THE GEOMETRY OF CONSERVATIVE SYSTEMS OF HYDRODYNAMIC TYPE. THE METHOD OF AVERAGING FOR FIELD-THEORETICAL SYSTEMS

S. P. NOVIKOV

Contents

Introduction

It is well known that in many problems of non-linear oscillation theory the so-called averaging method of Bogolyubov and others (see [6]) is very effective. This method is applied when the unperturbed system has some collection of cycles, that is, exact periodic solutions (the single-phase case) or of tori. that is, quasi-periodic solutions (the multiphase case), depending on several parameters. A phase particle located near this family of solutions oscillates "rapidly" along the tori of the family and "slowly" drifts along the parameters: this gives rise to a system averaged over the rapid oscillations for the slow drift over the family of parameters on which the tori depend.

The study of the slow drift in the first approximation, the estimation of the next terms of the expansion in powers of the ratio of the fast and slow scales, and the analysis of the resonance cases were the subject of many classical works (see the bibliography in [6]).

In principle, various field-theoretical analogues of the averaging method are possible. The version which we discuss here is not only a field-theoretical analogue of the averaging method of the type of Bogolyubov and others, but at the same time it is a non-linear analogue of the standard WKB-method in quantum mechanics (or eikonal in optics). In this version the system itself is not perturbed; it has a family of exact solutions of the form

$$\psi(Ux + Vt; u^1, \ldots, u^N), \qquad (*)$$

where $U(u)$ and $V(u)$ are m-component vectors and $\psi(\eta_1, \ldots, \eta_m; u^1, \ldots, u^N)$ is a 2π-periodic function in each variable η_i.

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We look for solutions of the original system having the form $(*)$ in the first approximation with respect to the natural small parameter — the ratio of the "rapid" and "slow" scales, where the remainder has zero mean over the rapid oscillations. Here $u^k(x,t)$ $(k = 1, 2, \dots, N)$ are not constants, but slowly varying functions of coordinates for $t = 0$. Under certain conditions on the family of solutions $(*)$ of the original system, in the first approximation there arises the so-called "Whitham equation of slow modulations"

$$(**) \qquad \frac{\partial u^k}{\partial t} = V_j^k(u)\frac{\partial u}{\partial x},$$

where the matrix $V_j^k(u)$ depends on the original system. This theory was originated by Whitham (see [7]) in the 1960's, then continued by Maslov (see [8]), Hayes [9], Ablowitz and Benney [10], Gurevich and Pitaevskii [3], Flaschka, McLaughlin and Forest (see [11]), Dobrokhotov and Maslov [13].

The program of research by the author, Dubrovin and Tsarëv (see [1] and [2]) is directed towards a solution of the following problems (for Hamiltonian field-theoretical systems):

Problem 1. To work out a universal Hamiltonian formalism of averaged systems in the first approximation (equations of Whitham type and more general systems of hydrodynamic type) taking into account the following:

It is natural to expect that the averaged system is also conservative: this is confirmed by partial results in [13] and [11], although the approach of these papers is not geometrically invariant and appeals to special coordinates of the type of Clebsch variables; already Riemann in the 19th century drew attention to the fact that equations of the type $(**)$ are a geometric object that is invariant under the substitution $u(w)$. Therefore, it is natural to expect a geometrically invariant Hamiltonian formalism.

Problem 2. If the original field-theoretical system is totally integrable (for instance, by the method of the inverse problem [3]), then it has a large collection of exact solutions of the form $(*)$, which are called "finite-zone" or "algebro-geometric". It is natural to expect that the system averaged over these families of tori is also totally integrable. However, nobody has succeeded to establish this in a single non-trivial case.

A solution of Problem 1 was given by the author and Dubrovin in [1], Problem 2 was solved by Tsarëv (see § 4 below). It turned out that the procedure of an exact solution of the averaged system cannot be stated without a geometrically invariant Hamiltonian formalism; therefore, Problems 1 and 2 turned out to be inseparable. Subsequent developments of the Hamiltonian formalism of systems of hydrodynamic type are discussed in [2]. This is also subject of studies of a number of the author's students that are under way now.

