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Dedicated to Boris Dubrovin's

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Homepage www.mi.ras.ru/~$snovikov$ (click publications),

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collaborators: P.Grinevich, A.Mironov

On the 2D Nonrelativistic Purely Magnetic Pauli Operator (spin 1/2) :

The Algebro-Geometric Theory of the Ground Level

In 1979-1980 three groups of authors completely calculated the ground states using following property of the 2D Pauli Operator with zero electric field (Avron-Seiler[AS], Aharonov-Casher[AC], Dubrovin and myself[DN]) using appropriate units and gauge conditions: $L_P = L \oplus \tilde{L} =$

$$= QQ^+ \oplus Q^+Q, \; Q = \partial_z + A$$

Here $\partial_z = \partial = \partial_x + i\partial_y$ and magnetic field $B = 1/2(A_{\bar{z}} - \bar{A}_z)$

The most interesting classes of magnetic fields are

1.AC: Rapidly decreasing fields with flux $|[B]| = |\int_{R^2} B dx dy| < \infty$. The ground states form a finite-dimensional space of dimension $m \in Z, m \leq [B] < m+1$

2.DN: Periodic fields with integer flux through the elementary cell $\int_{cell} B dx dy = [B] = m \in Z$. The ground states form an infinite dimensional subspace in the Hilbert Space $L_2(R^2)$ isomorphic to the Landau level for the same value of the magnetic flux.

In both cases ground states are The Instantons belonging to one spin-sector only:

a. They satisfy to the 1st order equations $Q^{+}\psi = 0$ for the case $[B] > 0$ and $Q\psi = 0$ for the case $[B] < 0$.

b. They belong to the Hilbert Space $L_2(R^2)$

In the latest literature started in 1980s this operator was associated with the "Super-Symmetry" operator $L \to \tilde{L}$ (known under the name "Laplace Transformation" in this case since XVIII Century). It implies only that all higher levels are double-degenerate having representatives in the both spin-sectors.

In the work of the present author with A.Veselov (1997) nontrivial periodic cases were found such that some higher levels are also infinitely degenerate similar to the ground level $\epsilon = 0$.

Question: Is Theory of Ground Level for the Purely Magnetic Pauli Operators related somehow to the Algebro-Geometric Theory of the scalar 2D Schrodinger Operators based on the Selected Energy Level? Do Corresponding 2D Soliton Hierarchies have Reduction of that kind?

The Algebro-Geometric Spectral Theory of the 2D Second Order Scalar Schrodinger Operators and Corresponding Soliton Hierarchies based on the selected energy level were started in 1976 by Manakov(M) and Dubrovin, Krichever and myself (DKN). Except the "reality conditions" for coefficients, Nontrivial Reductions were not considered till 1980s.

The Reduction Problem.

Nontrivial Reduction Problems were actively developed in 1980s. Several authors found them either for Nonlinear Systems or for Inverse Spectral Data (or for both). Solution of this problem for the Inverse Data is normally more valuable because it implies description of all hierarchy

Our Goal here is Quantum Mechanics and Spectral Theory.

The key problem is to find The Inverse Problem Data leading to the meaningful class of self-adjoint operators L. Let us mention following results here: 1.Cherednik found in 1980 Data leading to the Algebro-Geometric Periodic Operators with real smooth "topologically trivial" magnetic fields and electric potential (with applications to the finite-gap Sine-Gordon solutions developed by several Russian and US authors including Dubrovin, Natanzon and myself; 20 years later Grinevich and myself completely solved this problem)

2. The Algebro-Geometric Data leading to the zero magnetic field were found by Veselov and myself (1983). Extension of these results to the Rapidly Decreasing Potentials was studied in the collection of works by Manakov, Grinevich, R.Novikov and myself in the late 1980s. The so-called "Big Norm Problem " here was solved only for the levels below the Ground Level by Grinevich and myself (1989).

2D Soliton Hierarchies were studied and used for the different goals including 3D Geometry and problems of Θ-functions (Dubrovin, Taimanov and recently Krichever, but we do not discuss them

Our Problem: Which Reduction Leads to the Factorized Operators of the Form $L = QQ^+$?

