POISSON BRACKETS OF HYDRODYNAMIC TYPE, FROBENIUS ALGEBRAS AND LIE ALGEBRAS

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1. Poisson brackets of hydrodynamic type,

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + u^k_x b^{ij}_x(u(x))\delta(x-y), \tag{1}$$

were introduced and studied in [1] and [2] to construct a theory of conservative systems of hydrodynamic type and a Bogolyubov–Whitham method of averaging Hamiltonian field-theoretic systems.

Problem. Give a classification of Poisson brackets of the form (1), (2) depending linearly on the fields u^j relative to linear changes $u^k = A^k_j w^j$. Some examples were discussed in [3] and [4].

2. The simplest local Lie algebras arising from brackets of hydrodynamic type (1) are especially interesting, where, according to [1], we have

$$g^{ij} = C^{ij}_k u^k + g^{ij}_0, \quad b^{ij}_k = \text{const}, \quad g^{ij}_0 = \text{const}; \tag{2}$$

$$\begin{aligned} &[p,q]_k(z) = b^{ij}_k (p_i(z)q'_j(z) - q_i(z)p'_j(z)), \\ &b^{ij}_k + b^{ji}_k = C^{ij}_k = \partial g^{ij}/\partial u^k. \end{aligned} \tag{3}$$

Definition 1. A bracket (1) or a Lie algebra (3), linear in the fields, is called *symmetric* if $b^{ij}_k = b^{ji}_k$.

From the Jacobi identity we obtain by a direct computation

Lemma 1. *The tensor b^{ij}_k defines by* (3) *a local translationally invariant Lie algebra of first order if and only if the multiplication law* (4) *defines a finite-dimensional algebra B in which the following identities hold*:

$$\begin{aligned} &a, b, c \in B, \quad e^i e^j = b^{ij}_k e^k; \\ &a(bc) = b(ac), \quad (ab)c - a(bc) = (ac)b - a(cb). \end{aligned} \tag{4}$$

Here e^j is a basis of the space R^N. In the symmetric case $2b^{ij}_k = 2b^{ji}_k = C^{ij}_k$ this algebra is commutative and associative.

Definition 2. The Lie algebra (3) and the finite-dimensional algebra (4) are called *nondegenerate* if the pseudo-Riemannian metric $g^{ij}(u) = C^{ij}_k u^k + g^{ij}_0$ is nondegenerat at a "generic point" for some $g^{ij}_0 = g^{ji}_0$,

$$\det g^{ij}(u) = P_N(u^1, u^2, \ldots, u^N) \neq 0.$$

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We recall that a commutative associative algebra is called a Frobenius algebra if there is given a nondegenerate inner product $\langle \cdot, \cdot \rangle$ such that

$$\langle e^i e^j, e^k \rangle = \langle e^i, e^j e^k \rangle. \tag{5}$$

This means that the regular representation is "Frobenius", i.e., the operators of multiplication by any element are adjoint in this inner product. The necessary and sufficient conndition that an algebra with identity be Frobenius in our terms is that $2b^{ij}_k u^k = C^{ij}_k u^k$ be nondegenerate at a generic point (conversely, under this condition the algebra B has an identity).

Frobenius structures are nondegenerate inner products with properties (5). They all reduce to $C^{ij}_k u^k$ at some point $u = u_0$ if $\det(C^{ij}_k u^k) \neq 0$.

Proposition 1. *Classification of infinite-dimensional, nondegenerate, local, translationally invariant, symmetric Lie algebras of first order relative to linear changes* $p_k = A^j_k \tilde{p}_j$ *in the space of values of the fields is completely equivalent to the classification of finite-dimensional, commutative, associative algebras over* R *which admit the structure of a Frobenius algebra* (*possibly, without identity*).

Any nondegenerate metric (2) is reduced by the changes $u^i = \bar{u}^i + u^i_0$ to a purely linear metric $\bar{g}^{ij} = \gamma^{ij}(u) = C^{ij}_k u^k$, $\bar{g}^{ij}_0 = 0$, if

$$2b^{ij}_j = C^{ij}_k, \quad \det(C^{ij}_k u^k) \neq 0.$$

3. Since according to [1] the metric $g^{ij}(u) = C^k_{ij} u^k$ must have zero curvature, we appeal to changes $u(v)$, which are now nonlinear, where the metric in the new coordinates $(v^1, \dots, v^N)$ is constant:

$$g^{ij}(u(v)) = g^{\alpha\beta}_0 (\partial u^i / \partial v^\alpha)(\partial u^j / \partial v^\beta), \quad g^{\alpha\beta}_0 = \text{const}.$$

