ON POISSON BRACKETS COMPATIBLE WITH ALGEBRAIC GEOMETRY AND KORTEWEG–DE VRIES DYNAMICS ON THE SET OF FINITE-ZONE POTENTIALS

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I. **Some information regarding finite-zone potentials.** [1]. A finite-zone or quasiperiodic (in general, complex) potential $u(x)$ of the one-dimensional operator $L = -d^2/dx^2 + u(x)$ possesses a "Bloch" eigenfunction $\psi_\pm(x, \lambda)$ meromorphic in λ on a Riemann surface of genus g which is two-sheeted over the λ plane. In the periodic case ψ is the eigenvector of translation by a period. With the normalization $\psi_\pm(x_0, \lambda) \equiv 1$ there are g poles $\gamma_q(x_0) = (\tilde{\gamma}_q, \pm)$ and g zeros $\gamma_q(x)$, $q = 1, \dots, g$. The surface Γ can be written in the form

$$y^2 = R(\lambda) = \prod_{j=0}^{2g} (\lambda - \lambda_j), \quad \lambda_j \neq \lambda_k. \tag{1}$$

For real L we have $\lambda \in \mathbb{R}$, $\lambda_0 < \lambda_1 < \dots < \lambda_{2g} < \infty$, and the points λ_j, $\lambda_{2g+1} = \infty$ are end points of the lacunae of the spectrum.

For us the following facts are important.

1. A finite-zone potential is completely determined by an independent collection of data; a nonsingular Riemann surface (1) and a collection of distinct points $\gamma_1, \dots, \gamma_g \in \Gamma$. The collection of all finite-zone potentials is a $(3g + 1)$-dimensional complex space K_g in which there is distinguished a "real part" $K_g^{\mathbb{R}} \subset K_g$ where $\lambda_i \in \mathbb{R}$, and $\gamma_q = (\tilde{\gamma}_q, \pm)$, $\tilde{\gamma}_q \in \mathbb{R}$. For a real, smooth potential, $\gamma_q \in [\lambda_{2q-1}, \lambda_{2q}]$ or $\gamma_q \in a_q$ where a_q is a cycle on Γ over a lacuna.

2. The Abel transformation assigns to the symmetrized collection of points $\gamma = (\gamma_1, \dots, \gamma_g)$ a point of the complex torus $T^{2g} = J(\Gamma)$ (the Jacobi variety), i.e., a vector of the complex space $\mathbb{C}^g$ up to the lattice generated by $2g$ vectors. The image $A(\gamma)$ in $\mathbb{C}^g$ moves linearly as the potential varies according to all higher KdV equations.

3. In the real, smooth case $\lambda_i \in \mathbb{R}$ and $\gamma_q \in a_q$; a real torus $T^g \in J(\Gamma)$ is distinguished which can be geometrically represented as $T^g = A(a_1 \times \dots \times a_g)$. On the torus T^g there are coordinates $(\phi_1, \dots, \phi_g)$ defined mod 2π in which the dynamics of the KdV equation and its higher analogues are linear:

$$\begin{gathered} d\phi_q/d\tau_m = \omega_{mq}(\lambda_0, \lambda_1, \dots, \lambda_{2g}), \quad d\lambda_s/d\tau_m = 0, \\ \oint_{a_{q_1}} d\phi_{q_2} = 2\pi\delta_{q_1 q_2}, \quad m = 0, 1, 2, \dots. \end{gathered} \tag{2}$$

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The coordinates $\phi_1, \dots, \phi_g$ are uniquely determined, admitting only changes of the form

$$\phi_j \to \phi_j + \phi_{j0}(\lambda_0, \lambda_1, \dots, \lambda_{2g}). \tag{3}$$

II. **The Poisson brackets. The most important examples.**

Example 1. All the higher KdV equations are Hamiltonian systems (see [2] and [3]) on the space of rapidly decreasing or periodic functions with a given period (and also on the space of quasiperiodic functions with a given period group). The Hamiltonian structure can be restricted to the set K_g with coordinates $\lambda_0, \dots, \lambda_{2g}, \gamma_1, \dots, \gamma_g$. The annihilator (center) of this bracket is generated functionally by the quantities

$$F_0 = \bar{u}, \quad F_q = T_q(\lambda_0, \lambda_1, \dots, \lambda_{2g}), \quad q = 1, 2, \dots,$$
$$g\colon (F_l, \lambda_j)_1 = \{F_l, \gamma_q\}_1 = 0,$$

where

$$\bar{u} = \lim_{L \to \infty} \frac{1}{L} \int_0^L u(x)\, dx.$$

and T_q are the periods of the potential. Moreover, we have

$$\{\lambda_{j_1}, \lambda_{j_2}\}_1 = \{\gamma_{q_1}, \gamma_{q_2}\}_1 = 0,$$

and the Hamiltonians of the higher KdV equations can be expressed in terms of $\lambda_0, \dots, \lambda_{2g}$.