We have a very first Manakov's System $L_t = [H, L] + fL$. Here both L, H are the second order operators and f-scalar function: $L = \partial_x\partial_y + F\partial_y + S, H = \Delta + G\partial_y + A$. As Konopelchenko pointed out(1988), the reduction $S = 0$ is time-invariant (because this equation is linear for S). It looks like the 2D extension of the Burgers equation. He constructed also Backlund Transformations here. How to calculate this reduction for the Inverse Spectral Data?

Make replacement $x, y \longrightarrow z, \bar{z}$.

Which Data lead to the elliptic operators $L = \partial\bar{\partial} + G\bar{\partial} + S$ gauge equivalent to the self-adjoint operators ? The magnetic field $B = 1/2 G_{\bar{z}}$ should be real, in particular. For $S = 0$ this condition is sufficient for the nonsingular fields.

The Algebro-Geometric Data for self-adjoint Operators L: a.Riemann Surface Γ of genus g with 2 selected "infinite points" ∞_1, ∞_2, local parameters k_1^{-1}, k_2^{-1} and Divisor $D = P_1, ..., P_g$ of degree g define the operator L. b.Let an antiinvolution $\sigma : \Gamma \to \Gamma, \sigma^2 = 1$ be given such that $\sigma(D) + D \sim K + \infty_1 + \infty_2$, $\sigma^*(k_1) = -\bar{k}_2$, $\sigma(\infty_1) = \infty_2$ Such Data (generically) define a self-adjoint operator L.

We construct a "two-point Baker-Akhiezer function" $\Psi(z,\bar{z},k)$ meromorphic in the variable $k \in \Gamma$ outside of infinities with simple poles in the points of Divisor D and asymptotic at infinities:

$\Psi = \exp\{k_1 z + \sum_{j>1} k_1^j t''_j\}(1 + O(k_1^{-1}))$

$\Psi = c(z,\bar{z})\exp\{k_2\bar{z} + \sum_{j>1} k_2^j t'_j)\}(1+$
$+O(k_2^{-1}))$

It satisfies to the equations $L\Psi = 0, \Psi_{t'_j} = L'_j\Psi, \Psi_{t''_j} = L''_j\Psi$. So we have a Hierarchy of Nonlinear Systems in R^2 with 2 sets of "times" $t'_j, t''_j, j > 1$.

Here $L = \Delta + G\bar{\partial} + S, G = (\log c)_{\bar{z}}$, L'_j, L''_j are some linear differential operators in the variables $z, \bar{z}$. In order to construct self-adjoint deformations of L we need to put $t'_{2k} = -t'_{2k} \in iR$ and $t'_{2k+1} = t''_{2k+1} \in R$.
This reduction of the Schrodinger Hierarchy never has been discussed. Additional Reduction $S = 0$ is invariant under the whole Hierarchy: A Self-Adjoint 2D Burgers Hierarchy.

Our main result describes corresponding Inverse Problem Data:

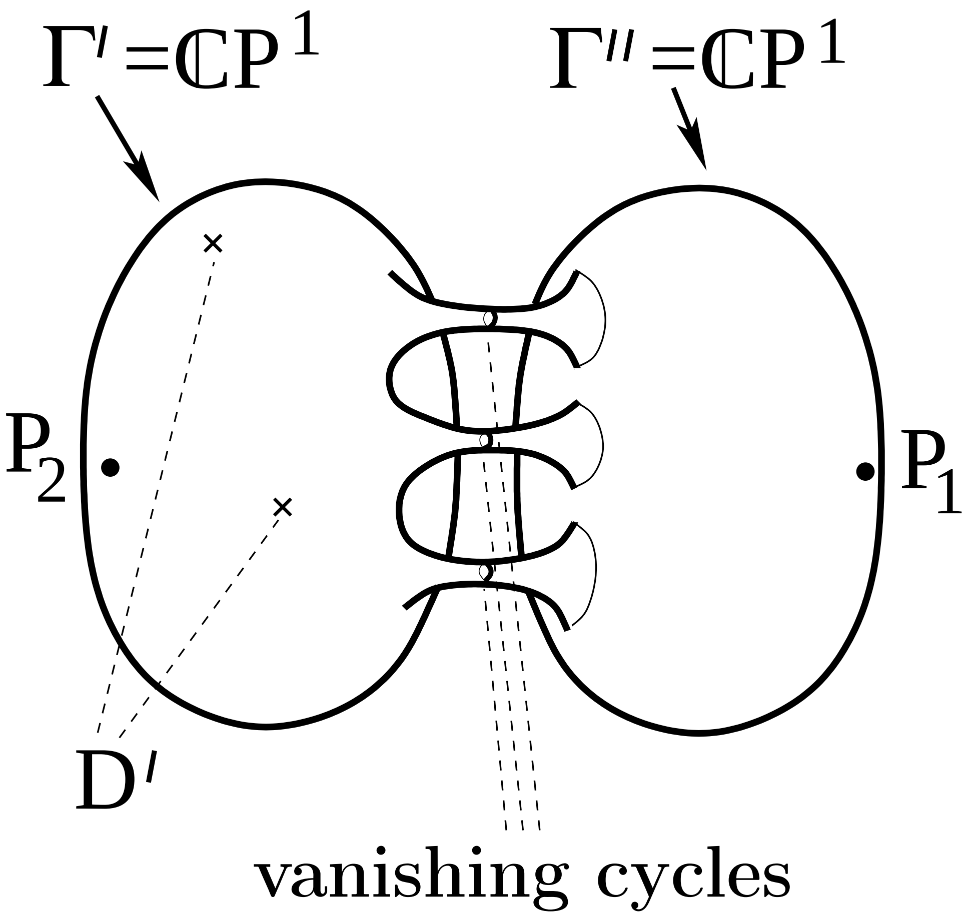
Take Riemann Surface splitted into the nonsingular pieces $\Gamma = \Gamma' \cup \Gamma''$ of genus $g' = g''$, intersecting each other in $k+1$ points $Q_0, ..., Q_k$(see Fig 1). Take infinities $\infty_1 \in \Gamma', \infty_2 \in \Gamma''$ with local parameters $k_1^{-1} = k'^{-1}, k_2^{-1} = k''^{-1}$. Let an antiinvolution $\sigma : \Gamma' \to \Gamma''$ be given such that $\sigma^*(k') = -\bar{k''}, \sigma(Q_s) = Q_{l_s}$. Take divisors D', D'' of degree $g'+k, g''$ correspondingly not crossing infinities and points Q_s. Assume that the total divisor $D = D' + D''$ satisfies to the "Degenerate Cherednik Type Equation" $D + \sigma(D) = K + \infty_1 + \infty_2$. Here $K = K' + K'' + Q_0 + ... + Q_k$ with condition on residues in the crossing points Q_s.

Such Data generate a symmetric scalar operator $c^{-1/2}Lc^{1/2} = = QQ^{+}, Q = \partial + 1/2(\log c)_z$, $c \in R$. Here $S = 0$. Therefore they generates a Purely Magnetic Pauli Operator
$L_P = QQ^{+} \oplus Q^{+}Q$. Magnetic Field is real $B = -1/2\Delta \log c$. It is nonsingular if $c \neq 0$, so the operator is self-adjoint. Otherwise we need to consider operator in the domain whose boundary consists of zeroes of c if they are non-degenerate. Here we have a ground state $c^{1/2}$ in one spin-sector only. This State belongs to the Hilbert Space with Dirichlet boundary condition.

How to find its ground states? Take $\psi_0 = c^{1/2}$ in the first spin-sector because $Q^+\psi_0 = 0$. Take $\phi_0 = c^{-1/2}$ in the second sector because $Q\phi_0 = 0$.‘ In the case of periodic $c \neq 0$ we have periodic ground states in both sectors but they do not belong to the Hilbert Space $L_2(R^2)$, so they belong to the bottom of continuous spectrum in both sectors. In Periodic Case our Ψ-function is a full complex family of the Bloch-Floquet functions. Magnetic Field is Topologically Trivial (i.e. its flux through Elementary Cell $[B]$ is equal to zero).