We consider the purely quadratic changes

$$u^i = \frac{1}{2} F^i_{\alpha\beta} v^\alpha v^\beta. \tag{6}$$

We have

Theorem 1. *For a change* (6) *to reduce the metric of zero curvature* $g^{ij} = C^{ij}_k u^k$ *such that* $b^{ij}_k = \Gamma^j_{sk} g^{si} = \text{const}$, $\det(C^{ij}_k u^k) \neq 0$, *to constant form it is necessary and sufficient that the following conditions hold:*

a) $b^{ij}_k = b^{ji}_k$;

b) F *and* g^{ij}_0 *determine a Frobenius representation of the algebra* (4), *where the* $F^i_{\alpha\gamma}$ *give a representation of the basis* e^i *of the algebra in the form of linear operators in* (v)*-space which are self adjoint in this inner product, so that*

$$e^i \to (F^i)^\alpha_\beta = g^{\alpha\gamma}_0 F^i_{\gamma\beta}, \quad F^i F^j = C^{ij}_k F^k / 2,$$
$$\det(F^i_{\alpha\beta} v^\beta) \neq 0 \quad \text{at a generic point } (v^1, \dots, v^N).$$

Summarizing previous observations, the proof of Theorem 1 essentially based on a simple tensor computation with the substitution (6), using the fact that the connection $\tilde{\Gamma}^\alpha_{\beta\gamma} \equiv 0$ in the new coordinates v.

4. In the general nonsymmetric case (3), where the finite-dimensional algebra B has the form (4), the classification is complicated. If $R_b(a) = ab = L_a(b)$, then from (4) we have

$$[L_a, L_b] = 0, \quad [R_a, R_b] = R_{[a,b]} = R_{ab-ba}. \tag{7}$$

If the algebra B is associative, then it possesses a left ideal $I \subset B$ such that

$$IB = 0, \quad [B,B] \subset I. \tag{8}$$

As a more general case, we consider algebras (4), (7) with an ideal $I \subset B$ such that $II = 0$. Then the quotient $A = B/I$ is an algebra of the type (4), (7), and the theory of extensions arises. Suppose, conversely, that there is given a two-sided A-module I such that for any triple of elements $a, b \in A$, $d \in I$ properties (4) and (7) hold, where $IA \subset I$, $AI \subset I$.

Proposition 2. *Any 2-cochain $d(a,b) \in I$ linear in a and b determines a new algebra B with ideal I, $I^2 = 0$, and identities (4) and (7) if and only if*

$$\begin{aligned}&\delta d(a,b,c) - \delta d(a,c,b) = \Delta_1 d(a,b,c) = 0,\\ &\delta d(a,b,c) - \delta d(b,a,c) - d(ab-ba,c) - [d(a,b) - d(b,a)]c = \Delta_2 d(a,b,c) = 0,\end{aligned} \tag{9}$$

where $\delta d(a,b,c) = d(a,bc) - d(ab,c) + ad(b,c) - d(a,b)c$.

Extensions with cocycles d_1 and d_2 are equivalent if d_1 and d_2 are cohomologous:

$$d_1 = d_2 + \delta h(a,b), \quad \delta h(a,b) = h(ab) + ah(b) - h(a)b. \tag{10}$$

For algebras A with identities (4) and (7) and corresponding A-modules I we have

$$\Delta_1 \delta \equiv 0, \quad \Delta_2 \delta \equiv 0.$$

The cocycles (9) with the equivalence (10) form the group of algebra extensions of (4), (7): $d \in H^2_\Phi(A,I)$.

From Proposition 2 there follows

Corollary 1. *If the algebra A is commutative and associative and the right action of A on I is trivial, $IA \equiv 0$, then the cochain $d(a,b)$ determines an algebra B of type (4), (7) if and only if the 3-cochain δd is symmetric relative to all permutations of a, b, c.*

Theorem 2. *Any associative algebra B of type (4), (7) can be obtained by extension of a commutative algebra where $IA \equiv 0$ according to (3). In the nondegenerate case the substitution (11) reduces the metric $g^{ij}(u) = (b^{ij}_k + b^{ji}_k)u^k + g^{ij}_0$ to constant form $\bar g^{\alpha\beta}_0$ (it is here assumed with no loss of generality that g^{ij}_0 is nondegenerate and $\bar g^{ij}_0 = g^{ij}_0$):*

$$\begin{aligned}&u^q = w^q + \frac{1}{2}F^q_{\alpha\beta} w^\alpha w^\beta,\\ &\alpha,\beta,q,i = 1,2,\ldots,n, \quad F^\gamma_{\alpha\beta} = F^\gamma_{\beta\alpha},\\ &F^{\alpha\gamma}_\beta = \bar g^{\kappa\alpha}_0 F^\gamma_{\alpha\beta} = (F^\gamma)^\alpha_\beta.\end{aligned} \tag{11}$$

The following relations hold:

$$F^\gamma F^\delta = F^\delta F^\gamma = b^{\gamma\delta}_\kappa F^\kappa = b^{\delta\gamma}_\kappa F^\kappa.$$

Here the vectors $f^1,\ldots,f^n$ are the basis unit vectors of the coordinates $w^1,\ldots,w^n$ in the notation (11). They generate a left regular representation of the algebra B

$$\bar g^{pq}_0 = \langle f^p, f^q\rangle = g^{pq}_0, \quad F^\alpha f^p = b^{\alpha p}_q f^q.$$

The substitution (11) is nondegenerate at a generic point $(w^1,\ldots,w^n)$ if and only if the metric $g^{ij}(u)$ is nondegenerate.

