Fourier Transform, Riemann Surfaces and Indefinite Metric

P. G. Grinevich, S.P.Novikov

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L.D.Landau Institute for Theoretical Physics RAS, Russia
and University of Maryland at College Park, College Park, USA

What is Fourier Transform in Riemann Surfaces? Which Problems need it?

Discrete Analog of The Fourier/Laurent bases in Riemann Surfaces was constructed by Krichever-Novikov (KN,1986-1990) for The Operator Quantization of The Closed String. The Ideas were borrowed from The Theory of Finite-Gap Potentials in The Theory of Solitons where Riemann Surfaces appear as Spectral Curves.

Continuous Fourier Transform on Riemann Surfaces was constructed in our work (GN, 2003).

As we found recently (GN, 2008-2009), this transform preserves an Indefinite Inner Product for genus $g > 0$, for the cases where Fourier Transform has good Multiplicative Properties. The Operators are Singular here.

The ordinary Fourier Transform:
Basic functions have 2 fundamental properties:

$\Psi_n(k) = (k)^n$, $x = n \in \mathbb{Z}, |k| = const$ (discrete)
$\Psi(x,k) = \exp(ikx)$, $x, k \in \mathbb{R}$ (continuous)

a) They form an orthonormal basis

b) They have graded multiplicative law:

$$\Psi_n(k)\Psi_m(k) = \Psi_{m+n}(k), \quad \Psi(x,k)\Psi(y,k) = \Psi(x+y,k)$$

Here Riemann Suface $\Gamma = S^2$ has genus $g = 0, \lambda = ik = z^{-1}, \tau = |\log z|$ belongs to the "Canonical Contour" κ_c on Riemann Surface $\tau = c$ in the discrete case. In the continuous case $Imk = c$, and especially interesting is the Special Canonical Contour κ_0.

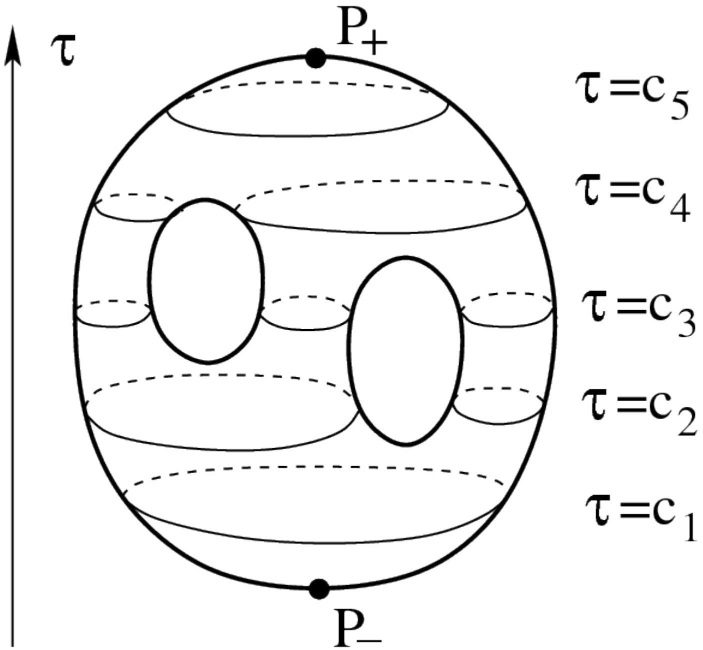
Only this contour will be considered for the Continuous Fourier Transform

Discrete Case: Fourier/Laurent (KN) bases on Riemann surfaces.

Krichever I.M.; Novikov S.P. 1. Algebras of Virasoro type, Riemann surfaces and the structures of soliton theory. 2. Virasoro type algebras, Riemann Surfaces and String in Minkowski Space. 3. Algebras of Virasoro type, energy-momentum tensor and decomposition operators on Riemann Surfaces, *Funkts. Anal. i ego Pril.* **21** (1987), no. 2, 46–63; **21** (1987), no. 4, 47-61; **23** (1989), no. 1, pp. 24-40.