The Case of Genus zero (Fig 1)

Fig 1



We take $l+1$ intersection points presented as $k' = k_s$ and $k'' = p_s$ in Γ', Γ'', and divisor $D' = (a_1, ..., a_l)$ of degree l in Γ'. We have $\Psi = e^{k'\bar{z}} \frac{w_0 k'^l + ... + w_l}{(k'-a_1)...(k'-a_l)}$, $\Psi|_{k'=k_s} = e^{p_s z}$.

As we can see, $c = w_0$.

So the solution is $c = \sum_{s=0}^{l} \kappa_s e^{W_s(z,\bar{z})}$,

Here W_s is a linear form. All complex coefficients are possible.

$W_s = \alpha_s x + \beta_s y, (\alpha_s, \beta_s) \in C^2_W$.

Transformation $c \to c' = c e^{\gamma + \alpha x + \beta y}$ leads to the gauge equivalent operator (the same magnetic field)

There exist 3 types of Real Solutions:

1.Purely Exponential Positive Case $\kappa_s, (\alpha_s, \beta_s) \in R, \kappa_s > 0$. 2.Purely Trigonometric Real Case. 3. Mixed exponential/trigonometric case.

Consider the case 1. Let "the Tropical Sum" of the forms in the set $\{W\}$ is nonnegative $I'_{\{W\}}(\phi) = \max_s(\alpha_s \cos\phi + \beta_s \sin\phi) \geq 0$.

Then $c^{-1/2}$ is bounded in R^2

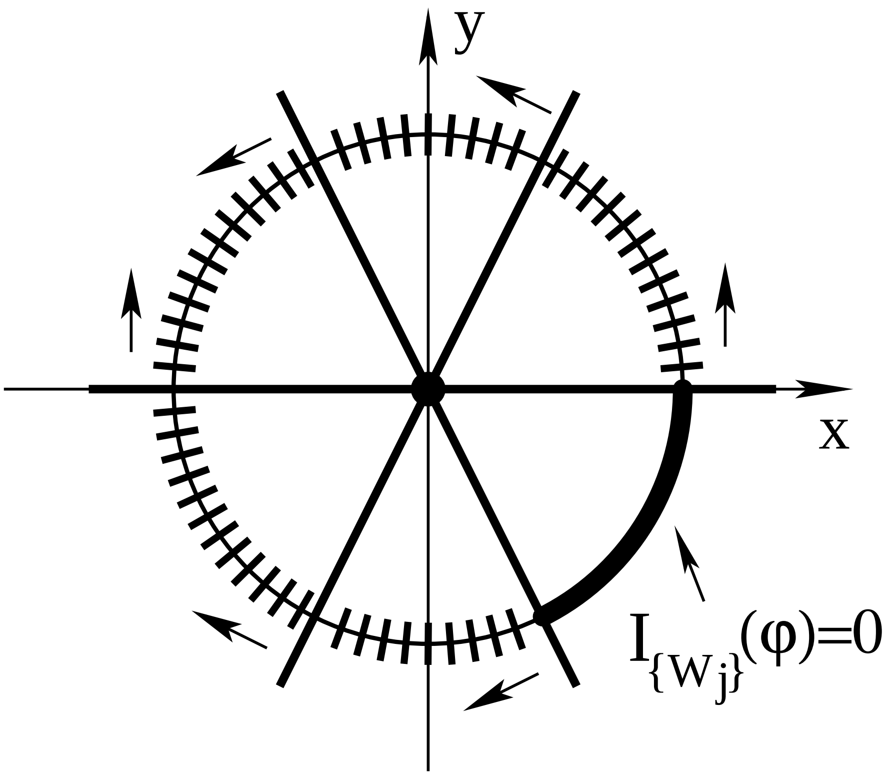
For the angles $I'_{\{W\}}(\phi) > 0$ we have a rapid decay