§ 1. Definitions

Systems of hydrodynamic type ("of the type of ideal fluids", possibly, multicomponent, with intrinsic degrees of freedom, but without viscosity) by definition have the form

$$(1) \qquad u_t^j = V_k^j(u)u_x^k \qquad (j, k = 1, 2, \dots, N)$$

(the spatially one-dimensional N-component case) or

$$u_t^j = V_k^{j,\alpha}(u)\partial_\alpha u^k, \quad \partial_\alpha = \frac{\partial}{\partial x^\alpha} \qquad (\alpha = 1, 2, \dots, n) \tag{2}$$

(the spatially n-dimensional N-component case). Systems of the form (1), (2) have been studied beginning with Riemann. Here $u = (u^1, \dots, u^N)$ are (possibly local) coordinates in some manifold M.

Hydronamic functionals by definition have the form

$$\mathcal{H} = \int h(u)\, d^n x \tag{3}$$

where h does not depend on the derivatives $\partial_\alpha u, \dots$. Work on creating a Hamiltonian formalism for systems (1), (2) has been started recently (see [1], [2]). The Poisson brackets of hydrodynamic type are defined apart from general requirements by the following formula:

$$\{u^i(x), u^k(y)\} = g^{ki,\alpha}(u(x))\partial_\alpha \delta(x-y) + b_j^{ki,\alpha}(u)\partial_\alpha u^j(x)\delta(x-y). \tag{4}$$

Hamiltonians of the form (3) by means of the brackets (4) generate conservative systems of hydrodynamic type (1), (2):

$$u_t^j(x) = \{u^j(x), \mathcal{H}\} = (g^{ij,\alpha}\partial_\alpha + b_k^{ij,\alpha}\partial_\alpha u^k)\frac{\partial h}{\partial u^i}. \tag{5}$$

An important fact: only conservative systems of the form (1), (2) can be realized physically!

§ 2. Geometry and algebra

The theory of systems and Poisson brackets of hydrodynamic type is invariant under local changes of coordinates of the form (6), which were already used by Riemann to simplify systems (1) for $n = 1$:

$$u = u(w). \tag{6}$$

However, precisely the Poisson brackets generate the geometry in the U-space (the manifold M) that has been studied by the author and Dubrovin in [1] and [2]; the results are especially complete in the spatially one-dimensional case $n = 1$, where the index α is absent. Namely, the following theorem is true:

Theorem. *If* $\det g^{ij} \neq 0$ *and* $b_k^{ij} = g^{is}\Gamma_{sk}^j$, *then under the changes* $u(w)$ *the quantity* $g^{ij}(u)$ *transforms like a pseudo-Riemannian metric, and* Γ_{sk}^j *like a collection of Christoffel symbols. The bracket* (4) *has all the properties of the Poisson bracket if and only if the connection* Γ_{sk}^j *is generated by the metric* g^{ij} *and also has zero curvature and torsion.*

Corollary. *There are coordinates* $(w^1, \dots, w^N)$ *such that* $g^{ij} = \text{const}$, $\Gamma_{sk}^j \equiv 0$. *The complete local invariant is the signature of the pseudo-Euclidean metric.*

A complete theory of Poisson brackets with degenerate "metric" has not yet been constructed even for $n = 1$, where there are now a number of results [14].

More general homogeneous geometric Poisson brackets of order m have the form $(n = 1)$:

$$\{u^j(x), u^k(y)\} = g^{kj}(u(x))\delta^{(m)}(x-y) + b^{kj}_s(u)u^s_x\delta^{(m-1)}(x-y)+ + [c^{kj}_s u^s_{xx} + d^{kj}_{sl}u^s_x u^l_x]\delta^{(m-2)}(x-y) + \dots \tag{7}$$

Every term in (7) has the total degree deg $= m$ where

$$\deg\delta^{(k)} = k, \quad \deg u = 0, \quad \deg\frac{\partial^k u}{\partial x^k} = k.$$

So far there is no answer, even under the condition $\det g^{ij} \neq 0$, to the question: when do coordinates $w^1, \dots, w^N$ exist such that after the substitution $u(w)$ the bracket reduces to the form (for $m = 2$, see [15])

$$\{w^j(x), w^k(y)\} = g^{kj}_0\delta^{(m)}(x-y), \quad g^{ij}_0 = \text{const}, \quad m \geqslant 3? \tag{8}$$

Non-homogeneous geometric Poisson brackets are linear combinations of homogeneous geometric Poisson brackets of various orders. For example, for the brackets of type $1 + 0$, where $\det g^{ij} \neq 0$, the following theorem is true.

