PERIODIC SOLUTIONS OF KIRCHHOFF'S EQUATIONS FOR THE FREE MOTION OF A RIGID BODY IN A FLUID AND THE EXTENDED THEORY OF LYUSTERNIK–SHNIREL'MAN–MORSE (LSM). I

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Introduction

Following Kirchhoff, we consider the potential motion of a finite rigid body in an ideal incompressible fluid, at rest at infinity, where the induced motion of particles of the fluid is completely determined by the motion of the body (cf. [1, Chap. 17]). Let M and p be the total moment and momentum in a fixed system of coordinates, rigidly connected with our body; $H(M,p)$ be the energy of the system. Kirchhoff's equations have the form

$$\dot{p} = p \times \omega, \quad \dot{M} = M \times \omega + p \times u, \tag{1}$$

where $\omega^i = \partial H/\partial M_i$, $u^i = \partial H/\partial p_i$ are the angular and translational velocities. Ordinarily the quantity $H(M,p)$ is an (arbitrary) positive quadratic form in the variables $(M_1, M_2, M_3, p_1, p_2, p_3)$, defined by the geometry of the body. For non-simply connected rigid bodies there are also possibly linear terms in H. Thers are known non-trivial (i.e., not reducing to the obvious symmetry group) integrable cases of Clebsch, Steklov, etc. (cf. [2]), but in general form the Kirchhoff problem is nonintegrable (although this is not rigorously proved). We always have the following integrals:

$$\begin{aligned} &f_1 = \sum p_i^2 = p^2, \quad f_2 = \sum M_i p_i = sp \quad \text{(Kirchhoff integrals)},\\ &2E = 2H = \sum a_{ij} M_i M_j + \sum 2b_{ij} M_i p_j + \sum c_{ij} p_i p_j \quad \text{(``energy'')}. \end{aligned} \tag{2}$$

Using some not complicated (but contemporary in its character) geometric and topological considerations, it turns out to be possible to carry out a qualitative study of the periodic motions by the methods of the calculus of variations in the large, the Lyusternik–Shnirel'man–Morse theory and its interesting generalizations (cf. Secs. 2 and 3), although "at first glance" this theory has no relation to Kirchhoff's equations (1). As far as the authors know, such a connection has not been discussed previously with one exception: the authors succeeded in finding the extraordinarily interesting work of V. V. Kozlov and M. P. Kharlamov (cf. [3, Chap. 6]), which investigated the problem of periodic motions of a rigid body in a vacuum with a fixed point in the gravitational force field. Starting from other ideas, these authors for the zeroth level of the "area integral" noted the reduction

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of the problem to a Lagrangian one with standard Lagrangian type ("kinetic energy minus potential") on the sphere S^2 and applied the LMS theory to find periodic orbits. We note that their problem can also (although this is not usually done) be represented on the Lie algebra of the group $E(3)$ with a certain Hamiltonian of special form, so that the mathematics of this problem is reduced to a special case of Kirchhoff type equations. The application of this kind of idea to Kirchhoff's problem was not discovered by them. They used geometric language and techniques essentially connected precisely with the fixed point (e.g., in the technique for reducing to a Lagrangian system on S^2, a large role was played by the group of rotations about this point and axis parallel to the force field). Incidentally, the area integral corresponds to the Kirchhoff integral f_2 after passage to the algebra of $E(3)$.

For example, this fact is established: *if* $f_2 = 0$*, then for any values of the Kirchhoff integrals* $(f_1 = p^2 \neq 0)$*, any energy* $E > E_0(p)$ *the system* (1) *has not less than two periodic orbits.*[1]

The motions obtained by us are periodic only in the variables (M, p), connected with the body. The realization of this motion in the external space τ^3 requires in addition the solution of linear equations with coefficients depending periodically on the time, expressed in terms of $M(t)$ and $p(t)$. Let $A^{-1}(t)$ be a rotation, carrying a fixed frame τ_0 into the frame $\tau(t)$, rigidly connected with the body, and $A(0) = 1$.

One has the equations

$$\dot{A} = -A\omega(t), \quad \dot{x} = A(t)u(t), \tag{3}$$

where $x^i(t)$ are the coordinates of the center of mass of the body, $\omega = (\omega^i)$, $u = (u^i)$ are the angular and translational velocities in the frame $\tau(t)$. In terms of the period T we have $\omega(t+T) = \omega(t)$, $u(t+T) = u(t)$. Quantities $A(T)$ and $x(T) - x(0)$ together determine for us a motion $B \in E(3)$, an element of the group of motions of the space τ^3. The position of the body at moments of time nT for all integral n is determined by the elements $B^n \in E(3)$. In the case of "general position" the element B is a helical rotation with translation along some axis in the space τ^3 by a finite vector δ and rotation by an angle Δ in the plane perpendicular to the vector δ. If the angle Δ is incommensurable with the number 2π, then we get a doubly periodic motion (neglecting parallel translation of the space). Thus, our theorem asserts the presence of a sufficiently large number of two-dimensional invariant tori T^2 in the phase space of motions of a rigid body in a fluid (modulo parallel translations). Thus, "in the mean" the body will be displaced along the direction of the vector δ and rotated around this axis, although in the course of a period T the motion can have complicated character.

The connection with global problems of the calculus of variations will be discussed by the authors more broadly than for the Kirchhoff equations. It is interesting, however, that precisely for Eqs. (1) on the surface $f_2 = 0$ there arises a situation, close to the classical Poincare problem, solved by Lyusternik and Shnirel'man in 1930 (cf. [4]), on periodic geodesics of a certain metric on the sphere S^2, or more generally, of periodic extremals of certain positive-definite functionals on the sphere S^2. We stress again that the connection of the Kirchhoff equations with the

[1]The completion of the investigation for $f_2 \neq 0$ and the further development of the "extended LSM theory" are carried out in a paper of S. P. Novikov which will soon appear in Funktsional'nyi Analiz [9].