$c^{-1/2} \to 0, R \to \infty$,

Let $I(\phi) = \max\{I'(\phi), 0\}$

Fig 2a

$c = e^{y} + e^{y-2x} + e^{-y-2x}$

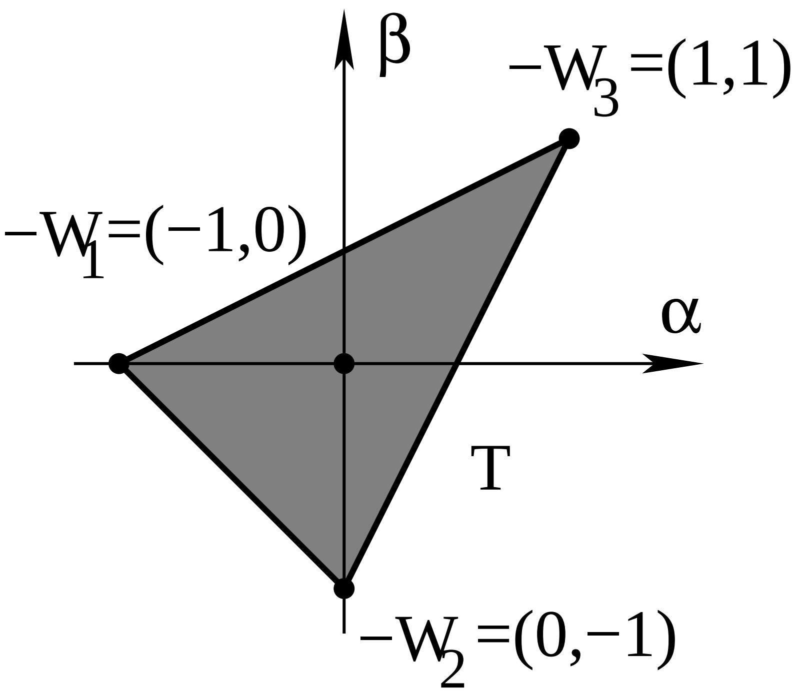


In every class $c' \in ce^W, W' \in R^2_W$, the set of representatives c' with nonnegative $I = I'_{\{W'\}}(\phi) \geq 0$ forms a convex polytop $\bar{T}_c$. Its inner part $T_c \subset \bar{T}_c$ consists of all c' such that $I_{\{W'\}} > 0$. Open part T_c is always nonempty for $l > 2$. $\bar{T}_c$ is nonempty for $l > 1$. (see Fig 2b for $l = 3$)