4.Krichever I.M., Novikov S.P.,Riemann Surfaces, Operator Fields, Strings. Analogues of Fourier-Laurent bases. In the Memorial Volume of V.Kniznik: Physics and Mathematics of Strings, pp 356-388, Editors L.Brink, D.Friedan, A.Polyakov, World Scientific, Singapore 1990

The String Diagram $(\Gamma, P_+, P_-, k_+, k_-)$:



Here $1/k_+$, $1/k_-$ are local parameters near "The Infinite Points" P_- ("in")and P_+ ("out") respectively,
dp is defined as meromorphic differential with 2 simple poles at P_+, P_-

$dp = dk_+/k_+ + O(1)$,

$dp = -dk_-/k_- + O(1)$

$\mathrm{Re} \oint dp = 0$ for all closed paths.

The "time" is $\tau = \mathrm{Re}\, p$

$\tau(P_+) = +\infty, \tau(P_-) = -\infty$

An analogue of discrete Fourier bases is defined for functions (tensor fields) at the contour $\kappa_c : -\infty < \tau = c < +\infty$

An analogue of Laurent basis for the holomorphic functions is defined for the domains between 2 contours $\kappa_{c'}$ and $\kappa_{c''}$ where $c' < c''$. All constructions are extended to the tensor fields with any tensor weights. The tensors with weights equal to 0,1,-1,2,1/2 are especially important for the string theory.

Krichever and Novikov introduced these bases to construct operator quantization of bosonic string. For this problem it was critical to have bases with good multiplicative properties.

These bases are defined by the following asymptotics at the points P_+, P_- where $k_\pm(\lambda) = \infty$:

$$\Psi_j(\lambda) = \begin{cases} k_+^{j+g/2}(c_j^+ + o(1)) & \lambda \to P_+ \\ k_-^{-j+g/2}(c_j^- + o(1)) & \lambda \to P_- \end{cases}$$

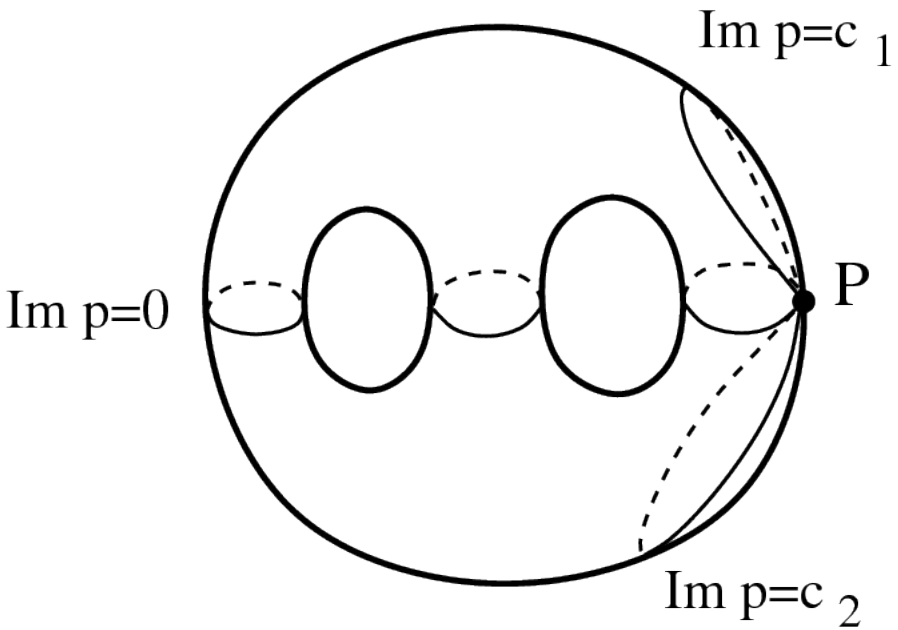
(We present here the case of scalar functions only, $j \in Z$ for $g = 2s$ and $j \in Z + 1/2$ for $g = 2s + 1$ and j big enough.)

The multiplication rule is Almost Graded:

$$\Psi_l(\lambda)\Psi_m(\lambda) = \sum_{n=l+m-N}^{n=l+m+N} C^n_{lm}\Psi_m(\lambda)$$

where $N = N(g)$, does not depend on l, m, C^n_{lm} do not depend on λ.

Continuous analogues of the Krichever-Novikov bases. Grinevich P.G., Novikov S.P. *Communications on Pure and Applied Mathematics*, **56**, Issue 7 (2003), pp. 956-978.