Poincare problem on geodesics on the sphere S^2 is the fundamental observation, lying at the base of the present paper, whose subsequent investigation led to a distinctive extension of the LSM theory.

1. Hamiltonian Formalism for Kirchhoff's Equations

We recall that (1) from a modern point of view is a Hamiltonian system on the Lie algebra $E(3) = L$ of the group of motions of Euclidean three-dimensional space, although up to the most recent time it seems that this has nowhere been noted explicitly. More precisely, the phase space is the dual space $L^* = \tau^6$ with basis $(e_1' e_2', e_3', e_1'', e_2'', e_3'')$ where a typical element $q \in L^*$ has the form

$$q = \sum_{i=1}^{3} M_i e_i' + \sum_{i=1}^{3} p_i e_i''. \tag{4}$$

The Poisson bracket $\{f, g\}$ is defined for functions $f(q), g(q)$ on the phase space L^*. The simplest linear functions $f_i(q) = M_i$ or $g_i(q) = p_i$ already by definition represent elements of the algebra $L = E(3)$. One introduces the obvious definition: the Poisson bracket of linear functions on L^* is equal to their commutator in the algebra L,

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, p_j\} = \varepsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0. \tag{5}$$

Together with the Leibniz formula and the other standard properties of the Poisson bracket (cf. [5, Part I, Chap. 5])

$$\{fg, h\} = f\{g, h\} + g\{f, h\} \tag{6}$$

formula (5) completely determines the Poisson bracket of any smooth functions. Equation (1) has standard Hamiltonian form in the Liouville form (this is easy to verify):

$$\dot{M}_i = \{M_i, H\}, \quad \dot{p}_i = \{p_i, H\}, \tag{7}$$

where $H(M, p)$ is the Hamiltonian. General quadratic Hamiltonians have form (2). Passage to an equivalent system is effected only by motions of three-dimensional space, i.e., by the natural action of the group $E(3)$ on the space L^*,

$$(Aq, l) = (q, AlA^{-1}), \tag{8}$$

where $A \in E(3)$, $q \in L^*$, $l \in L$, the action is indicated in a matrix realization. By a transformation (8) one can reduce form (2) to the canonical form

$$H = \frac{1}{2}\left(\sum a_{ii} M_i^2 + 2\sum b_{ij}\left(\frac{M_i p_j + p_i M_j}{2}\right) + \sum c_{ij} p_i p_j\right) \tag{9}$$

(i.e., the matrix a_{ij} is diagonal, and b_{ij} is symmetric). Similar Hamiltonian systems have "trivial" integrals of motion, besides the energy E: there exist nontrivial functions on the phase space L^*, having zero Poisson bracket with the entire phase space,

$$f_1 = \sum p_i^2 = p^2, \quad f_2 = \sum M_i p_i = sp, \quad \{f_1, g\} = \{f_2, g\} = 0, \tag{10}$$

for any function g on L^* (f_1, f_2 are Kirchhoff integrals).

We turn our attention to the following very simple but important circumstance.

Lemma 1. *Any level surface of the Kirchhoff integrals $f_1 = p^2 \neq 0$, $f_2 = ps$ is topologically equivalent with the space of the tangent bundle over the two-dimensional sphere S^2.*

Proof. The sphere S^2 is defined by the equation $f_1 = p^2 \neq 0$. The second equation $f_2 = \sum M_i p_i = sp$ for fixed $\{p_1, p_2, p_3\} \in S^2$ determines a plane in the M-space, parallel to the tangent plane to the sphere at this point. Whence follows Lemma 1. □

We introduce variables $q_i = M_i - \gamma p_i$ from the orthogonality conditions

$$\sum q_i p_i = \sum (M_i - \gamma p_i) p_i = ps - \gamma p^2 = 0, \quad \gamma = sp^{-1}. \tag{11}$$

We have the Poisson brackets $\{q_i, q_j\}$, following from (5). We note an important circumstance: since the functions f_1, f_2 have identically zero Poisson brackets with the entire phase space, the Poisson brackets of functions on the level surface ($f_1 = p^2$, $f_2 = sp$) are calculated from the same formulas (5) taking formal account of the conditions $f_1 = \text{const}$, $f_2 = \text{const}$. Here on the level surface the matrix of Poisson brackets of (local) coordinates is nondegenerate; its skew-symmetric inverse matrix defines a closed 2-form Ω, giving a symplectic structure on the four-dimensional structure ($f_1 = p^2$, $f_2 = ps$).[2] The fact that the 2-form Ω, obtained from the Kirchhoff theory, for now does not have the standard form $\sum_{l=1}^{2} d\xi_l \wedge dx^l \neq \Omega$, where x^l are local coordinates on the sphere S^2 and ξ_l are the corresponding impulses in the cotangent space $T^*(S^2)$ (the coordinates x^l are given on the sphere $\sum p_i^2 = p^2$) is a serious difficulty for us. This fact is especially important for us: we have worked up until now in the Hamiltonian formalism. The desire to apply the methods of "the calculus of variations in the large" requires passage to the Lagrangian formalism, where the Lagrangian is a uniquely defined scalar. This is possible only in the case when the form Ω is exact and has standard form [here globally on $T^*(S^2)$]. As we know, from any Lagrangian, by the Legendre transformation one gets a Hamiltonian formalism in explicit canonical variables, and the symplectic structure on $T^*(S^2)$ is standard.