Theorem [2]**.** *There are coordinates $w^k, \dots, w^N$, where a bracket of the type $1+0$ has the form*

$$\{w^j(x), w^k(y)\} = g^{kj}_0\delta'(x-y) + a^{kj}(w)\delta(x-y), \tag{9}$$

$g^{kj}_0 = \text{const}$ *and a^{kj} is an ordinary finite-dimensional Poisson bracket in the w-space (of the manifolds M^n). The tensor a^{kj} has the form*

$$a^{kj}(w) = C^{kj}_s w^s + h^{kj}_0, \tag{10}$$

where C^{jk}_s is the tensor of structure constants of the semisimple Lie algebra with the Killing metric g^{kj}_0 and h^{kj}_0 is a cocycle on that algebra.

The study of non-homogeneous brackets of the form $2+1+0$ and $3+1+0$ leads to extremely interesting infinite-dimensional Lie algebras, whose classification is determined by certain finite-dimensional algebras. The term of type 1 depends linearly on the fields w, which generates an infinite Lie algebra; the leading term has the form (8) in coordinates w. Starting from other formulations of the problem, individual examples of this type were discussed (although very incompletely) in [4], [5], and [16].

Homogeneous geometric brackets of order m determine on the manifold M^N a geometry of its own. In particular, the tensor $g^{ij}(u^1, \dots, u^N)$ defines a "metric", that is, an inner product of tangential (co)-vectors, while the symbols $g_{ij}b^{kj}_s(u)$, $g_{ij}c^{(k)}_s(u), \dots$ specify "Christoffel symbols" of certain connections. For even $m = 2k$ this inner product is skew-symmetric. If $m = 0$ and $\det g^{ij} \neq 0$, then the form $\Omega = g_{ij}du^i \wedge du^j$ is always closed: $d\Omega = 0$. The case $m = 2$ is studied in [15].

Theorem [15]**.** *If $m = 2$ then the connection $g_{ij}c^{kj}_s(u) = \Gamma^k_{si}$ is symmetric, has zero curvature, and coincides with the symmetric part of the connection $g_{ij}b^{kj}_s = \widetilde{\Gamma}^k_{si}$, that is $\widetilde{\Gamma}^k_{si} + \widetilde{\Gamma}^k_{is} = 2\Gamma^k_{si}$. The torsion tensor of the connection $\widetilde{\Gamma}^k_{si}$, where $T^k_{si} = \widetilde{\Gamma}^k_{si} - \widetilde{\Gamma}^k_{is}$ has the following properties*:

a) *T_{ksi} is absolutely skew-symmetric*;

b) *the form $T = T_{ksi}\,du^k \wedge du^s \wedge du^i$ is the differential of the form Ω, that is, $d\Omega = (\text{const}) \cdot T$.*

Moreover, the form Ω satisfies some differential identities. In particular, the theorem implies that a bracket of type 2, where $\det g^{ij} \neq 0$, reduces to the form $g_0^{ij}\delta''(x-y)$ if and only if $d\Omega = 0$.

In [15] a classification is given for the brackets of type 2 with a non-degenerate "metric" g^{ij}. For arbitrary $m \geqslant 3$ there is no classification so far. Apparently the "last" connection $\Gamma_{si}^k = g_{ij} D_s^{ki}$ is symmetric and has zero curvature, where $D_s^{kj}(u)$ is the coefficient of the derivatives of order m in (7) for the Poisson bracket, that is, it is of the form

$$(D_s^{kj}(u) u_{xx\ldots x}^s + \ldots)\delta(x-y).$$

When $D_s^{kj} \equiv 0$, this fixes the coordinates uniquely up to affine transformations. Other identities have not yet been studied.

In the author's opinion, it is of great independent interest to study "skew-symmetric geometries" of order $m = 2$, which consist of a pair $(\Omega, \widetilde{\Gamma})$, where $\Omega = g_{ij}\, du^i \wedge du^j$ is a non-degenerate form and $\tilde{\Gamma}_{sj}^k$ is a connection such that its torsion tensor $T_{sj}^k = \widetilde{T}_{sj}^k - \widetilde{\Gamma}_{js}^k$ after dropping the superscript is completely skew-symmetric; the differential form

$$T = T_{ijk}\, du^i \wedge du^j \wedge du^k$$

is such that $T = (\mathrm{const}) \cdot d\Omega$. In the most interesting case we must require that the symmetric part of the connection $\widetilde{\Gamma}_{sj}^k$ has zero curvature. It is natural to require that $\widetilde{\nabla}_k g_{ij} = 0$, by virtue of the connection $\widetilde{\Gamma}_{ij}^k$, that is, that the metric and connection are compatible in this "geometry".