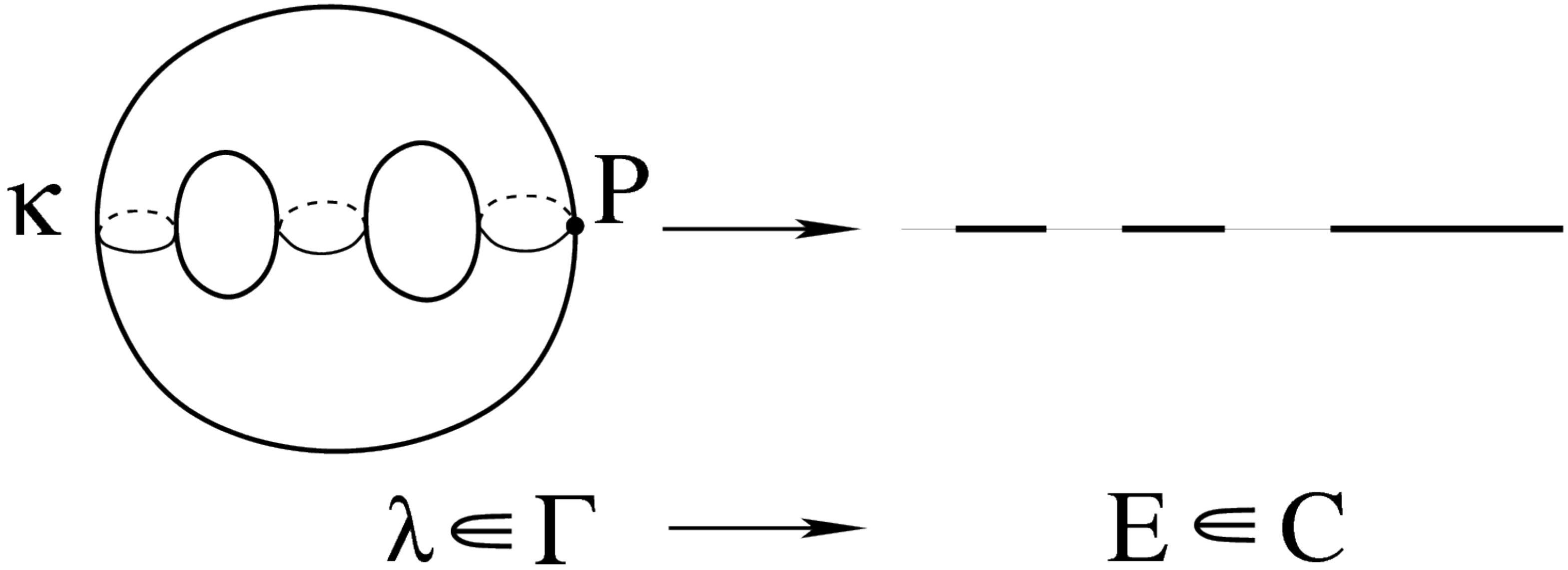
Let $z = 1/k$ be local parameter near P, define dp as a meromorphic differential with a second-order pole at P
$dp = dk + O(1)$,
$\mathrm{Im} \oint dp = 0$ for all closed paths.
$\tau = \mathrm{Im}\, p$ is well-defined.

The Special Canonical Contour is $\kappa_0 : \tau = \mathrm{Im}\, p = 0$.

The standard finite-gap inverse spectral data:

1) A compact Riemann surface Γ of genus g with an "infinite" point P and local parameter $z = 1/k$ near $P, z(P) = 0$. 2) A collection of points $\gamma_1, \ldots, \gamma_g$ (the poles of ψ-function), $D = \gamma_1 + \ldots + \gamma_g$.

3)The "reality conditions" should be added for Γ and poles. These data were found in 1974 by S.Novikov and others for the Finite-Gap 2nd Order Operators and KdV Solutions where Γ is hyperelliptic (2-sheeted over λ-plane, see survey article B.Dubrovin, V.Matveev, S.Novikov ....(1976), and generalized to the general Riemann Surfaces by I.Krichever in 1976 discovering the KP solutions, see survey article I.Krichever.....(1977). Let us point out that Theta-functional calculation of hyperelliptic Ψ-function on Riemann Surface was incorrectly attributed to Baker in this article. In fact, Baker in his note published in 1928 described its analytical properties and mentioned that it can be calculated using technic developed in his book but actually never realized this possibility. It was calculated first time by A.Its in the Appendix to the previous survey article.