One has the following

Lemma 2. *We consider the natural projection $T^*(S^2) \xrightarrow{\pi} S^2$ of any level surface of the Kirchhoff intergals ($f_1 = p^2 \neq 0$, $f = ps$) onto the sphere S^2, where $\pi(p, q) = p$. For any pair of functions g and h on the sphere S^2 the Poisson bracket $\{\pi^* g, \pi^* h\}$ is equal to zero.*

The proof of Lemma 2 follows immediately from (5) and the fact that the Poisson bracket on the level surface is obtained by simple restriction.

We consider the inclusion $S^2 \to T^*(S^2)$, where $q_i = 0$, giving a basic cycle $z \in H_2(T^*(S^2)) = \mathbb{Z}$. An easy calculation leads to this fact.

Lemma 3. *The cohomology class of the 2-form Ω, defining the symplectic structure, has the form*

$$(\Omega, z) = \iint_{S^2} \Omega = 4\pi s \tag{12}$$

and vanishes only on level surfaces $f_2 = 0$, $f_1 \neq 0$.

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[2] In the theory of group representations this form is called the "Kirillov form" on an orbit of the action of the group G on the space L^*, coinciding with a level surface of the Kirchhoff integrals for the case of the group $E(3)$.

We shall indicate an extraordinarily useful change of variables into spherical coordinates on the sphere S^2, reducing the form Ω to simple canonical form. We introduce spherical coordinates $-\pi/2 \leqslant \theta \leqslant \pi/2$, $0 < \psi \leqslant 2\pi$ and variables p_θ, p_ψ, starting from the formulas

$$
\begin{aligned}
&p_1 = p\cos\theta\cos\psi, \quad q_1 = M_1 - \frac{s}{p}p_1,\\
&p_2 = p\cos\theta\sin\psi, \quad q_2 = M_2 - \frac{s}{p}p_2,\\
&p_3 = p\sin\theta, \qquad\quad q_3 = M_3 - \frac{s}{p}p_3,\\
&q_1 = p_\psi \operatorname{tg}\theta\cos\psi - p_\theta\sin\psi, \quad q_2 = p_\psi\operatorname{tg}\theta\sin\psi + p_\theta\cos\psi,\\
&q_3 = -p_\psi, \quad \theta = x^1, \quad \psi = x^2, \quad p_\theta = \xi_1, \quad p_\psi = \xi_2.
\end{aligned} \tag{13}
$$

Lemma 4. *The Poisson brackets and the form Ω have the form*

$$
\begin{aligned}
&\{\theta,\psi\} = \{\theta, p_\psi\} = \{p_\theta,\psi\} = 0, \quad \{p_\theta,\theta\} = \{p_\psi,\psi\} = 1,\\
&\{p_\theta, p_\psi\} = s\cos\theta,\\
&\Omega = \sum_{l=1}^{n} d\xi_l \wedge dx^l + s\cos\theta\, d\theta\wedge d\psi.
\end{aligned} \tag{14}
$$

The proof of Lemma 4 is extracted by direct calculation from the formula (13) for the change of variables.

We note that the form Ω is a special case of forms of the form

$$
\Omega = \sum_{l=1}^{2} d\xi_l \wedge dx^l + \pi^*\Omega_0, \tag{15}
$$

where $\pi\colon T^*(M^n) \to M^n$ is the projection, Ω_0 is some closed 2-form in the base M^n; and (x^l) is a collection of local coordinates in the base M^n.

2. Lagrangian and Hamiltonian Formalisms. Periodic Problem of the Extended Calculus of Variations in the Large. Magnetic Field and Poisson Bracket

Thus, we postulate the following situation: there is a symplectic manifold with 2-form Ω, diffeomorphic without considering the symplectic structure to the constant bundle $T^*(M^n)$ of some manifold. Let $\pi\colon T^*(M^*) \to M^n$ be the natural projection.

Definition 1. The symplectic structure Ω on $T^*(M^n)$ is said to be "variationally admissible" if it has the following property: for any pair of functions f, g on the base M^n, the Poisson bracket $\{\pi^* f, \pi^* g\}$ is equal to zero.

Example. By virtue of Lemmas 1 and 2 of Sec. 1, the Poisson bracket occurring in the theory of Kirchhoff's equations are variationally admissible.

For variationally admissible symplectic structures all the fibers $\pi^{-1}(Q)$, $Q \in M^n$ are Lagrangian submanifolds, like the ordinary momentum space. Thus, $T^*(M^n)$ is fibered by Lagrangian manifolds. We note that more general "Lagrangian fibrations" of symplectic manifolds were considered earlier (cf. [6]), although for other goals also. Now we consider a covering of the manifold M^n by domains U_α with the following properties:

$$
M^n = \bigcup_\alpha U_\alpha,
$$

a) the complement $M^n \setminus U_\alpha$ has codimension 2; b) for any smooth closed curve γ in M^n one can find a domain U_α such that γ lies entirely in U_α; c) the form Ω, restricted to the domain U_α for any α, is exact: $\Omega = d\omega_a$.

As usual, by the "phase Lagrangian" L_α and "action" S_α we mean the following quantities for curves in the domain $\pi^{-1}(U_\alpha)$:

$$L_\alpha \, dt = \omega_\alpha - H \, dt, \tag{16}$$

where H is a given function on $T^*(M^n)$, the "Hamiltonian,"

$$S_\alpha(\gamma) = \int_{P_1}^{P_2} (\omega_\alpha - H \, dt), \quad \gamma(0) = P_1, \quad \gamma(1) = P_2 \tag{17}$$

[the integral is taken along the curve $\gamma(t) \subset \pi^{-1}(U_\alpha)$ between the points P_1 and P_2 of the manifold $T^*(M^n)$].