Example 2. The finite-zone potentials satisfy the commutativity equation $[L, A] = 0$, where A is an operator of order $2g + 1$ having the Lagrangian form (see [1], [4] or [5])

$$[L, A] = 0 \leftrightarrow \delta J / \delta u = 0,$$

$$J(u, u_x, \dots, u^{(g)}, c_1, c_2, \dots, c_{g+1}) = I_g + \sum_{i=1}^{g+1} c_i I_{g-i},$$

where $I_{-1}, I_0, \dots, I_g$ are the integrals of Kruskal and others (see [1]),

$$c_1 = \sum_{j=0}^{2g} \lambda_j, \quad c_k = \sum_{i_1 < i_2 < \dots < i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

Thus, the commutativity equations are Hamiltonian systems relative to the standard Poisson bracket generated by the variational problem (6). The quantities c_i form the annihilator of this bracket on K_g:

$$\{c_i, \lambda_j\}_2 = \{c_i, \gamma_q\}_2 = 0.$$

Moreover, in a similar way

$$\{\lambda_{j_1}, \lambda_{j_2}\}_2 = \{\gamma_{q_1}, \gamma_{q_2}\}_2 = 0.$$

All the higher KdV equations are Hamiltonian systems in the bracket $\{\cdot, \cdot\}_2$ with Hamiltonians which can be expressed explicitly in terms of $\lambda_0, \dots, \lambda_{2g}$ (see [7] and [6]).

Example 3. An important Poisson bracket is generated by the cryptoisomorphism (of Moser and Trubowitz, see [9] and [8]) between the KdV dynamics on the space K_g and the classical Neumann systems for the motion of a particle on the sphere $S^g \subset \mathbb{R}^{g+1}$ under the action of a constraint and a quadratic potential $U(\vec{x}) = \sum_0^g d_s x_s^2$ depending on $g + 1$ parameters $d_0, \dots, d_g$. After this isomorphism the standard Poisson bracket on $T^*(S^g)$ induces a bracket $\{\cdot,\cdot\}_3$ on K_g such that all the higher KdV equations are Hamiltonian and

$$\{\lambda_{2j},\lambda_i\}_3=\{\lambda_{2j},\gamma_q\}_3=0,\quad \lambda_{2j}=d,\quad \{\lambda_{j_1},\lambda_{j_2}\}_3=\{\gamma_{q_1},\gamma_{q_2}\}_3=0.$$

The Hamiltonians of all higher KdV equations in this bracket are explicitly computed in [9].

We shall consider some other examples below.

III. **The main results.** Suppose that there is a submanifold $K'_g\subset K_g$ given by equations only in the variables λ_j, while the dimension of K'_g is not less than $2g$,

$$\dim K'_g=2g+r,\quad r\geq 0.$$

On K'_g we shall consider Poisson brackets $\{\cdot,\cdot\}$ possessing certain of the following properties ("analytic brackets"),

a) $\{\lambda_{j_2},\lambda_{j_2}\}=\{\gamma_{q_1},\gamma_{q_2}\}=0.$

b) There exists a function $Q(\lambda_0,\dots,\lambda_{2g},\lambda)$, $z^{-1}=\sqrt{\lambda}$ which is analytic in z in a punctured neighborhood of $z=0$ and such that

$$\{Q(\lambda_0,\lambda_1,\dots,\lambda_{2g},\gamma_{q_1}),\gamma_{q_2}\}=\delta_{q_1q_2}.$$

Definition 1. We call a Poisson bracket possessing property a) *compatible with algebraic geometry.*

Definition 2. We call a Poisson bracket *compatible with KdV dynamics* if it possesses property a) and all the higher KdV equations are Hamiltonian systems. It is assumed that their Hamiltonians $h_m(\lambda_0,\dots,\lambda_{2g})$ can be chosen so that the series

$$h_m(\lambda_0,\lambda_1,\dots,\lambda_{2g},z)=2\sum_{m\geq 0}\left(\frac{z}{2}\right)^{2m+3}h_m(\lambda_0,\lambda_1,\dots,\lambda_{2g})$$

has nonzero radius of convergence. (It is likely that the last condition is not essential, but without it the proofs become more involved.)