1)The eigenfunction $\Psi(\lambda, x)$, $\lambda \in \Gamma$, $x \in \mathbb{R}$ is meromorphic in $\Gamma \backslash P$ with simple poles $\gamma_1, \ldots, \gamma_g$, $\Psi(\lambda, x_0) = 1$.
2. $\Psi(\lambda, x) = (1 + o(1)) \exp(ik(x - x_0)), \lambda \to \infty$.

Let $g = 0$ and $\Gamma = C \bigcup \infty, P = \infty$. Here k is the standard coordinate. Then $p = k$, $\Psi(\lambda, x) = \exp(ikx)$ is the standard Fourier basis on the real line $\mathrm{Im}\, k = 0$

A continuous analog of the Fourier bases (Grinevich-Novikov, 2003).
Let $\gamma_1 = \ldots = \gamma_g = P, x_0 = 0$; ψ-functions form an almost-

graded algebra (here $c_0 = 1, c_1 = \zeta(x+y) = \sigma'/\sigma$ for $g = 1$):

$$\Psi(\lambda, x)\Psi(\lambda, y) = \sum_{j=0}^{g} c_j(x, y)\partial_z^j \Psi(\lambda, z)\big|_{z=x+y}$$

We study functions of λ, and x is a parameter numerating our basic functions.

The functions $\Psi(\lambda, x)$ are singular in x. They have a pole at $x = 0$. For example, the classical periodic Lame' operator $-\partial_x^2 + g(g+1)\wp(x)$ is exactly a special case here for all $g > 0$.
Physical Soliton Theory (KdV) deals with regular finite-gap operators $-\partial_x^2 + g(g+1)\wp(x + i\omega')$ where $2\omega'$ is an imaginary period ("traveling wave" for $g = 1$).

Our aim is to answer the following question:

Do such singular operators have a reasonable spectral theory on the whole real line x?

Classical people since Hermit considered their spectrum only at the interval $[0, T = 2\omega]$ with zero boundary conditions. We need the whole line for the Fourier Transform with good multiplicative properties.

The Baker-Akhiezer ψ-functions for regular real periodic operators never form an almost-graded multiplicative system for $g > 0$. We need singular operators for Good Fourier Transform

Consider real finite-gap periodic (regular or singular) operator

$$L = -\partial_x^2 + u(x).$$

Γ is real and defined by: $\mu^2 = (E - E_1)\cdots(E - E_{2g+1})$
Permutation of sheets is defined $\sigma(E,\mu) = (E,-\mu), \sigma^2 = \mathrm{id}$

The regular case corresponds to the following data:

1) All E_j are real. Let $E_1 < E_2 < \ldots < E_{2g+1}$

2) Each closed interval $[E_{2j}, E_{2j+1}]$, $j = 1,\ldots,g$ contains exactly one pole: $\lambda_j \in [E_{2j}, E_{2j+1}]$, here λ_j denotes the projection of γ_j to the E-plane.

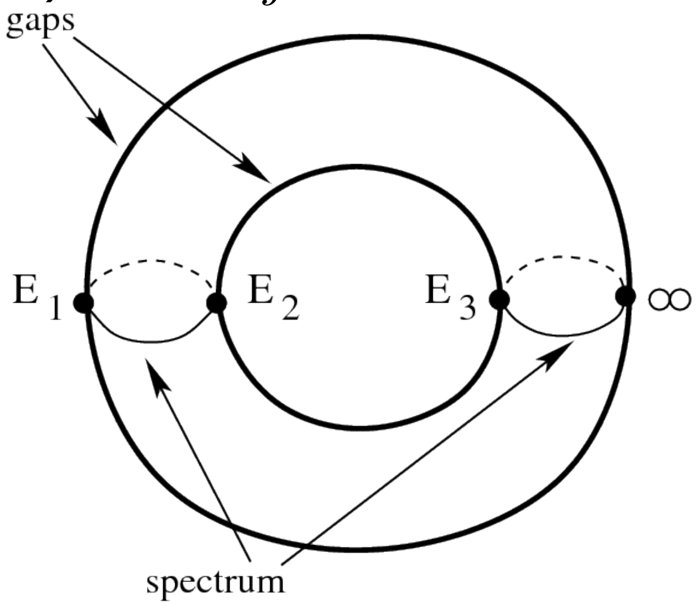
The real singular case corresponds to the following data:

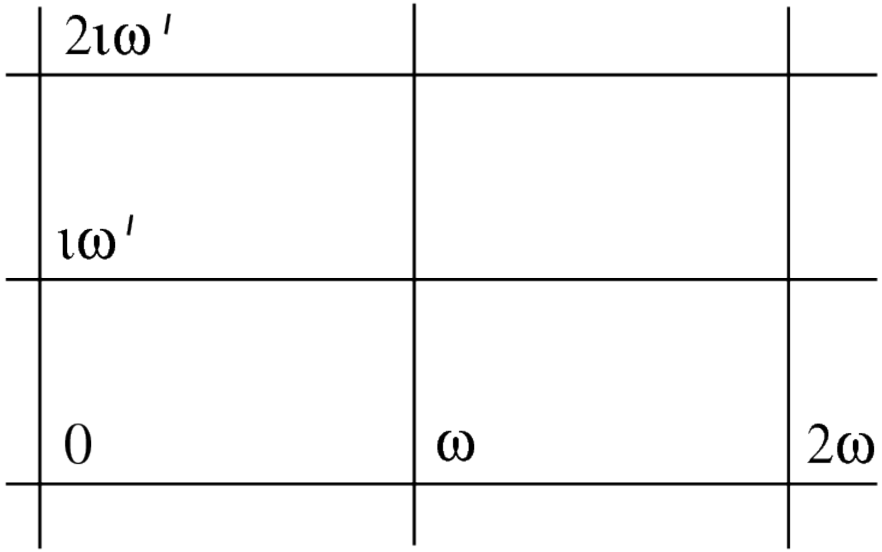
1) Γ is real i.e. collection of branching points is invariant under complex conjugation. Let $\tau(E,\mu) = (\bar{E},\bar{\mu})$, $\tau^2 = \mathrm{id}$.

2) Collection of poles is also "real": the whole collection is invariant under τ.

Basic Example: Let $g = 1$ (Γ is a torus):

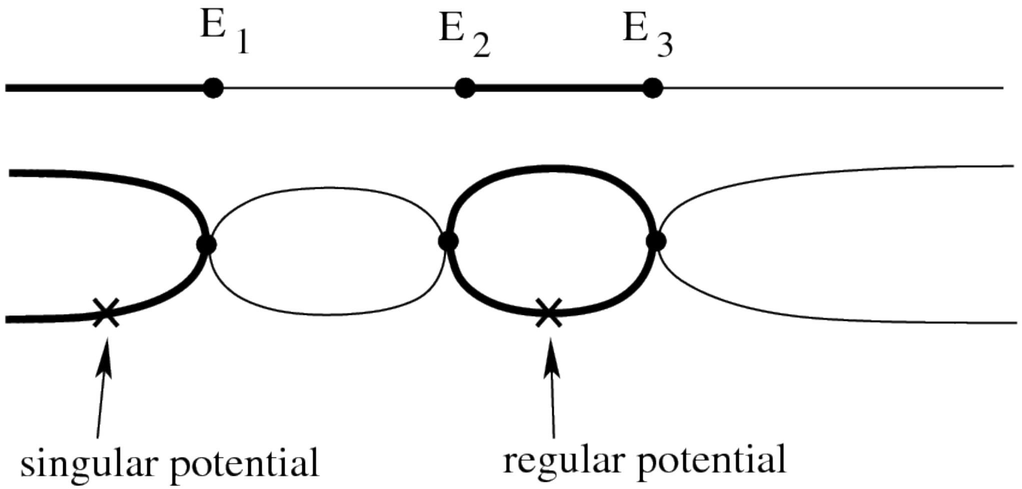
1) All E_j are real, $j = 1, 2, 3$:





The lattice of periods of the Weierstrass $\wp$ -function is **rectangular with periods** $2\omega, 2i\omega'$.