In the intersection of domains $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$ the difference of the two Lagrangians is a closed form,

$$L_\alpha \, dt - L_\beta \, dt = \omega_\alpha - \omega_\beta, \quad d(\omega_\alpha - \omega_\beta) = 0. \tag{18}$$

Here the Lagrangian L_α itself is defined modulo closed 1-forms,

$$L_\alpha \, dt \to L_\alpha \, dt + \varphi_\alpha, \quad d\varphi_\alpha = 0. \tag{19}$$

Thus, the collection of all Lagrangians L_α defines for us a 1-cocycle $(\omega_\alpha - w_\beta)$ for the covering U_α of the manifold M^n with coefficients in the sheaf $\mathbb{Z}^{(1)}$ of germs of closed 1-forms. We note the following:

$$0 \to \tau \xrightarrow{i} O_\tau \xrightarrow{d} \mathbb{Z}^{(1)} \to 0,$$

where τ is the sheaf of constants, and the isomorphism is established by the coboundary operator in the well-known exact sequence of sheaves

$$H^1(T^*(M^n), \mathbb{Z}^{(1)}) = H^1(M^n, \mathbb{Z}^{(1)}) \xrightarrow{\delta} H^2(M^n, \tau) = H^*(T^*(M^n), \tau),$$

where O_τ is the sheaf of germs of real smooth C^∞-functions on $T^*(M^n)$, i is the inclusion of the constants. The cocycle $\kappa = (\omega_\alpha - \omega_\beta)$ is connected with the cohomology class of the form $\Omega \in H^2(M^n, \tau)$ as follows: $\kappa = \delta^{-1}(\Omega)$.

In what follows we shall always use the local coordinates $(x^1, \dots, x^n)$ in the domains U_α or any parts of them and the corresponding coordinates $\pi^* x^i$ in the preimages $\pi^{-1}(U_\alpha)$, again denoted by x^i: $(x^1, \dots, x^n, y^1, \dots, y^n)$ are the coordinates in $\pi^{-1}(U_\alpha)$, where the form Ω has the following form by virtue of the requirement of variational admissibility:

$$\Omega = \sum d_{ij} dx^i \wedge dy^i + \sum e_{ij} dy^i \wedge dy^j, \tag{20}$$

where the matrix d_{ij} is nondegenerate, d_{ij} and e_{ij} depend on (x, y), $d\Omega = 0$.

Definition 2. The Hamiltonian $H(x, y)$ is said to be strongly nondegenerate in all the domains $\pi^{-1}(U_\alpha)$, if the equation

$$\dot{x}^j = \{x, H\} = F_j(x, y) \tag{21}$$

is solvable uniquely and globally in the form

$$y^i = y^j(x, \dot{x}). \tag{22}$$

Now we consider the second-order equation in the variables $x, \dot{x}$, following along with (22) from the second group of Hamilton's equations,

$$\dot{y}^i = \{y^i, H\}. \tag{23}$$

One has the following easy lemma.

Lemma 5. *Suppose given a strongly nondegenerate Hamiltonian on the manifold $T^*(M^n)$ with variationally admissible symplectic structure Ω and that the manifold $T^*(M^n)$ is provided with a covering U_α, with properties* a)–c) *indicated above. In this case the second-order equation* (23) *in the variables $x, \dot{x}$, considered as functions* (*local coordinates*) *on the base M^n, is obtained from the Lagrange variational principle*

$$\delta S_\alpha = 0.$$

With the help of the local Darboux theorem, introducing canonical coordinates in $T^*(M^n)$ for fixed (x) we easily reduce the proof of Lemma 5 to the classical lemmas of Lagrange and Hamilton.

Now we consider an arbitrary smooth closed curve $\gamma(t) \subset M^n$, which is situated entirely in the domain U_α as well as entirely in the domain U_β, $\gamma \subset U_\alpha \cap U_\beta$. There are defined two actions $S_\alpha(\gamma)$ and $S_\beta(\gamma)$. If the cohomology class γ in the group $H_1(U_\alpha \cap U_\beta, \tau)$ is nontrivial, then the following situation is possible

$$S_\alpha(\gamma) \neq S_\beta(\gamma), \quad S_\alpha(\gamma) - S_\beta(\gamma) = \oint_\gamma (\omega_\alpha - \omega_\beta). \tag{24}$$

The set of closed curves W_α of the type of interest to us, lying in the domain U_α, forms an open domain in the space of all closed curves of this type on the manifold M^n, denote by $\hat{\Omega}(M^n) = \bigcup_\alpha W_\alpha$ (e.g., smooth curves of a given homotopy class on M^n, non-self-intersecting curves for the case $M^n = S^2$, τP^2, etc.). On the domain $W_\alpha \subset \hat{\Omega}(M^n)$ there is defined the function (functional) $S_\alpha(\gamma)$. By virtue of (24), the first variations δS_α and δS_β coincide in the intersections of domains $W_\alpha \cap W_\beta$. Thus, we have an infinite-dimensional analog of a closed 1-form (δS_α) on $\hat{\Omega}(M^n)$. There is a locally well-defined family of "level surfaces" $S_\alpha = \text{const}$, independent of the choice of the index α in the intersections $W_\alpha \cap W_\beta$.

Lemma 6. *If the collection of Lagrangians L_α arising in Lemma* 5 *satisfies for all α the positivity condition*

$$(\partial^2 L_\alpha / \partial x^i \, \partial x^j)\xi^i \xi^j > 0, \tag{25}$$

then in a neighborhood of any smooth or piecewise-smooth curve $\gamma \subset \hat{\Omega}(M^n)$ there is a well-defined "local LSM theory," just as for germs of smooth functions on finite-dimensional manifolds: *for extremals the Morse index theorem is true, there is well-defined an* (*orthogonal*) *flow for the deformation of the family of level surfaces $S_\alpha = \text{const}$ with respect to the gradient. In the entire collection* (S_α) *there is given a well-defined family of fibers, level surfaces with the usual properties for the finite-dimensional case of uniqueness and continuous dependence on the initial data.*

The proof of Lemma 6 follows directly from the classical theorems of the global calculus of variations.