Definition 3. A Poisson bracket is said to be *analytically compatible with the real structure* if it possesses properties a) and b), while $Q(\lambda_0,\lambda_1,\dots,\lambda_{2g},\lambda)\,d\lambda$ can be extended as a single-valued meromorphic 1-form without poles on a_j to a covering $\hat\Gamma\xrightarrow{P}\Gamma$ of the Riemann surface Γ where all the cycles a_q remain closed in $\hat\Gamma$, and for $\lambda_j\in\mathbb{R}$ it is required that

$$Q(\lambda_0,\lambda_1,\dots,\lambda_{2g},\bar\lambda)=\bar Q(\lambda_0,\lambda_1,\dots,\lambda_{2g},\lambda).$$

We have the following simple result.

Lemma 1. *If a Poisson bracket is compatible with algebraic geometry, and the Hamiltonian H depends only on $\lambda_0,\dots,\lambda_{2g}$, then the Hamiltonian system is linearized by the Abel transformation.*

In correspondence with general Liouville theory [10] we can go over to action-angle variables. The angles have already been described above (see (2)); the arbitrariness (3) makes it possible to satisfy the condition $\{\phi_{q_1},\phi_{q_2}\}=0$.

Lemma 2. *If a Poisson bracket is analytically compatible with the real structure, then the action variables J_q conjugate to the angles $\phi_1, \ldots, \phi_g$ can be expressed solely in terms of λ_j and have the form*

$$J_q = \frac{1}{2\pi} \oint_{a_q} Q(\lambda)\, d\lambda, \quad z = \lambda^{-1/2}.$$

We shall consider the examples of brackets presented above.

Example 1. It is shown in [11] that $Q(\lambda) = \pm 2ip(\lambda)$, where $p(\lambda)$ is the "quasimomentum" defined by the Abelian differential dp:

$$\oint_{a_q} dp(\lambda) = 0, \quad dp(\lambda) = \frac{dz}{z^2} + 0(1), \quad z = \lambda^{-1/2}.$$

If $u(x)$ is periodic, then

$$\psi_\pm(x+T,\lambda) = \exp(\pm ip(\lambda)T) \quad \psi_\pm(x,\lambda).$$

Expanding the quantity $p(\lambda)$ in z at $\lambda = \infty$, we obtain the Hamiltonians of the higher KdV equations with index m as the coefficients of z^{2m+3} (see [1]), and

$$h(z) = -2 \sum_{m\geq 0} \left(\frac{z}{2}\right)^{2m+3} h_i(\lambda_0, \lambda_1, \ldots, \lambda_{2g}) = 2p(z) - \frac{2}{z} + z\bar{u}.$$

Example 2. Using the results of [12], we obtain

$$Q(\lambda) = \sqrt{-R(\lambda)}, \quad R(\lambda) = \prod_{i=0}^{2g} (\lambda - \lambda_i).$$

Example 3. Comparing with [9], [8] and [12], we obtain

$$Q(\lambda) = \frac{1}{2}\sqrt{-R(\lambda)} \prod_{i=0}^{2g} (\lambda - \lambda_{2g})^{-1}.$$

Lemma 3. *If a Poisson bracket is compatible with KdV dynamics and possesses property* b), *then the difference $Q - h = P$ is a Laurent series in z and z^{-1} with coefficients in the annihilator*:

$$Q - h = \sum_{-\infty<\alpha<\infty} p_\alpha(\lambda_0, \lambda_1, \ldots, \lambda_{2g}) z^\alpha = P(\lambda_0, \lambda_1, \ldots, \lambda_{2g}, z),$$

$$\{p_\alpha, \lambda_j\} = \{p_\alpha, \gamma_q\} = 0.$$

Proof. We consider the time derivatives of $\gamma_q^{(m)}$ on the basis of the higher KdV equations with index m and the series

$$\{h, \gamma_q\} = 2 \sum_{m\geq 0} \left(\frac{z}{2}\right)^{2m+3} \gamma_q^{(m)}. \tag{4}$$

It follows from the results of [12] that the series (4) possesses the property

$$\{h(\lambda_0, \lambda_1, \ldots, \lambda_{2g}, z(\gamma_{q_1})), \gamma_{q_2}\} = 2 \sum_{m\geq 0} (2^{-1} z(\gamma_{q_1}))^{2m+3} \gamma_{q_2}^{(m)} = \delta_{q_1 q_2}.$$

Hence $\{P(\lambda_0, \ldots, \lambda_{2g}, z(\gamma_{q_1})), \gamma_{q_2}\} = 0$.