The gaps are $[-\infty, E_1]$ and $[E_2, E_3]$

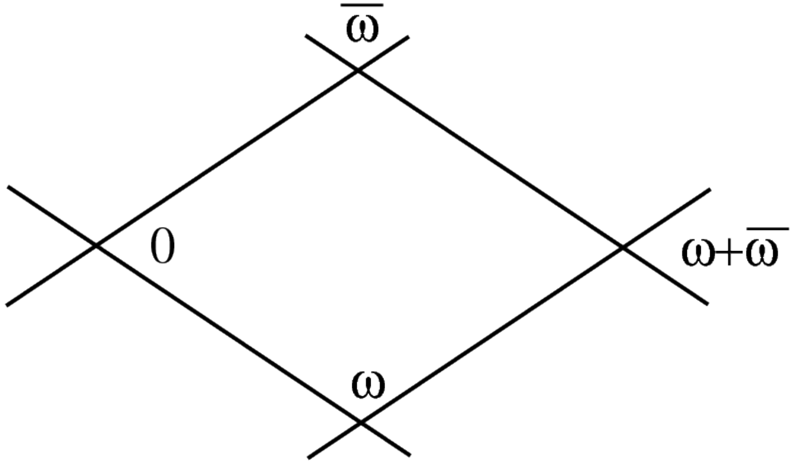


κ_0 is represented by fine lines.

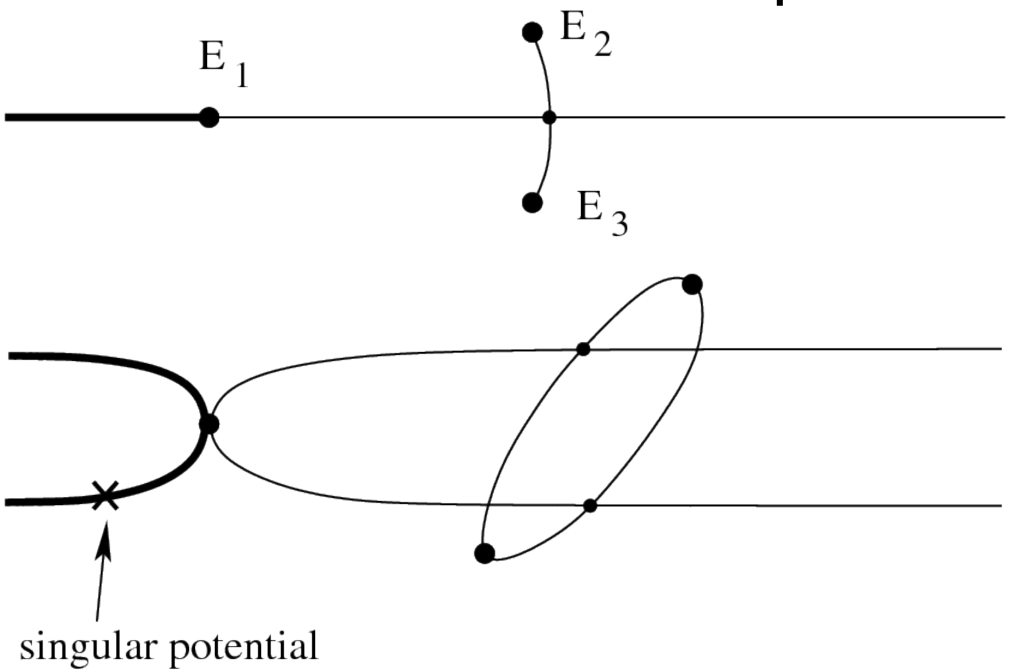
The contour κ_0 has 2 components here: infinite and finite. There is only one pole γ: For regular case it belongs to the finite gap, for the singular case it belongs to the infinite gap (The Shifted Hermit -Lame Operators).

In both cases the spectrum on the whole line is a union of 2 real sets $[E_1, E_2] \bigcup [E_3, \infty]$ (projection of κ_0) but eigenfunctions and functional spaces on the x-line are drastically different

2)Let E_1 is real, $E_3 = \overline{E_2}$:



The lattice of periods is **rombic**.



κ_0 given by fine lines.

The spectrum on the whole line coincides with the projection of the contour κ_0 on the $E - line$. It contains complex (nonreal) arc joining $E_2, E_3 = \bar{E}_2$. Spectral meaning of singular operators on the whole line was not discussed before.