Under the hypotheses of Lemma 6 one has:

Corollary 1. *One can find a regular covering $\hat{\hat{\Omega}} \xrightarrow{\pi} \hat{\Omega}(M^n)$ with free Abelian monodromy group $T = \mathbb{Z} \times \cdots \times \mathbb{Z}$ such that the collection of $\pi^* S_\alpha$ for all α "fits together" into a single-valued function S on Ω. In particular, if $H_1(\hat{\Omega}(M^n), \tau) = 0$, then $\hat{\hat{\Omega}} = \Omega$. In the case when the symplectic structure was defined by a globally exact form $\Omega = d\omega$ on $T^*(M^n)$, one also has $\hat{\hat{\Omega}} = \hat{\Omega}$. To the third case (of Lyusternik–Shnirel'man), when $\hat{\hat{\Omega}} = \hat{\Omega}$, corresponds the choice of $\hat{\Omega}$ in the form of the space of non-self-intersecting smooth curves on the sphere S^2 modulo single-point curves or on the projective plane τP^2, where $H_1(\tau P_2, \tau) = 0$.*

The applicability of the LSM theory to estimating the number of critical points of functionals on spaces of the type of closed curves $\hat{\Omega}(M^n)$ [or on spaces of the type of $\Omega(M^n, x_0, x_1)$ of curves joining the points x_0 and x_1] is based on, in addition to the well-definedness of the gradient descent by levels and the Morse index theorem, the following important fact: upon descent along any "gradientlike" field any point of the function-space sooner or later "lands" on some critical point. This fact follows from the very important local properties of functionals. Namely, we consider "additive" functionals, obtained by integrating the Lagrangian L along a smooth (or piecewise-smooth) curve. Upon descent according to level surfaces of the functional, there occurs not only the monotone lowering of the level of the functional, but also the monotone improvement of the local properties of the curve; the curve locally becomes smoother and smoother (e.g., for "polygonal arcs" of extremals the total angle between segments decreases "in the mean"). Precisely from the monotone improvement of the properly formulated local properties of the curves upon gradient descent, the fact cited should probably follow.

Problem. Find a class of cases where, under the hypothesis of Lemma 6, any trajectory of the gradient descent for the multivalued action function $\{S_\alpha\}$ in a finite time approaches unboundedly closely to some stationary point, a periodic extremal.

In these cases we arrive at multivalued functions on spaces of closed curves $\hat{\Omega}(M^n)$, described in Corollary 1 and having in addition an important property of lines of gradient descent.

For certain classes of curves we are able to prove this conjecture simply because there arises in the end a simply connected space of curves, on which the multivalued function $S(\gamma)$ reduces to a single-valued one and is positive-definite here. An important such class is the Lyusternik–Shnirel'man class $\hat{\Omega}_1(S^2)$ of non-self-intersecting piecewise-smooth curves $\gamma \subset S^2$. As is known [4], the class of $\hat{\Omega}_1(S^2)$, where the subset of single-point curves $S^2 \subset \hat{\Omega}_1$ lies, is contractible to the subset $Q \subset \hat{\Omega}_1$ of curves which are planar intersections with the sphere S^2. This subclass Q, obviously, is isomorphic with the space of nontrivial $O(1)$-bundles with base τP^2, with fiber the closed segment $I(-1 \leqslant x \leqslant 1)$, where the single-point curves form the boundary. Thus, $\hat{\Omega}_1(S^2)$ is homotopically equivalent with τP^2, and $\hat{\Omega}_1/S^2$ is homotopically equivalent with τP^3.

In our problems, as a rule, there will arise functionals (or multivalued functionals) $S_\alpha(\gamma)$, depending not only on the curve γ itself, but also on its direction. In connection with this we must consider the space of non-self-intersecting directed curves on the sphere $\hat{\Omega}_1^+(S^2)$.

Essentially, the class of functionals of interest to us has the form

$$L = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j - A_i(x)\dot{x}^i - U(x), \tag{26$'$}$$

which are trajectorially isomorphic (for fixed energy E) to "Maupertuis–Fermat" functionals

$$L_E = \sqrt{(E - U(x))g_{ij}\dot{x}^i\dot{x}^j \cdot 2} - A_i\dot{x}^i, \tag{26$''$}$$

where the 1-form $A_i\,dx^i = \omega'_\alpha$ can be defined only in the domain $U_\alpha \subset M^n$. Here (x) are local coordinates in the domain U_α, g_{ij} is the metric and U is a scalar function, defined globally on all of M^n. The 2-form $d\omega'_\alpha = H_{ij}\,dx^i \wedge dx^j$ must also be globally defined on the entire manifold M^n. For $n = 3$ the Lagrangian (26$'$) corresponds to a particle in a magnetic field, where the strength tensor of the magnetic field is equal to $d\omega'_\alpha = H_{ij}\,dx^i \wedge dx^j$.

One has the following simple assertion.

Assertion. [3] We consider the Hamiltonian in the absence of a magnetic field to be the form $d\omega'_\alpha$,

$$H(x,p) = \frac{1}{2} g^{ij}\xi_i\xi_j + U(x). \tag{27}$$

The equation of motion of a particle in a magnetic field, given by the Lagrangian (26$'$), is determined by the same Hamiltonian, but with the new (nontrivial) Poisson brackets

$$\dot{x}^i = \{x^i, H\},\ \dot{\xi}^j = \{\xi^j, H\}, \quad \{x^i, x^j\} = 0, \quad \{\xi_j, x_j\} = \delta_{ij},\ \{\xi_i, \xi_j\} = H_{ij}, \tag{28}$$

where $H_{ij}\,dx^i \wedge dx^j = d\omega'_\alpha = (\partial A_i/\partial x^j - \partial A_j/\partial x^i)dx^i \wedge dx^j$. Bracket (28) is defined by a 2-form of type (14), (15) $\Omega = -dx^i \wedge d\xi_i + H_{ij}\,dx^i \wedge dx^j$.

