SPECTRAL THEORY OF COMMUTING OPERATORS OF RANK TWO WITH PERIODIC COEFFICIENTS

S. P. NOVIKOV AND P. G. GRINEVICH

In the Novikov–Krichever formula for a fourth-order operator L_4, occurring in a commuting pair of pank two and genus one, there is an arithmetical error; the operator L_4 has the form

$$L_4 = (d^2/dx^2 + u)^2 + a\, d/dx + d/dx\, a + b \tag{1}$$

(one of the terms $d/dx\, a$ has been omitted), where

$$a = \frac{\lambda_1'}{\mu_1}(\lambda_1 - \lambda_2), \quad b = -\lambda_1 - \lambda_2, \quad \lambda_i = \wp(\gamma_i), \quad \mu_i = -\wp'(\gamma_i).$$

For all the quotations and formulas see the paper by Grinevich in this issue.

Theorem 1. *The operator L_4 is formally symmetric if and only if $a = 0$, i.e., $\lambda_1 = \lambda_2$ (the constant $\gamma_0 = 0$ modulo the semiperiods of the function $\wp$).*

In this case the formulas for the coefficients are strongly simplified:

$$a = 0, \quad b = -2\lambda, \text{ where } \lambda = \wp(c(x)),$$
$$u = \frac{1}{4}\frac{\lambda''^2}{\lambda'^2} - \frac{1}{2}\frac{\lambda'''}{\lambda'} - \frac{1}{4}\frac{P_3(\lambda)}{\lambda'^2}. \tag{2}$$

Theorem 2. *Assume that the following conditions hold*: 1) *the Riemann surface Γ is real* (*i.e., g_1, g_2 are real*); 2) *the function $\lambda(x)$ is real*; 3) *at the points where $\lambda' = 0$ we have $\lambda'' \neq 0$, $\lambda''^2 = P_3(\lambda)$. Then, the operator L_4, defined by* (1), (2), *has nonsingular real periodic coefficients and is a semibounded self-adjoint operator in $L_2(\mathbb{R})$.*

For any linear ordinary differential operator L (with respect to x) with periodic coefficients there is defined a "monodromy matrix" $\hat{T}(\lambda)$, i.e., a shift operator on the period of solutions of the equation $L\psi = \lambda\psi$, written in a certain basis. The order of the matrix $\hat{T}(\lambda)$ is equal to the order k of the operator L. The eigenvectors $\psi_q(x, \lambda)$ of the operator $\hat{T}(\lambda)$ are called the Bloch eigenfunctions (or the Floquet functions). The function $\psi_q(x, \lambda)$ is meromorphic in λ on the Riemann surface $\tilde{\Gamma}$ over the λ-plane of k sheets, whose points are the pairs $Q_q = (\lambda, q)$, $q = 1, \dots, k$.

Theorem 3. *Assume that the operator L of order $k = nl$ occurs in the commuting pair $[A, L] = 0$ of rank l with Riemann surface Γ of finite genus $\{P(A, L) = 0\}$. Then, the monodromy matrix $\hat{T}(\lambda)$, $\lambda \in \mathbb{C}$, written in the basis of the Baker–Akhiezer eigenfunctions $\varphi_\alpha(x, \mathcal{P}_i)$, $\alpha = 1, \dots, l$, defines a matrix $\hat{T}^*(\mathcal{P})$ of order*

Date: Original article submitted May 22, 1981.

UDC. 517.43.

Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 16, No. 1, pp. 25–26, January–March, 1982.

l on the n-sheeted Riemann surface Γ over the λ-plane, where $\mathcal{P}_j = (\lambda, j)$, $j = 1, \ldots, n$. In this case the matrix $\hat{T}(\lambda)$ is in block form,

$$\hat{T}(\lambda) = \begin{pmatrix} \hat{T}^*(\mathcal{P}_1) & & \\ & \ddots & \\ & & \hat{T}^*(\mathcal{P}_n) \end{pmatrix}, \tag{3}$$
$$n = k/l, \quad \pi(\mathcal{P}_j) = \lambda, \quad \pi\colon \Gamma \to \mathbb{C}, \ \pi(\lambda, j) = \lambda,$$
$$\mathcal{P}_1 = (\lambda, 1), \ \ldots, \ \mathcal{P}_n = (\lambda, n).$$

For operators of rank $l = 2$ and order $k = nl = 4$ the matrix $\hat{T}^*(\mathcal{P})$ on a two-sheeted surface Γ is unimodular. In the self-adjoint case the spectrum in $L_2(\mathbb{R})$ of the operator L_4 lies in the "real" subset $\Gamma_{\mathbb{R}} \subset \Gamma$, invariant relative to the complex conjugation (anti-involution) $\sigma\colon \Gamma \to \Gamma$, $\sigma(\Gamma_{\mathbb{R}}) = \Gamma_{\mathbb{R}}$, where the matrix $\hat{T}^*$ is real. For the genus $g = 1$, the image of $\pi(\Gamma_{\mathbb{R}})$ on the λ-line coincides with the set $P_3(\lambda) = 4\lambda^3 + g_2\lambda + g_3 > 0$. In analogy with the usual second-order Schrödinger (Sturm–Liouville, Hill) operator, the spectrum is isolated in the set $\pi(\Gamma_{\mathbb{R}})$ by the condition $\operatorname{Sp} \hat{T}^* \leq 2$.

Conclusions. Thus, the spectral theory of periodic operators of order $k = nl$ of rank l is similar to the spectral theory of order l over a Riemann surface instead of the λ-plane (the spectral parameter becomes "effectively" not rational but algebraic). Conversely, probably, if the monodromy matrix $\hat{T}^*(\lambda)$ reduces to the block form (3), where $\hat{T}^*(\mathcal{P})$ is defined on an algebraic n-sheeted Riemann surface Γ, then one has a non-trivial differential operator A of some order $s = ml = mk/n$, which commutes with L, $[A, L] = 0$. In this case the pair (A, L) has rank $l = k/n$. The converse statement has not been proved so far. The direct statement is proved similarly to the fundamental theorem of [1] for $l = 1$ by the scheme presented in a more general, suitable and invariant form in the survey [2] (Chap. II).

Problem. Investigate the spectral properties of the operator L in $L_2(\mathbb{R})$, if the coefficients are polynomial or rational for the rank $l > 1$ (e.g., for the "Dixmier operator").

Conjecture. We fix the order k only for one operator L (we do not fix the order of A). Then the operators L of rank $l = 1$ are everywhere dense among all periodic operators of order k.

The authors are grateful to I. M. Krichever for his help in the proof of the unimodularity of the matrix $\hat{T}^*$

References

[1] S. P. Novikov, "The periodic problem for the Korteweg–de Vries equation. I," Funkts. Anal. Prilozhen., **8**, No. 3, 54–66 (1974).

[2] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, "Nonlinear equations of Korteweg–de Vries type, finite-zone linear operators and Abelian varieties," Usp. Mat. Nauk, **31**, No. 1, 55–136 (1976).

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR.
Moscow State University