Fig 2b

$$\text{b) } \mathbb{R}^2 = (\alpha, \beta)$$

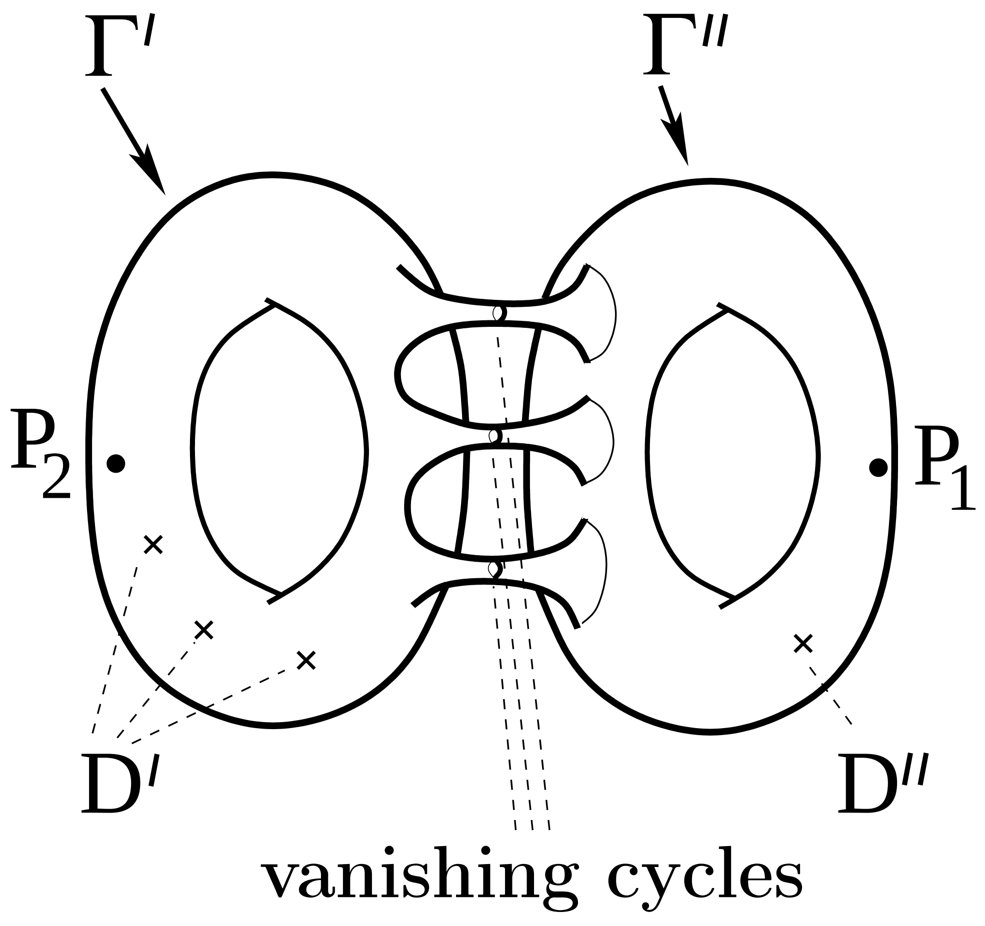
$$W_j = \alpha_j x + \beta_j y$$



Magnetic field is decaying for $R \to \infty$ except some selected angles, it is a Lump Type Field analogous to the KP "Lump Potentials". A linear sum under the $1/2\Delta\log()$ reflects linearization of the Burgers Hierarchy in the variable c. $[B] = \int\int_{D_R^2} B dx dy =$
$= -1/2R \oint_{S^1} I_{\{W\}}(\phi) d\phi + O(R^{-1})$
All points in T_c define ground states in the Hilbert Space $L_2(R^2)$. The boundary points define the bottom of continuous spectrum.

The case of genus 1.

Fig 3



We take elliptic curve $\Gamma' = \Gamma'' = C/\Lambda$ with euclidean local parameters k, p (the point 0 is "infinity"), periods $1, 2i\omega \in iR$, n intersection points $Q_0, Q_1, ..., Q_n \in \Gamma'$ and $R_0, ..., R_n \in \Gamma''$. Divisors $D' = (P_1, ..., P_n), D'' = P$ have degree $n+1, 1$ correspondingly. We have $\psi' = e^{-\bar{z}\zeta(k)} \frac{\prod_s \sigma(k-Q_s)}{\prod_l \sigma(k+P_l)} \times$ $\times (\sum_j w_j \frac{\sigma(k+\bar{z}+\tilde{P}+\tilde{Q}-Q_j)}{\sigma(k-Q_j)})$. Here $\tilde{P} = P_1 + ... + P_n, \tilde{Q} = Q_0 + ... + Q_n$, sum as in C

$\psi'' = e^{-z\zeta(p)}\sigma(p+z+P)/(\sigma(z+P)\sigma(p+P))$, $\psi'(Q_s) = \psi''(R_s)$. All singularity of the quantity c disappear after multiplication $\tilde{c} = c\sigma(\bar{z}+\tilde{Q}+\tilde{P})\sigma(z+P)$. Take $n = 1, Q_0 = -Q_1, R_0 = Q_1, R_1 = Q_0$ and solution to the equation $\omega\zeta(Q_0) = \eta_1 Q_0$ We have $P = \tilde{Q}+\tilde{P}$ in this case, so $-1/2\Delta|\sigma|^2 = -2\pi\delta(z)$.

So our Conclusion is:
The magnetic field $\tilde{B} = -1/2\Delta\tilde{c}$ is periodic nonsingular with magnetic flux equal to ONE QUANTUM UNIT. The magnetic field $B = -1/2\Delta c$ is singular; it has magnetic flux equal to zero through the elementary cell and δ-singularity in the point P. So this field corresponds to the "Bohm-Aharonov" (BA) situation. So adding BA singularity we can obtain Algebro-Geometrically Integrable cases here.