The proof of the assertion is carried out by direct elementary calculation.

Thus, we get the following conclusion: the systems on the sphere S^2 ($f_1 = p^2 \neq 0$, $f_2 = ps$) arising in the Kirchhoff problem are equivalent to a particle (with constraints) in a potential field, and also in an external magnetic field. Here there are two cases: a) If the "crossed" matrix b_{ij} in Hamiltonian (9) is equal to zero, then this "effective" magnetic field vanishes for $s = 0$. The magnitude of the flux is equal to the level $f_2 f_1^{-1/2} = 4\pi s$. b) If $b_{ij} \neq 0$, then in the Lagrangian there appears an additional everywhere uniquely defined 1-form, equivalent to the variable magnetic field $\delta H_{ij}\,dx^i \wedge dx^j$, orthogonal to S^2 and having flux zero $0 = \iint_{S^2} \delta H_{ij}\,dx^i \wedge dx^j$. Hence even at the level zero $f_2 = ps = 0$ the Lagrangian has the form (26$'$), where the form $A_i\,dx^i \neq 0$, although globally defined on the sphere S^2.

Now we consider a Lagrangian of the form (26$''$) on any Riemannian manifold M^n with complete positive metric $(E - U)g_{ij}$ in both cases a) and b). According to the general results of Morse [7] for such functionals, the Morse index theorem is true ($\partial^2 L/\partial\dot{x}^i\partial\dot{x}^j > 0$) and there is a well-defined gradientlike descent in various spaces of piecewise-smooth directed curves (closed, with fixed ends, etc.). However, a functional of form (26$''$) can be not positive-definite due to the presence of the linear term $A_i\dot{x}^i$.

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[3] In quantum mechanics, where the Poisson bracket is replaced by the commutator, this fact is obvious and known to many physicists. In its classical aspect one can refer, e.g., to [8].

We consider such an example: suppose given the plane τ^2 with the standard metric $g_{ij} = \delta_{ij} \cdot m$ ($m = \text{const}$) in the homogeneous magnetic field directed along the third axis $H = \text{const}$,

$$
\begin{aligned}
L &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{eH}{2c}(\dot{x}y - \dot{y}x), \\
L_E &= \sqrt{2m(E-u)(\dot{x}^2 + \dot{y}^2)} + \frac{eH}{2c}(\dot{x}y - \dot{y}x).
\end{aligned} \tag{29}
$$

The energy is equal to $E = m(\dot{x}^2 + \dot{y}^2)/2$. Motion takes place on a circle of radius R, where $R^2 = 2mc^2e^2EH^{-2}$. This means that for fixed E it is impossible to join two points of the plane τ^2 by an extremal of functional (29). The reason is quite simple: we consider two points of the plane x_1 and x_2 and we join them by a very long curve γ_1, which we close with a "short" segment γ_0 between x_1 and x_2. We parametrize the curves γ_0, γ_1 by the natural parameter τ. Then the action variable on the closed curve

$$
S_E(\gamma_0^{-1}\gamma_1) = \oint_{\gamma_0^{-1}\gamma_1} L_E d\tau \sim S_E(\gamma_1)
$$

is a sum of two pieces: a quantity proportional to length—the integral of the kinetic energy, and a quantity proportional to the area, due to the 1-form (of the magnetic field). Extending γ_1 and choosing its sign, we deduce immediately that the action S_E is not separated from minus infinity. This is also the reason for the inapplicability of an LSM type theory.

We consider the following spaces of non-self-intersecting piecewise-smooth directed curves: $\hat{\Omega}_1^+(S^2)$ on the sphere S^2, $\hat{\Omega}_1^+(\tau P^2) = \hat{\Omega}_{11}^+ \cup \hat{\Omega}_{12}^+$ on the projective plane, which splits into two connected components depending on the homotopy class of the path, trivial for paths from $\hat{\Omega}_{11}^+$ and nontrivial in $\hat{\Omega}_{12}^+$.

Lemma 7. *On all these spaces the multivalued action functional $S_\alpha(\gamma)$ becomes single-valued.*

There are three cases.

Case 1. The form $H_{ij}\,dx^i \wedge dx^j = d(A_j\,dx^j)$ in Lagrangian (26) is not exact. Then the multivalued action functional $\{S_\alpha\}$ defines a single-valued function on the two point completion of the space $\hat{\Omega}_1^+(S^2)$. These points correspond to the family of single-point curves contractible to the point. However this one point turns into two points due to the multivaluedness of the action on the space of all closed curves. We denote the two-point completion of the space by $K \supset \hat{\Omega}_1^+(S^2)$. The homotopy type of the space K is obtained by compactifying the space of planar (not single-pointed) sections of the sphere $S^2 \subset \tau^3$, with the indicated direction on the curve. This space is $S^2 \times I$, where I is the open interval $(-1 < \tau < 1)$. The compactification by a pair of points has the homotopy type of the suspension of the sphere S^2, $K \sim S^3$, where the one-point curves represent the upper and lower poles.

Case 2. The form $H_{ij}\,dx^i \wedge dx^j$ is exact, i.e., the form $A_j\,dx^j$ is globally single-valuedly defined on the sphere S^2. Then we have $K \sim S^3$, as also in Case 1, but with the additional important property that the action function $S(\gamma)$ has one and the same value at both completing points, corresponding to single-point curves. Identifying the upper and lower poles in S^3, we get the homotopy type of the completion of the space of curves of interest to us, which we denote by K', $H_1(K') =$

$\mathbb{Z}$, $H_3(K') = \mathbb{Z}$. Thus, the homology modulo single-point curves here is nontrivial (only H_1, H_3).