A systematic theory of infinite Lie algebras arising from hydrodynamic (geometric) Poisson brackets of type 1 that are linear in fields has been constructed by the author and Balinskii in [17]; some isolated interesting algebraic observations about these brackets were first made in [16], p.26, in examples (these observations are partly due to S. I. Gel'fand). For these brackets

$$m = 1, \qquad g^{ij} = g_0^{ij} + c_k^{ij} u^k, \qquad b_k^{ij} = \mathrm{const},$$
$$\{u^i(x), u^i(y)\} = g^{ij}(u(x))\delta'(x-y) + b_k^{ij} u^k \delta(x-y).$$

It is convenient to introduce a finite-dimensional algebra B with basis $e^1, \ldots, e^N \in B$, where

$$e^i e^j = b_k^{ij} e^k, \qquad C_k^{ij} = b_k^{ij} + b_k^{ji}.$$

The Lie algebra L is defined on the space of functions $p(x) \in B$ as follows:

$$(*) \qquad [p(x), q(x)] = p'q - q'p$$

(right multiplication is taken in the algebra B). The skew-symmetry and Jacobi identity in L imply the following identities in B (and vice versa):

$$(**) \qquad \begin{cases} \text{a)} & (ab)c - a(bc) = (ac)b - a(cb), \\ \text{b)} & a(bc) = b(ac). \end{cases}$$

A Lie algebra L is said to be non-degenerate if there is a tensor g_0^{ij} such that the "metric" $g^{ij}(u) = g_0^{ij} + c_k^{ij} u^k$ is non-degenerate and has zero curvature for almost all $u^1, \ldots, u^N$, where $c_k^{ij} = b_k^{ij} + b_k^{ji}$ and the Poisson bracket of type 1 is well-defined by (4).

In this case a finite-dimensional algebra B with inner product g_0^{ij} is called "quasi-Frobenius". If B is a commutative algebra with unit element, then we have the classical Frobenius algebras. Their non-commutative and even non-associative generalizations arising in this theory differ from those already available in algebra. B is a quasi-Frobenius algebra if and only if all the operators of left multiplication $L_a(b) = ab$ are symmetric in the metric g_0^{ij}, where $\det g_0^{ij} \neq 0$. The coordinates $u^1, \dots, u^N$ are determined to within affine changes $u^i = a^i_j w + u^i_0$. In [17] non-linear changes of coordinates are studied that reduce the metric $g^{ij}(u)$ to a constant. These changes go beyond the limits of pure algebra, and are so far not completely studied. In [17] also groups of extensions of finite-dimensional algebras B with the properties $(**)$ are constructed. As Zel'manov has recently shown, every finite-dimensional algebra with the identities $(*)$ has a non-trivial ideal with zero multiplication. Therefore, all these algebras are constructed by successive extensions starting from commutative and associative algebras by means of the generalized cocycles of [17].

Local translation invariant Lie algebras L defined by the identity $(*)$ in terms of an algebra B of type $(**)$ are a natural generalization of the algebra of vector fields on a line and a circle. Of natural interest are analogues of the Virasoro algebra, that is, central R-extensions of Lie algebras L. These extensions have a common form given by a 2-cocycle of the form

$$(***) \qquad \gamma(p,q) = -\gamma(q,p) = \sum_{\tau=0}^{3} \int \gamma^{ij}_\tau p_i^{(\tau)}(x)\, q_j(x)\, dx,$$

where $p(x) = p_i(x)e^i$, $q(x) = q_j(x)e^j \in L$. There cannot be terms with $\tau > 3$. The number τ is called the "weight" of the cocycle. Obviously, $\gamma^{ij}_\tau = (-1)^i \gamma^{ji}_\tau$. The cocycles of the form

$$\gamma^{ij} = (b^{ij}_k + (-1)^{\tau+1} b^{ji}_k) u^k_0$$

are called natural (intrinsic). It is easy to find conditions under which $\gamma^{ij}_\tau(u_0)$ is a cocycle of weight τ for all points u_0 (see [17]). A cocycle is said to be non-degenerate if $\det \gamma^{ij}_\tau \neq 0$. A non-degenerate cocycle of weight $\tau = 3$ exists if and only if B is a commutative and associative Frobenius algebra. If this cocycle is intrinsic, then the algebra has a unit element. A non-degenerate cocycle of weight $\tau = 1$ generates a metric $g^{ij}(u) = g^{ij}_0 + (b^{ij}_k + b^{ji}_k)u^k$ of zero curvature that is non-degenerate at a generic point. In this case the algebra L is non-degenerate (see above). The Poisson brackets of one-dimensional gas dynamics with the fields $u^1 = p$, $u^2 = \rho$, $u^3 = s$ have such cocycles, where the algebra B is non-commutative (see Example 5 in § 5). Sometimes there also arise interesting cocycles of weight $\tau = 2$ or $\tau = 0$, where γ^{ij}_τ is skew-symmetric.