On the level surfaces of the annihilator this Poisson bracket is nondegenerate. From the foregoing it therefore follows that

$$P(\lambda_0, \ldots, \lambda_{2g}, \gamma_i) = G_i(a, \gamma_1, \ldots, \gamma_g)$$

where $a(\lambda_0, \dots, \lambda_{2g})$ belongs to the annihilator. From the form of $G_i = P(\lambda_0, \dots, \lambda_{2g}, \gamma_i)$ and the independence of the groups of variables λ_j and γ_q on the full phase space it follows that $G_i = G(a, \gamma_i)$. Finally, again using the independence of λ_j and γ_i, we make the reverse substitution $\gamma_i \to \lambda$ and obtain finally

$$P(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{2g}, \lambda) = Q - h = G(a, \lambda),$$

where $a(\lambda_0, \dots, \lambda_{2g})$ belongs to the annihilator. The proof of the lemma is complete. □

Summarizing the preceding lemmas, we obtain the following result.

Theorem. *If a Poisson bracket is compatible with KdV dynamics and is analytically compatible with the real structure, then the following assertions are true*:

a) *The action variables conjugate to the angles on the compact real torus $T^g = A(a_1 \times \dots \times a_g) \subset J(\Gamma)$ have the form*

$$J_q = \frac{1}{2\pi} \oint_{a_q} Q\, d\lambda = -\frac{1}{2\pi} \oint_{a_q} \lambda\, dQ. \tag{5}$$

b) *In the expansion*

$$Q = \sum_\alpha \frac{z^\alpha}{2} g_\alpha(\lambda_0, \dots, \lambda_{2g}),$$

of Q in the Laurent series in $z = \lambda^{-1/2}$ near $z = 0$, the coefficients $h_m = \frac{1}{2} g_{2m+3}$, $m \geq 0$, are the successive Hamiltonians of the higher KdV equations; *the remaining g_α belong to the annihilator of the Poisson bracket.*

Remark. It follows from (5) that in Example 1, where $Q = \pm 2ip(\lambda)$, the action variables (and hence the quasiclassical Bohr–Sommerfeld quantization conditions) can be expressed in terms of integrals of the type of the Peierls functions in the "jelly" model (see [13]) but over forbidden zones with imaginary quasimomentum.

IV. **Other examples of Poisson brackets.**

Example 4. The restriction to K_g of the bracket of [14] is not unique; here $O_{\alpha,\beta}(\lambda) = \pm 2ip(\lambda)(\alpha\lambda + \beta)^{-1}$; the annihilator of the Poisson brackets obtained consists of all periods $T_q(\lambda_0, \dots, \lambda_{2g})$ and any quantity functionally independent of the coefficients of the series $Q_{\alpha,\beta}(\lambda)$ and all the T_q (for example, $\sum_0^{2g} \lambda_j$). These brackets possess all the "good" properties.

Example 5. Already in [1] for $g = 2$ a cryptoisomorphism of the KdV dynamics with the dynamics of a solid body in the integrable case of Kovalevskaja was discussed; this induces on K_2' new Poisson bracket. Here

$$Q(\lambda) = \sqrt{-R(\lambda)}(\lambda - \lambda_3)^{-1}(\lambda - \lambda_4)^{-1} = z^{-1} + 3Hz + \cdots,$$

where $H = \frac{1}{3} \sum \lambda_i$ is the gyroscope Hamiltonian giving the dynamics corresponding to translation in x (see, for example, [15]). This bracket is already not compatible with KdV dynamics (see Lemma 3).

Other algebro-geometric brackets on K_2 or $K_2' \subset K_2$ can be obtained by proceeding from the integrable cases of Clebsch and Steklov in hydrodynamics and of Gorjacev and others in the theory of a solid body with a fixed point [15].

Remark. A parallel theory can also be developed for the nonlinear Schrödinger equation, the Toda lattice, etc. The analytic properties of $Q(\lambda)$ are to be altered with the consideration that $\lambda = \infty$ consists of 2 points. For the nonlinear Schrödinger equation the number of phase variables increases. The special case of the nonlinear Schrödinger equation and the sine-Gordon equation, where reality conditions are nontrivial, will be investigated in a subsequent paper by B. A. Dubrovin and S. P. Novikov.

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