Case 3. The form $H_{ij}\,dx^i \wedge dx^j$ is exact, the functional is invariant with respect to reflection of the sphere (i.e., actually on τP^2) $S^2 \to S^2$, $p \to -p$, $M \to +M$ ($s \to -s$). In this case we turn to the space $\hat{\Omega}^+_{12}$ of directed non-self-intersecting piecewise-smooth curves on τP^2, nonhomotopic to zero (here there are no single-point curves). The space $\hat{\Omega}^+_{12}$ has the homotopy type of S^2.

Lemma 8. *On all the spaces* $\hat{\Omega}^+_1(S^2)$, $\hat{\Omega}^+_{12}$ *and their completions* K, K' *by single-point curves the functional* ($26''$) *is separated from minus infinity* (*semibounded below*).

Proof. By virtue of the classical results of many authors it suffices for us to consider only the contribution of the 1-form $A_i\,dx^i$, possibly defined only in the domain U_α, where $H_{ij}\,dx^i \wedge dx^j$ is an exact form. The integral of the form $A_i\,dx^i$ along the curve γ is equal (in modulus) to the integral of $H_{ij}\,dx^i \wedge dx^j$ over the area inside this curve or (up to the single constant $\iint_{S^2} H\,dx^1 \wedge dx^2$) the area outside the curve in S^2. Whence follows Lemma 8. □

One can also indicate certain spaces of curves on the plane τ^2 with analogous properties. We consider a magnetic field directed along the z axis of one of two types: a) the field $H\,dx \wedge dy$ is localized, i.e., decreases rapidly as $x^2 + y^2 \to \infty$; b) the field $H\,dx \wedge dy$ is doubly periodic with some lattice of periods.

In case a) we consider the two spaces $\hat{\Omega}^+_1(\tau^2)$ and $\hat{\Omega}^-_1(\tau^2)$ of closed non-self-intersecting curves in both directions.

In case b) we consider only the space $\hat{\Omega}^+_1(\tau^2)$ of non-self-intersecting curves, which are directed in the direction from the rotation of particles in the homogeneous magnetic field $\bar{H}$, where the sign of the charge is fixed and

$$\bar{H} = \frac{1}{|K|} \iint_K H\,dx \wedge dy,$$

K is an elementary cell of the lattice.

Lemma 9. *In both cases* a) *and* b) *the action functional* $S_E(\gamma)$ *is separated from minus infinity.*

We shall not prove this assertion, since we cannot seriously use it in view of the homotopy triviality of these spaces of curves (here the homology modulo single-point curves is trivial).

If the mean magnetic field $\bar{H}$ vanishes in case b), then we have a single-valued functional defined on closed curves lying on the torus T^2. We consider, e.g., non-self-intersecting directed curves $\hat{\Omega}^+_{1(m,n)}(T^2)$ on the torus T^2 of a fixed homotopy class in $\pi_1(T^2) = \mathbb{Z} + \mathbb{Z}$, where m and n are relatively prime integers, $(m, n) = 1$. The space $\hat{\Omega}^+_{1(m,n)}(T^2)$ has the homotopy type of a circle, $\hat{\Omega}^+_{1(m,n)} \sim S^1$.

Completely analogously to Lemmas 7 and 8 one proves Lemma 10.

Lemma 10. *The action functional* S_E *is semibounded* (*separated from minus infinity*) *on the space* $\hat{\Omega}^+_{1(m,n)}(T^2)$.

3. Application of the Extended Lyusternik–Shnirel'man–Morse (LSM) Theory to Periodic Solutions of Kirchhoff's Equations

We fix the energy E_0 with the following properties: 1) at levels $E > E_0$ the Kirchhoff system (1) has no stationary points on a given level surface of the Kirchhoff integrals $f_1 = p^2 \neq 0$ and $f_2 = ps$, 2) one can find a number $\varepsilon_E > 0$ (in the standard metric), such that any two points of the sphere S^2, mutually situated at a distance not greater than ε_E, can be joined by a unique "short" trajectory of Kirchhoff's equations with energy $E > E_0$ and given f_2, f_1.

Under these conditions one has the following theorems.

Theorem 1. a) *If the Hamiltonian $E = H(M, p)$ of the Kirchhoff problem* (9) *has block form $b_{ij} = 0$, then at any level of the Kirchhoff integrals $f_1 = p^2$, $f_2 = ps = 0$ and for all values $E > E_0(|p|)$, larger than some threshold, there are not less than six periodic orbits which are non-self-intersecting under projection onto the p-sphere S^2, symmetric with respect to the reflection of the sphere $p \to -p$.*

b) *If Hamiltonian* (9) *does not have block form $b_{ij} \neq 0$, then for the zero level of the integral $f_2 = 0$ and any value $f_1 = p^2 \neq 0$ there are at least two* (*non-self-intersecting in p*) *orbits for any energy $E > E_0(p)$, greater than some threshold.*

c) *In the case of general position, when all non-self-intersecting periodic extremals are nondegenerate, their number is necessarily finite and even* (*an extremal is taken together with a direction*; *geometrically there are only half of them for the case $s = 0$ and $b_{ij} = 0$, where the functional S_E is invariant with respect to time reversal*).