Of interest is the development of representation theory (modules of Verma type) for such algebras, depending on all cocycles of weight $\tau = 0, 1, 2, 3$ as parameters. Also of interest are "super"-generalizations of such algebras, where L and B acquire a $\mathbb{Z}_2$-grading.

A theory of Poisson brackets of hydrodynamic type in dimensions $n \geqslant 2$ has been constructed in [2] for the case $\det g^{ij,\alpha} \neq 0$. For $n = 2$, $N = 2$ there is a rather complicated classification. For $N > 2$ and all $n \geqslant 2$ the following theorem has been proved.

Theorem [2]**.** *For $N > 2$ in the coordinates $u^1, \dots, u^N$, where $g^{ij,1} = g^{ij,1}_0(u) =$ const, $\det g^{ij,1}_0 \neq 0$, all the coefficients of the Poisson bracket* (4) *depend linearly* (*possibly non-homogeneously*) *on the fields $u^j(x)$.*

This theorem yields the entire class of examples of Lie algebras of type $(*)$ above. Of interest are their multidimensional analogues, where $g^{ij,\alpha} = g_0^{ij,\alpha} u^k$ $b_k^{ij,\alpha} = \text{const}$; their structure theory, extensions of the type of Virasoro algebras, and modules of Verma type have not yet been considered.

Returning to the case $n = 1$, $m > 1$, of special interest are the Poisson brackets of type m that depend linearly on the fields $u^j(x)$. Such brackets have the form

$$\{u^i(x), u^j(y)\} = \sum_{q=0}^{m} \delta^{(m-q)}(x-y)[b_{k,q}^{ij} u_q^k + g_q^{ij}],$$

$$g_q^{ij} = \text{const}, \qquad b_{kq}^{ij} = \text{const}, \qquad u_q^k = u_{xx\ldots x}^k,$$

$$g^{ij}(u) = b_{k,0}^{ij} u^k + g_0^{ij}, \qquad b_{k,q}^{ij} + (-1)^{m-q} b_{kq}^{ji} = b_{k,q-1}^{ij} - (-1)^{m-q} b_{k,q}^{ji}.$$

In this case, there arise on the space R several operations

$$(e^j e^i)_q = b_{k,q}^{ij} e^k \qquad (q = 1, 2, \ldots, m).$$

Basic among them is the last, $q = m$,

$$e^i e^j = b_{k,m}^{ij} e^k.$$

For this operation and all $m \geqslant 1$ the identity $(**)$ b) above turns out to be true:

$$(****) \qquad a(bc) = b(ac).$$

There arises the class of finite-dimensional algebras B with the identity $(****)$, where for distinct m there are distinct additional structures. The cases $m = 2$ and 3 are especially interesting.

Yet another interesting geometric problem related to Poisson brackets of hydronamic type 1 is as follows. We say that $(u^1, \ldots, u^N)$ are *Liouville coordinates* if the Poisson bracket has the form (14)

$$g^{ij}(u) = \gamma^{ji}(u) + \gamma^{ji}(u), \qquad b_k^{ij} = \partial\gamma^{ij}/\partial u^k.$$

In particular, if the bracket is linear with respect to the fields $u^i(x)$, then the (u^j) are Liouville coordinates. It would be interesting to give a classification of the Liouville coordinates. Their importance will become clear in the next section.