Direct and Inverse Spectral Transform

We invent following "spectral measure" for $\lambda = (E, \pm) \in \Gamma$

$\Psi^*(\lambda, x) = \Psi(\sigma\lambda, x)$;$\gamma_j = (\lambda_j, \mu_j)$=poles,

$$d\mu = \frac{(E-\lambda_1)\dots(E-\lambda_g)dE}{2\sqrt{(E-E_1)\dots(E-E_{2g+1})}},$$

For every smooth function $\phi(\lambda), \lambda \in \kappa_0$, with decay sufficiently fast at $\lambda \to P$, we define A Spectral Transform:

$$\tilde{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\kappa_0} \phi(\lambda) \Psi^*(\lambda, x) d\mu(\lambda) \tag{1}$$

and Inverse Spectral Transform:

$$\phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\phi}(\lambda) \Psi(\lambda, x) dx \tag{2}$$

We call it Fourier Transform if all $\lambda_j = \infty$;
Here $d\mu^{Fourier} = dE/2\sqrt{(E - E_1) \ldots (E - E_{2g+1})}$,
and our basis has good multiplicative properties
In the regular case Spectral Transform is an isometry between the spaces $L^2(\kappa_0)$ and $L^2(\mathbb{R})$ with inner

products

$$< \psi_1, \psi_2 >_{\kappa_0} = \int_{\kappa_0} \psi_1(\lambda)\overline{\psi}_2(\lambda) d\mu(\lambda)$$

and

$$< f_1, f_2 >_{\mathbb{R}} = \int_{\mathbb{R}} f_1(x)\overline{f}_2(x) dx.$$

(The “measure” $d\mu$ is the standard ”spectral density” $1/2\chi_R(E, x_0), \chi = \chi_R + i\chi_I = \Psi'/\Psi$.)

The Singular Potentials:

1) Formula for the Spectral Transform remains valid; For the Inverse Transform it remains valid after a natural regularization.

2) Spectral Transform is an isometry between the spaces with

indefinite metric described below.
All singularities have a form

$$u(x) = n_j(n_j+1)/(x-x_j)^2 + O(1), n_j \in Z. \quad (3)$$

The function $\Psi(\lambda, x)$ is meromorphic in x. For $\lambda', \lambda'' \in \Gamma$ all residues of the product $\Psi(\lambda', x)\Psi(\lambda'', x)$ are equal to 0.
The Scattering Data for potentials with singularities of the type $2/x^2$ were constructed in:
Arkad'ev, V.A., Pogrebkov, A.K., Polivanov, M. K., *Journal of Soviet Mathematics*, **31**, Number 6, (1985), pp. 3264-3279.
The Indefinite Inner Product was not discussed.
All residues in the formula for the Inverse Spectral

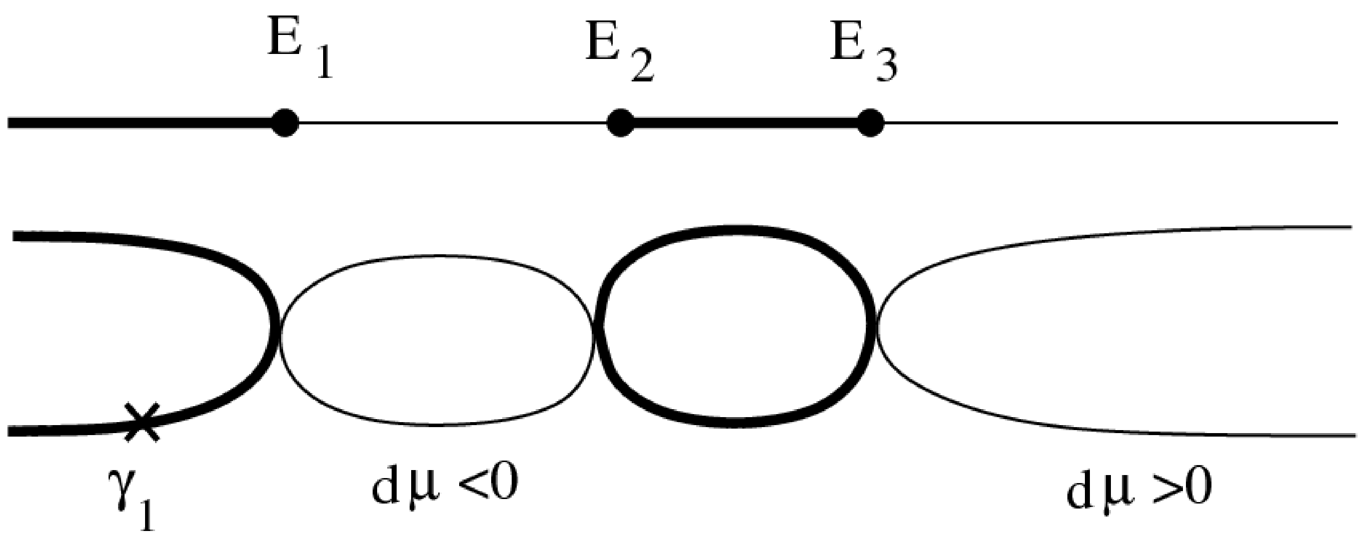
Transform are equal to 0. Our Regularization: If we meet singularity under the integral, we go around it in the complex domain. We can go above or below, but the result is the same.