Theorem 2. *We consider a particle in a magnetic field $H(x, y)$, directed along the z axis, doubly periodic in (x, y), with zero mean $H = \iint_K H\,dx \wedge dy = 0$, where K is an elementary cell, and in a doubly periodic potential field $v(x, y)$ with the same periods. Then for all energies $E \geqslant \max v(x, y)$ and for any nonzero element of the group $\pi_1(T^2) = \mathbb{Z} + \mathbb{Z}$ there are not less than two non-self-intersecting trajectories* (*up to translation by a vector of the lattice*), *which are periodic on the torus T^2 and represent this element. This means that the difference of the trajectory $\gamma(t)$ and some line with integral slope* (*passing through a pair of points of the lattice*) *is a periodic function, in particular, is bounded on the entire line $-\infty < t < \infty$.*

b) *Let the magnetic field be localized. Then on the plane* (*in the case of general position, when all periodic trajectories are nondegenerate*) *there can be only an even* (*necessarily finite*) *number of non-self-intersecting periodic orbits of each direction of rotation.*

c) *Let the magnetic field be periodic. In the case of general position there can be only an even* (*finite*) *number of periodic orbits, directed opposite to the rotation of particles in a homogeneous field with the same flux as the magnetic field through an elementary cell.*

[The even number in points a) and b) can be equal to zero, so that this is not a very profound assertion.]

Remark. The energy threshold $E_0(p)$ in Theorem 1 can be determined effectively and quite simply: The Lagrangian on the sphere S^2 in the coordinates (θ, ψ) has form (26′), where g_{ij} and U are globally defined smooth quantities (tensor and scalar). We set: $E_0(p)$ is any number larger than $\max U(\theta, \psi)$ on the sphere S^2.

For energy $E \leqslant \max U(\theta, \psi)$ one should consider the boundary of the domain

$$E - U \leqslant 0. \tag{30}$$

Probably, using arguments of the type given in Chap. 6 of [3], one can get in this case theorems on the so-called librational motions, concerning the boundary of domain (30).

The proof of Theorems 1 and 2 is extracted from Lemmas 6-10 of Sec. 2. To prove Theorem 1 it is only necessary to see the positivity of the Lagrangians L_α and the fact that they have the form (26). For this one should operate according to the algorithm indicated in Secs. 1 and 2; we choose a collection of domains U_α in the form of a sphere with a pair of opposite points (poles) punctured, defining the "index" α. We shall calculate in convenient coordinates (13), choosing the form ω_α in the form $L\,dt = \omega_\alpha - H\,dt$,

$$\omega_\alpha = p_\theta\, d\theta + p_\psi\, d\psi + s\cos\theta\, d\psi, \tag{31}$$

$x^1 = \theta$, $x^2 = \psi$ are spherical coordinates. Further, we express p_ψ, p_θ from the equations

$$\dot\psi = \{\psi, H\}, \quad \dot\theta = \{\theta, H\} \tag{32}$$

and we calculate the Lagrangian L_α in terms of $\psi, \theta, \dot\psi, \dot\theta$. Carrying out this procedure, we get the Lagrangian needed, for which the matrix $L_{\dot x^i \dot x^j}$ of second order is calculated without difficulties:

(33)

$$2H = \sum a_{ii}(q_i + sp_ip^{-1})^2 + \sum b_{ij}[(q_i + sp^{-1}p_i)p_j + (q_j + sp^{-1}p_j)p_i] + \sum c_{ij}p_ip_j,$$

$$L = \frac{1}{2} g_{lm}\dot x^l \dot x^m - A_l \dot x^l - U(x),$$

$$g^{lm}(\theta, \psi)\xi_l\xi_m = \sum a_{ii}q_i^2 > 0,$$

$$A_0^l \xi_l = s\left(\sum q_ip_ip^{-1}a_{ii}\right) + p\left(\sum b_{ij}[p_jp^{-1}q_i + p_ip^{-1}q_j]\right),$$

$$A_l\dot x^l = A_l^{(0)}\dot x^l + s\sin\theta\dot\psi, \quad A_l^{(0)} = g_{lm}A_0^m,$$

$$g_{lm}g^{mk} = \delta_l^k, \quad x^1 = \theta, \quad x^2 = \psi, \quad \xi_1 = p_\theta, \quad \xi_2 = p_\psi,$$

$$2U(x) = s^2\left(\sum a_{ii}p_i^2p^{-2}\right) + 2sp\left(\sum b_{ij}p_ip_jp^{-2}\right) + p^2\left(\sum c_{ij}p_ip_jp^{-2}\right) - g_{lm}A_0^lA_0^m.$$

From the formulas given follows the positivity of the form $L_{\dot x^i \dot x^j}$. By the same token Theorem 1 is proved [$L \to L_E$ (cf. (26″))].

The proof of Theorem 2 follows automatically from Lemmas 9 and 10 of Sec. 2.

In the Kirchhoff problem on the motion of a rigid body in an ideal incompressible fluid it would be useful to analyze the stability of spatial motions in τ^3 by virtue of (3) in the case when the trajectory is stable in (M, p)-space.

If there is an elliptic (stable) trajectory γ, then in its phase neighborhood of sufficiently small radius there is a countable set of "longer" now self-intersecting periodic trajectories $\gamma_k \to \gamma$ with multiple periods $k \to \infty$, and also a countable sequence of invariant tori $T_j^2 \to \gamma$ $(j \to \infty)$ containing γ and contractible to γ. (Birkhoff situation, justified by KAM, Kolmogorov, Arnol'd, Moser.) There is essential interest in the picture of the motions in τ^3, which we get by virtue of (3) near the initial trajectory $M(t), p(t)$. The initial curve γ defines in τ^3, as a rule, probably an unstable motion with the help of (3) for the following reasons: in time

kT we get a motion $g_k \in E(3)$ (cf. Introduction), having the form of translation along the axis l_k (crossing one another in τ^3 and the axes δ for γ "in general position"), and of angles Δ_k around these axes. The axes l_k and δ are sufficiently slightly nonparallel that for a long time the body went along side the initial motion. Thus (if this picture is correct), for "random" initial data near the trajectory γ, we get motion always close to γ in the frame connected with the body, but defining a completely random walk on the group $E(3)$, considering the situation through a period T and its multiples.

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