§ 3. The averaging method. Conservativity

For evolution systems

$$(11) \qquad \psi_i = K(\psi, \psi_x, \psi_{xx}, \ldots),$$

having an N-parameter family of exact quasiperiodic solutions of the form

$$(12) \qquad \psi(Ux + Vt, u^1, \ldots, u^N), \quad U(u), \quad V(u),$$

the development of an analogue of the classical averaging method of Bogolyuboy—Krylov and others began in the 1960's (Whitham, Maslov, ...). By averaging over the "rapid" oscillations we obtain the averaged field-theoretical system of hydrodynamic type for the approximate solutions of (11), where the u^j are assumed to be not constant, but slowly varying functions of coordinates and time, that is, the Whitham equation

$$(13) \qquad u_t^j = V_k^j(u) u_x^k.$$

The principle of hereditary conservativity under averaging can be stated as follows (see [1]): suppose that, firstly, the system (11) is Hamiltonian with an arbitrary local Poisson bracket (not depending explicitly on x) and Hamiltonian, and, secondly, there is an involutory collection of local field integrals $I_k = \int j_k(\psi, \psi_x, \dots)\, dx$ ($k = 1, \dots, N$) such that $u^k = \overline{j_k}$ on solutions of (12). Then the averaged system (13) is conservative of hydrodynamic type, where $u^k = \overline{j^k}$ and the Poisson bracket (4) is defined by the formula

$$g^{kj} = \gamma^{kj} + \gamma^{jk}, \qquad b^{kj}_s = \partial\gamma^{kj}/\partial u^s \tag{14}$$

(that is, the $u^k = \overline{j_k}$ are Liouville coordinates), where

$$\gamma^{kj} = \overline{B^{kj}}, \qquad \frac{dB^{kj}(\psi, \psi_x, \dots)}{\partial x} = \{j_k, I_j\}$$

(the bar above indicates averaging over the rapid oscillations). The density of the Hamiltonian (momentum) of the averaged system is obtained directly from that of the original system by averaging.

§ 4. Total integrability

For the two-component case $N = 2$ already Riemann showed that by a change of coordinates $u(w)$ the matrix $V^k_j(u)$ can always be made diagonal in the domain if the eigenvalues are real and distinct. For $N > 2$ this is not always the case. If the system (1) in some coordinates u has a diagonal matrix V^k_j. then the coordinates $u^1, \dots, u^N$ are called *Riemann invariants.*

For $N = 2$ it has been known since the 19th century that for the inverse functions $x(u^1, u^2)$, $t(u^1, u^2)$ (1) turns into a linear equation (the "hodograph method"). No analogues of the hodograph method have been known for $N > 2$.

However, in recent years a number of experts in soliton theory have started studying equations of slow modulations of Whitham type (13) for integrable field-theoretical systems of the form (11), for example, KdV or sine-Gordon and others, using the families of finite-zero or algebro-geometric solutions of the form (12) that arise from the method of the inverse problem for these systems (see [3]). It has become clear, in particular, that the Whitham equations (13) in the cases of KdV and sine-Gordon reduce to diagonal form (Whitham, Flaschka, McLaughlin). However, despite the integrability of the original system of the KdV or sine-Gordon type, nobody has succeeded in establishing the integrability of the averaged Whitham equations.

The author has conjectured that the combination of two properties: reducibility of the system (1) to diagonal form together with the conservativity (Hamiltonian property) give rise to increased integrability of systems of hydrodynamic type (1).

This conjecture has recently been proved by Tsarëv [18]. He has shown that systems that are both diagonal and Hamiltonian have a large enough family of involutory integrals of hydrodynamic type (3). All these integrals generate new systems of hydrodynamic type, which commute with the original one. He found the following analogue of the hodograph method for $N > 2$. Suppose that two commuting Hamiltonian systems of hydrodynamic type are given:

$$u^j_t = V^j_k(u)u^k_x, \qquad u^j_t = W^j_k(u)u^k_x. \tag{15}$$

If V_j^k is diagonal, then the collection of equations

$$W_k^j(u) = V_k^j(u) + t\delta_k^j \tag{16}$$

gives exactly N equations for the N unknown functions $u^j(x,t)$. The theorem of Tsarëv asserts that a solution $u^j(x,t)$ of (16) is a solution of (1):

$$u_t^j = V_k^j(u)u_x^k.$$

The proof uses essentially the commutativity and Hamiltonian property of the both flows (15). We see that instead of the method of the inverse problem for the averaged system the integration procedure includes only inversions of the finite-dimensional map (16), which depends on (x,t) as parameters.

§ 5. Most important examples

Example 1. For the classical hydrodynamics of compressible fluid we have the following fields:

$u^1 = p,$	the density of momentum,
$u^2 = \rho,$	the density of mass,
$u^3 = s,$	the density of entropy,
$\varepsilon = (p^2/2\rho) + \varepsilon_0(\rho, s),$	the density of energy,
$\mathcal{H} = \int \varepsilon\, dx,$	the Hamiltonian.