5. Lie algebras (3) sometimes possess central R-extensions by means of the simplest cocycles of the type of the Gel'fand–Fuks cocycle for an algebra of vector fields [5]. The 2-cocycles of order τ are given by the formula

$$\gamma_\tau(p,q) = \int \gamma_\tau^{ij} p_i^{(\tau)} q_j \, dx = -\gamma_\tau(q,p). \tag{12}$$

The cocycles (12) generate additions to the Poisson brackets (1), (3) of the form

$$\gamma_\tau^{ij} \delta^{(\tau)}(x-y).$$

Such cocycles are possible for $\tau \leqslant 3$. Formula (12) defines a cocycle on the Lie algebra (3) and an addition to the bracket (1) without violating the Jacobi identity if and only if

$$\begin{aligned} &\tau = 0\colon p^{ijk} = t^{ijk} - t^{kji}, \quad p^{\sigma(ijk)} = (-1)^\sigma p^{ijk}, \\ &\tau = 1\colon t^{ijk} = t^{ikj}, \\ &\tau = 2\colon t^{ijk} = -t^{ikj}, \quad t^{ijk} + t^{kij} + t^{jki} = 0, \\ &\tau = 3\colon t^{ijk} = t^{\sigma(ijk)}, \end{aligned} \tag{13}$$

where $t^{\alpha\beta\gamma} = b_k^{\alpha\beta}\gamma^{k\delta}$, σ is any permutation, and $(-1)^\sigma$ is its sign. We have

Proposition 3. *The forms $\delta_\tau^{ij}(u) = (b_k^{ij} + (-1)^{\tau+1} b_k^{ji})u^k$ are cocycles and define a central extension of the Lie algebra* (3) *for all $(u^1, \dots, u^n)$ if and only if the algebra B possesses the following properties*:

$\tau = 0$: *in the algebra the identity $[ab+ba,c]/2 = (ca)b - c(ab)$ holds*;

$\tau = 1$: *the form $\delta_1^{ij}(u)$ always defines a cocycle on the Lie algebra which is a coboundary*;

$\tau = 2$: *the algebra B is associative*;

$\tau = 3$: *the algebra B is such that $a([b,c]) = -[b,c]a$, where $[b,c] = bc - cb$, $a,b,c \in B$.*

Proposition 4. *If there exists a nondegenerate form $\gamma_\tau^{ij} = (-1)^{\tau+1}\gamma_\tau^{ji}$ defining an "exterior" cocycle of order τ on the Lie algebra* (3), *then*

a) $\tau = 1$: *If the algebra B is associative, then the metric $g^{ij}(u) = \delta_1^{ij}(u) + \gamma_1^{ij}$ reduces to constant form by the substitution* (11). *The transformations $\gamma_1^{ij} \to \gamma_1^{ij} + \delta_1^{ij}(u_0)$ replace the cocycle by a homologous cocycle.*

b) $\tau = 2$: *Such a nondegenerate form γ_2^{ij} is possessed by a Lie algebra with associative algebra B and ideal $I \subset B$, where $[B,B] \subset I$, $IB = 0$, the quotient $A = B/I$ is a Frobenins algebra with identity. Here $A = I$ acts on I by the left regular representation and the cocycle $d \in H^2(A,I)$ giving the extensions is arbitrary* (*the converse has not been proved*!).

c) $\tau = 3$: *The algebra B is a Frobenius algebra*; *$\gamma_3^{ij} = \delta_3^{ij}(u_0)$ if the algebra has an identity.*

Example. For the Poisson brackets of one-dimensional hydrodynamics we have $p = u^1$, $\rho = u^2$, $s = u^3$ and $b_1^{11} = b_2^{12} = b_3^{13} = 1$ (the remaining $b_k^{ij} = 0$). The algebra B is such that any symmetric form γ_1^{ij} defines a cocycle on the Lie algebra of order $\tau = 1$. The cohomology classes of such cocycles correspond to arbitrary symmetric 2×2 matrices—restrictions of the form γ_1^{ij} to the space of variables (u^2, u^3). The corresponding central extensions (14) of the Lie algebra for $\tau = 1$ are nonequivalent.

For vector-valued functions periodic in x, by passing to an expansion in Fourier series, we obtain a basis (L^i_n, z) with the relations

$$\begin{aligned} &[L^i_n, L^j_m] = (mb^{ij}_k - nb^{ji}_k)L^k_{m+n} + \sum_\tau \gamma^{ij}_\tau n^\tau \delta_{m+n,0} \cdot z, \\ &[z, L^i_n] = 0, \quad \gamma^{ij}_\tau = (-1)^{\tau+1}\gamma^{ji}_\tau, \end{aligned} \tag{14}$$

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