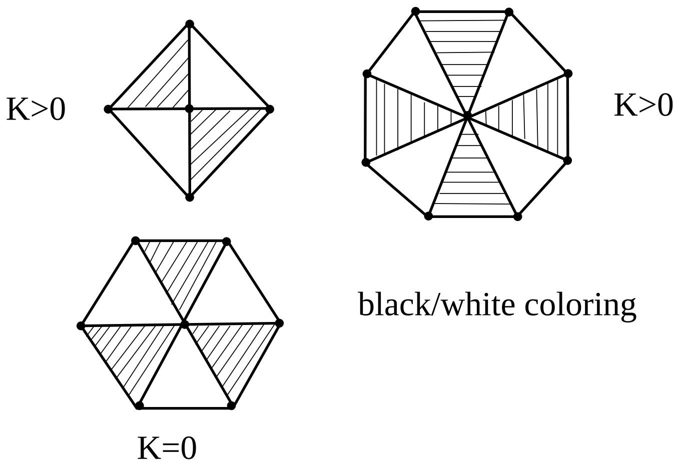
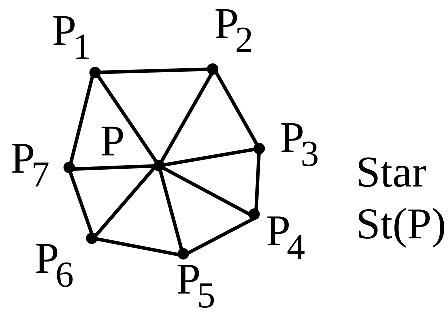


Fig 5



Theory of curvature was developed recently.

Part IV: New Discretization of Complex Analysis. Classical discretization of complex analysis is based on the square lattice (Lelong-Ferrant, 1940). Our ideas are based on the properties of equilateraL triangle lattice.

For every 2-manifold with black/whit triangulation and $b_{T:P} = 1$ we define d.(i.e. discrete) holomorphic functions as real functions satisfying to the equation: $Q^b\psi = 0$ and d.anti-holomorphic functions $Q^w\psi = 0$. Here ψ is real.

"The Covariant Constants" are such functions that $Q\psi = 0$:

$$Q^b\psi = 0, Q^w\psi = 0$$

For the standard Laplace-Beltrami Operator $L_0 = \partial\partial^*$ we have

$$-2L_0 + m_P = Q^+Q = 2Q^{b+}Q^b = 2Q^{w+}Q^w$$

where m_P is equal to the number or edges (triangles) entering P. So for $m_P = const$ the zero modes of Q^+Q coincide with maximal modes of Laplace-Beltrami L_0. For compact manifolds they are exactly Covariant Constants: $Q^{+}Q\psi$ = implies $(Q\psi, Q\psi) = 0$ and $Q\psi = 0$. So every d-holomorphic function on compact manifold is covariant constant: $Q^b\psi = 0$ implies $Q^{b+}Q^b\psi = 0$ implies $Q^{+}Q\psi = 0$ implies $(Q\psi, Q\psi) = 0$ implies $Q^b\psi = 0, Q^w\psi = 0$. We call it Liouville Principle. We assume now that the space of covariant constants is exactly 2-dimensional. (May be finite covering is needed for that).

Continuous limit: Take covariant constant f_0 whose values in every triangle are $1, \zeta, \zeta^2$ where $\zeta^3 = 1$. Make the transformation

$$L \to f_0^{-1} L f_0, \psi \to f_0^{-1} \psi$$

After that our theory should be considered over the complex field C. One of covariant constants became the ordinary constant. In the continuous limit one half of our theory converges to the ordinary complex analysis but second half of this discrete theory is divergent for the small scales. We are working in the most symmetric purely discrete gauge form over the field R imitating all complex analysis.

Maximum Principle

Let ψ be holomorphic function in a finite domain D consisting of black triangles whose vertices all belong to D. A Boundary Triangle is such that at least one of its vertices belongs to some black triangle outside of D. The Evaluation Map $E_\psi : T \to R^2$ assigns to black triangle with vertices P, P', P'' a vector in the space of covariant constants R^2 defined by $\psi(P), \psi(P'), \psi(P'')$.