The Poisson brackets have the form (see [5])

$$\begin{aligned} \{p(x), p(y)\} &= 2p(x)\delta'(x-y) + p'(x)\delta(x-y), \\ \{p(x), \rho(y)\} &= \rho(x)\delta'(x-y), \\ \{p(x), s(y)\} &= s(x)\delta'(x-y), \\ \{\rho(x), \rho(y)\} &= \{s(x), s(y)\} = \{\rho(x), s(y)\} = 0. \end{aligned} \tag{17}$$

In the barotropic case the entropy density as a field variable is absent, and $\varepsilon = (p^2/2\rho) + \varepsilon_0(\rho)$. In this case $N = 2$ and $\det g^{ij} \neq 0$, where according to (17)

$$g^{ij} = \begin{pmatrix} 2p & \rho \\ \rho & 0 \end{pmatrix}, \qquad \gamma^{ij} = \begin{pmatrix} p & \rho \\ 0 & 0 \end{pmatrix}.$$

Example 2. A one-dimensional relativistic fluid. Here $N = 2$ since there are only two fields:

$$u^1 = p, \quad \text{the density of momentum,} \qquad u^2 = \varepsilon, \quad \text{the density of energy,} \qquad \mathcal{H} = \int \varepsilon\, dx.$$

The equations of motion have the form

$$p_t + (\varepsilon - 2q)_x = 0, \qquad \varepsilon_t + p_x = 0. \tag{18}$$

The energy tensor of momentum has the form ($c \equiv 1$)

$$T^{ij} = \begin{pmatrix} \varepsilon & p \\ p & \varepsilon - 2q \end{pmatrix}, \tag{19}$$

where $2q = \varepsilon_0 - P$ is the metric trace of the tensor in the Minkowski metric, P is the pressure, ε_0 is the energy density in the accompanying reference system, and the tensor T^i_j is diagonal and has the form

$$T^i_j = \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & -P \end{pmatrix}, \qquad T^i_i = \varepsilon_0 - P. \tag{20}$$

The equation of motion (18) is closed by the state equation, a single constraint on the tensor T^{ij}. By virtue of the requirement of Galilean (Lorentz) invariance this constraint is imposed only on the invariants of the tensor T^{ij}:

$$\Phi(\varepsilon_0, P) = 0.$$

The Poisson bracket is given in the form (14), where

$$\gamma^{ij} = \begin{pmatrix} p & \varepsilon - 2q \\ \varepsilon & p \end{pmatrix}. \tag{21}$$

The integration procedure is discussed in detail in [12] (following Khalatnikov's work), but its Hamiltonian aspects have not yet been analyzed.

Example 3. The original field system has the form of the non-linear Klein—Gordon equation ($c \equiv 1$)

$$\Box\psi = \psi_{xx} - \psi_{tt} = V'(\psi). \tag{22}$$

The family of single-phase (periodic) solutions has the form

$$\psi = \varphi(x - vt); \qquad d\varphi\sqrt{1 - v^2} = dx\sqrt{2(\varepsilon - V(\varphi))}. \tag{23}$$

The momentum and energy integrals give rise to parameters u^1 and u^2 on which the family (23) of exact solutions depends,

$$\begin{cases} I_1 = \displaystyle\int \psi_t\, dx, \qquad p = \overline{\psi}_t, \\ I_2 = \displaystyle\int [(\psi_t^2 + \psi_x^2)/2 + V(\psi)]\, dx. \end{cases} \tag{24}$$

After averaging the densities of these integrals we obtain the averaged energy and momentum densities:

$$u^1 = p, \quad u^2 = \varepsilon.$$

The Poisson bracket is given, according to § 3 above, by a matrix that can conveniently be written in the form (14)

$$\gamma^{ij} = \begin{pmatrix} p & \varepsilon - 2q \\ \varepsilon & p \end{pmatrix}. \tag{25}$$

Here the function $V(\psi)$ determines the state equation $\Phi(\varepsilon_0, P) = 0$ on the invariants of the tensor T^{ij}

$$T^{ij} = \begin{pmatrix} \varepsilon & p \\ p & \varepsilon - 2q \end{pmatrix}.$$

For the first time the Hamiltonian property of equations of Whitham type in this particular case and in special (non-physical) variables of Clebsch type was established in [9], and the hidden isomorphism with the relativistic hydrodynamics was discovered in [8]. In our formalism this follows immediately from the fact that the connection Γ^k_{ij} defined by (14) in § 3 is torsion-free. This implies that the constraint on the tensor T^{ij} is imposed only on the invariants.