The Inner Product on the Riemann Surface (in the space of functions in κ_0)

$$< \psi_1, \psi_2 >_{\kappa_0} = \int_{\kappa_0} \psi_1(\lambda)\overline{\psi}_2(\tau\lambda) d\mu$$

1)All branching points are real: τ acts identically on κ_0, the form $d\mu$ is negative somewhere. For Fourier Transform $d\mu^{Fourier}/dp < 0$ in $[(g+1)/2]$ finite components of the contour κ_0.

2)Some pair of branching points is complex adjoint, τ is not identity in κ_0 : the inner product is nonlocal and therefore indefinite.



The Inner Product on the space of functions in $\mathbb{R}$

$$< f_1, f_2 >_{\mathbb{R}} = \int_{\mathbb{R}} f_1(x)\overline{f}_2(\bar{x})dx$$

These functions belong to the image of the Spectral Transform. For the Generic case all singularities have the form $u(x) \sim 2/(x - x_j)^2$. Locally the functions f_1, f_2 have the form:

$$f(x) = d_{-1}/(x - x_j) + d_1(x - x_j) + \dots$$

We write $\bar{x}$ instead of x for inner product to make both terms holomorphic. Residues in all singularities are equal to 0 for the products under integral, therefore we go around it in the complex domain either above or below. This scalar product is indefinite.

Pontryagin-Sobolev (PS)spaces: Every function $f(x)$ can be written for real x as

$$f(x) = \int_0^{2\pi/T} \hat{f}(p,x)dp,$$

where $f(p, x + T) = \exp(ipT)f(p, x)$. Therefore $L^2(\mathbb{R})$ is represented as a direct integral of Bloch-Floquet spaces $B_\varkappa$:

$$f(x) \in B_\varkappa \quad \text{if} \quad f(x+T) = \varkappa f(x), \quad |\varkappa| = 1.$$

Our inner product has finite number r of negative squares in $B_\varkappa$, so it is PS. For Fourier case $r = [(g+1)/2]$.

This result allows to estimate the number r' of singularities for $u(x)$ at the real line.

Example: KdV deformations. Take:

$$u(x,0) = g(g+1)\wp(x), u(x,0) = g(g+1)/x^2, g \in Z$$

Apply KdV dynamics to this potentials. We obtain

$$u(x,t) = \sum_{j=1}^{g(g+1)/2} 2\wp(x - x_j(t))$$

Question: Calculate the number r' of real poles $x_j(t)$. Result remains the same taking $g(g+1)/x^2$ instead of $g(g+1)\wp(x)$

Our Argument:

For $t = 0$ the number r of negative squares of our inner product is equal to $[(g+1)/2]$. The number of negative squares is stable, therefore we have al least $r' \geq [(g+1)/2]$ real poles. We checked in many cases numerically that $r' = r$ for all g.

Existence of one real pole was proved many years ago: Adler M., Moser Ju. On a class of polynomials connected with the Korteweg-de Vries Equation. *Comm. Math. Phys.* (1978) pp 1-30.

Final remark: Singular Bloch-Floquet eigenfunctions are known also for the $k+1$-particle Moser-

Calogero operator with Weierstrass elliptic pairwise potential if coupling constant is equal to $n(n+1), n \in Z$. They form a k-dimensional complex algebraic variety. The Hermit-type result is not obtained here yet for $k > 1$: no one function was constructed until now serving the discrete spectrum in the bounded domain inside of the poles. Our case corresponds to $k = 1$. We believe that for all $k > 1$ this family of eigenfunctions also serves spectral problem in some indefinite inner product in the proper space of functions defined in the whole space $\mathbb{R}^k$.