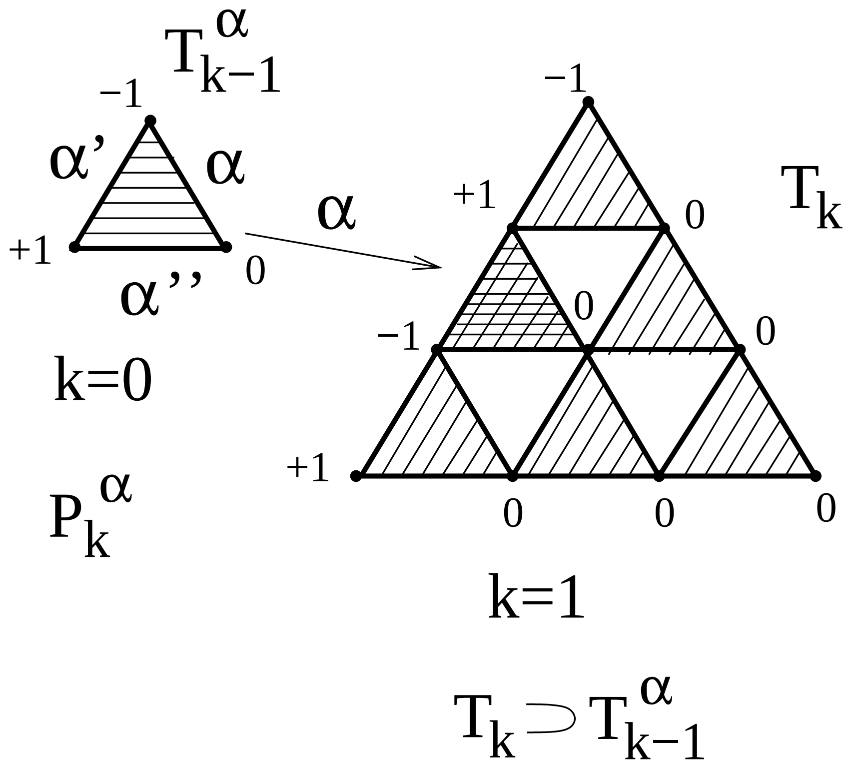
Theorem. The image $E_\psi(D))$ coincides with the convex hull of the image $E_\psi(\partial D)$.

D-Holomorphic Polynomials and Taylor Series

Consider now equilateral triangle lattice in the plane with shifts T_1, T_2 and natural b/w coloring as above (see Fig 6). Our operators Q^b, Q^w map here the space of functions of vertices into itself. We have We call d.holomorphic function ψ Holomorphic Polynomial of degree k if $(Q^w)^{k+1}\psi = 0$ Consider big equilateral triangle T_k whose edges are black from inside and contain exactly $2k+2$ vertices (see Fig 6).

Theorem (The Taylor Approximation). For every d.holomorphic function ψ and big triangle T_k there exists exactly one holomorphic polynomial P_k of degree k such that $\psi - P_k = 0$ in the triangle T_k. The space H_k of holomorphic polynomials has dimension $2k+2$ over R.

Fig 6



The choice of basis of holomorphic polynomials depends on T_k. There are 3 functions $P_k^{\alpha}(T_k), \alpha = 1, 2, 3$, equal to zero in T_k except one boundary edge with number α. Along this edge P_k^{α} is equal to $1, -1, 1, -1, ..., -1$. We have (see Fig) Lemma.

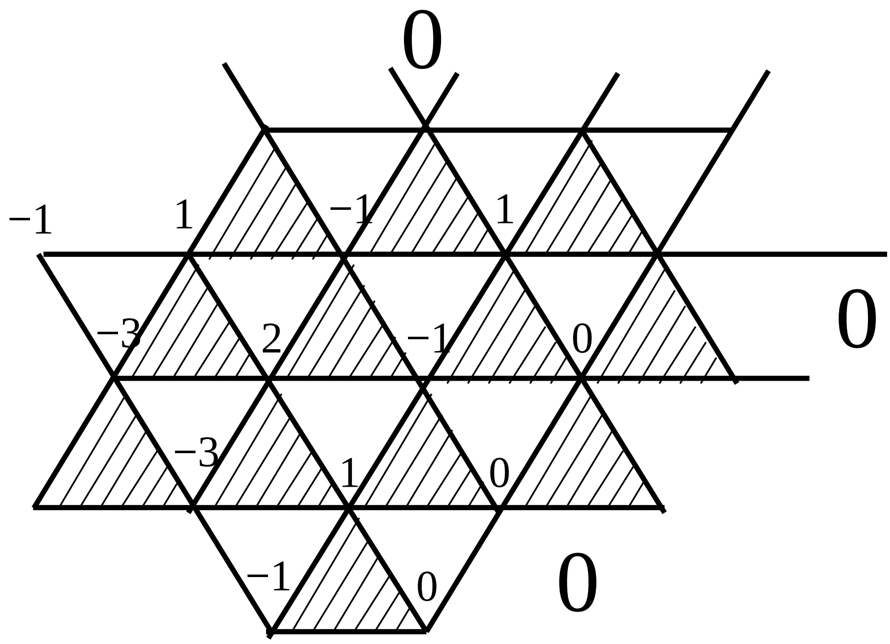
$$P_k^1 + P_k^2 + P_k^3 = P_{k-1} \in H_{k-1}, \alpha = 1, 2, 3$$

Here $P_{k-1} = P^{\alpha}_{k-1}$ corresponds to the triangle T^{α}_{k-1} (see Fig 6).

Cauchy Formula.