Example 4. Systems of KdV type have the form

$$\psi_t = \frac{d}{dx}\frac{\delta\mathcal{H}}{\delta\psi(x)}, \qquad cH = \int\left[\frac{\psi_x^2}{2} + V(\psi)\right]dx \tag{26}$$

with the Gardner—Zakharov—Faddeev bracket

$$\{\psi(x);\psi(y)\} = \delta'(x-y).$$

There are three general integrals (in involution):

$$\begin{aligned} I_0 &= \int \psi\,dx, \quad \text{bracket annihilator,} \\ I_1 &= \int \frac{1}{2}\psi^2 dx, \quad \text{momentum,} \\ I_2 = \mathcal{H} &= \int\left[\frac{\psi_x^2}{2} + V(\psi)\right]dx, \quad \text{energy.} \end{aligned}$$

After averaging their densities we have the quantities

$$\overline{\psi} = u, \qquad \frac{\overline{\psi}^2}{2} = p, \qquad \overline{\frac{\psi_x^2}{2} + V(\psi)} = \varepsilon, \tag{27}$$

where the family of single-phase exact solutions is given in the form $\psi = \varphi(x-vt)$ and depends on three parameters. The Poisson bracket is give by the matrix

$$\gamma_0^{ij} = \begin{pmatrix} 1 & u & -vu-d \\ 0 & p & -vp+2f \\ 0 & \varepsilon & -v\varepsilon - vf + d^2 \end{pmatrix}, \tag{28}$$

$$(\varphi')^2 = 2v(\varphi) + v\varphi^2 + 2d\varphi + 2f, \quad \psi = \varphi(x-vt).$$

Here the quantities v, d and f can be expressed in terms of (u,p,ε). We introduce the quantities

$$p = p_+ = \frac{\psi^2}{2}, \qquad p_- = \frac{\overline{(\psi-\overline{\psi})^2}}{2} = \frac{u^2}{2} - p.$$

Their brackets have the following form, by virtue of (28):

$$\{p_+,p_-\} = 0, \quad \{p_\pm(x), p_\pm(y)\} = 2p_\pm(x)\delta'(x-y) + p'_\pm\delta(x-y).$$

This shows that the both variables $p_\pm$ are analogous to momentum ("transfer momentum" p_+ and "fluctuation momentum" p_-). Thus, the averaged system of KdV type is an interesting form of a "two-fluid" hydrodynamics.

Example 5. In the particular case of an ordinary KdV equation, $V(\psi) = \psi^3$ the system (26) is integrable by the method of the inverse problem [3] and has the form

$$\psi_t = -\psi_{xxx} + 6\psi\psi_x.$$

As is known, in this case the averaged KdV equation has a number of additional properties: 1) it has Riemann invariants, that is, the matrix $V_j^i(u)$ is diagonalizable (Whitham for the single-phase case; see [3], [5], and [11] for the general multiphase case). 2) It is Hamiltonian with respect to another Poisson bracket of hydrodynamic type. In the single-phase case in the same variables (27) that bracket is given by the matrix

$$\gamma_1^{ij} = \begin{pmatrix} 2u & -vu-d & 10\varepsilon \\ 2p & -vp+2f & \frac{5}{3}(-v\varepsilon - vf + d^2) \\ 2\varepsilon & -v\varepsilon - vf + d^2 & (-v^2-2d)\varepsilon + 2c^2f - vd^2 + 2df \end{pmatrix}. \tag{29}$$

It is obtained by the procedure indicated in § 3 from the second local bracket, where the KdV equation is given by the Hamiltonian $\mathcal{H}_1$:

$$\mathcal{H}_1 = \int \frac{\psi^2}{2} dx \{\psi(x), \psi(y)\}_1 = \delta'''(x-y) + \frac{3}{2}(2\psi\delta' + \psi'\delta). \tag{30}$$

Multiphase Whitham equations for KdV and sine-Gordon equations always lead to Hamiltonian systems having Riemann invariants. Therefore, the integration procedure indicated in § 4 is always applicable to them. We obtain involutory integrals of hydrodynamic type when averaging the integrals of Kruskal and others (see [3]) for these systems. Thus, according to 4, a countable series of exact solutions of the averaged Whitham system arises, which it is natural to call "finite-zone in average". The properties of these solutions even in the single-phase case are still being investigated.

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