Let ψ be d.holomorphic in the bounded domain D in the equilateral triangle lattice. We can easy construct fundamental solution $G(x-y)$ such that $Q^b G(x-y) = \delta_y(x)$ where $x = (m, n)$ and $\delta_y(x) = 1, y = x$ and zero otherwise. For $y = 0$ such function is given in Fig 7. It is equal to zero for all $x = (m, n)$ where $m > 0$ or $n > 0$. Its values at the boundary are $(-1)^m$ in the points $(-m, 0)$ and $(-1)^n$ in the points $(0, -n)$ and $G = (-1)^{m+n}\frac{(m+n)!}{m!n!}$ for $m < 0, n < 0$ (The Pasqual Triangle).

Fig 7



"Pascal Triangle" G(x) x=(m,n)

Take function $\tilde{\psi} = \psi$ in D and zero outside. The function $Q^b\tilde{\psi}$ is concentrated along the boundary ∂D which is a "strip".

Theorem. Following Cauchy Formula is valid for $x \in D$:

$$\sum_y (Q^b\tilde{\psi}(y))G(x-y) = \psi(x)$$

Any Green function can be used here. Our function looks more hyperbolic than elliptic. Recently Grinevich and R.Novikov found "really elliptic" function $G(x-y)$ decreasing for $|x-y| \to \infty$. Such Green function (The Cauchy Kernel) is unique. It was missed in previous work of the present author and I.Dynnikov: Fourier Transform is convergent in our case:

$$G_{m,n} = \int_0^{2\pi} \int_0^{2\pi} dk_1 dk_2 \times$$

$$\exp\{imk_1 + ink_2\}/(1 + e^{ik_1} + e^{ik_2})$$

They obtained a number of results using this function.

Hyperbolic (Lobachevski) Plane.

Recently we started to develop d-complex analysis for the equilateral lattices on hyperbolic plane. Neither analogs of Taylor Polynomials nor Grinevich-R.Novikov type Green function are known here. We have negative curvature if number of edges entering every vertex is $m_P > 6$. In our case it should be even number. For the homogeneous triangulations with

$m_P = 8, 10, 12, ...$ we have a big group preserving triangulation. Let us concentrate on the minimal case $m_P = 8$.

How to describe boundary of r-ball for every integer r?

A picture is presented below for $r = 0, 1, 2$.

Fig 8

We define a class of the Right-Convex oriented simplicial paths–see Fig 9a,b,c,d. Their local picture from the right side is following by definition

Fig 9

We are coding right-convex oriented paths by the words in 2 symbols b, w assigning bw to fig 9a, wb to fig 9b, bb to fig 9c and ww to fig 9d. Let us introduce Structural Transformation T on the space of infinite periodic right-convex oriented paths by the formulas $T: bw \longrightarrow bwbw$, $wb \longrightarrow wbwb$, $bb \longrightarrow bwb$,$ww \longrightarrow wbw$ We apply T to every pair of neighboring letters in the word and after that delete old letters.

For the word $R_1 = ...bwbwbwbw...$ which is a boundary of 1-ball,we have $T(R_1) = ...bwbwwbwbb...$ $wbwwbwbbwbwwbwbbwbwwbwb... = R_2$

Lemma 1. The T-image of right-convex path coincides with the right-convex path which is a closest neighbor from the left side. In particular,

$$T^r(R_1) = R_{r+1} = \partial D^{r+1}, r \geq 1$$

for r-balls D_r

Such maps are standard for experts in symbolic dynamics. Mike Boyle from the University of Maryland helped me a lot to investigate it.

Lemma 2. For every word A we have: $|T(A)|/|A|$ asymptotically equal to $2+\sqrt{3}, |A| \to \infty$. This asymptotic behavior is almost exact for $r \geq 4, A = R_r$

We have $|R_1| = 8, |R_2| = 32, |R_3| = 120, |R_4| = 448, |R_5| = 1604, \dots$

Construct basis of d-holomorphic functions $z_P^r(x)$ such that $z_P^r = 0$ for all points x in $R_k, k < r$ and for all points in R_r except of the selected place $P \subset R_r$. Here $P = wbbw$ or $P = wbw$ (see Fig 10 for the values of these functions in P)

Fig 10

Conjecture: There exists basis of d-holomorphic functions z_P^r which are globally bounded in the Hyperbolic Plane. Their linear combinations are similar to polynomials $\sum_{k=0}^{n} a_k z^k$ in the unit disc in the continuous case). Another basis: Fix zero point and right-convex line $l = ...bbbbbb...$ passing through 0. Construct specific d-holomorphic function $h^{0,l}$ equal to zero from the right side of l and equal to ± 1 along the line

l (see Fig 11). Its continuation to the left side of l is nonunique. Make an "optimal" continuation to the left side (i.e. with minimal possible growth). What kind of growth is it? One can easily construct basis of d-holomorphic functions using this specific function $h^{0,l}$ and its group shifts.

Fig 11

Theorem. Dimension of the space of d-holomorphic functions in the r-ball D_r is equal to $1 + |R_r|/2$

It is quite similar to the continuous case. Similar basis of d-antiholomorphic functions can be also constructed $\bar{z}^r_P(x)$ replacing b by w in the previous definition and in the Fig 10. On the boundary R_r these spaces generates all space of functions; their intersection is exactly space of covariant constants.