HOLOMORPHIC BUNDLES OVER RIEMANN SURFACES AND THE KADOMTSEV–PETVIASHVILI EQUATION. I

I. M. KRICHEVER AND S. P. NOVIKOV

Introduction

The Kadomtsev–Petviashvili (KP) equation was first derived in [5] as a physically natural two-dimensional analog of the well-known KdV equation; it arises in the study of solitons and other KdV solutions which are subject to slow perturbations in the direction transverse to that of the main wave. As a physical model, the KP equation has the same degree of universality as the KdV.

The KP equation has a Lax commutator representation (see 3, 4])

$$\left[\frac{\partial}{\partial t} - A, \frac{\partial}{\partial y} - L\right] = 0 \iff \frac{\partial \tilde{L}}{\partial t} = [A, \tilde{L}]; \quad \tilde{L} = \frac{\partial}{\partial y} - L, \tag{1}$$

where $L = \frac{\partial^2}{\partial x^2} + U(x, y, t)$, $A = \frac{\partial^3}{\partial x^3} + \frac{3}{2}U + W(x, y, t)$ which after elimination of $W(x, y, t)$ leads to

$$0 = \frac{3}{4}\frac{\partial^2 U}{\partial y^2} + \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial t} + \frac{1}{4}\left(6U\frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3}\right)\right). \tag{2}$$

This is a special one of the Zakharov–Shabat equations (1). There is no comprehensive theory of these equations. One knows a whole series of finite-dimensional classes of exact solutions with remarkable mathematical properties (see [1, 4, 6–8]). However, during discussions of [4], S. P. Novikov and V. E. Zakharov conjectured that the KP equation has "algebrogeometric" exact solutions which generalize the known finite-gap or multisoliton solutions of KdV (see [2]) in that they depend on several arbitrary functions of one variable. This conjecture originated as follows: paper [4] presented particular, solitonlike solutions, which contained an arbitrary function as parameter. The present paper is concerned with the discovery of solutions depending on arbitrary functions. We use here techniques developed by Krichever in [9], which are devoted to commuting ordinary differential operators of not necessarily relatively prime order.

1. Matrix Analog of Multiparameter Baker–Akhiezer Functions. Stable Bundles over Riemann Surfaced

Recall that the scalar Baker–Akhiezer function $\psi(\boldsymbol{x}, P; \boldsymbol{x}_0)$ is defined on a Riemann surface Γ of genus g, $P \in \Gamma$ with distinguished point $P_0 = \infty$ and local parameter $z = k^{-1}$ near P_0; it has the following characterization.

a) ψ is meromorphic $\Gamma \setminus P_0$ and has g poles $\gamma_1 > \cdots > \gamma_g$, which do not depend on x.

Date: Original article submitted March 22, 1978.
UDC. 513.015.7+517.944.
Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 12, No. 4, pp. 41–52, 1978.

b) $\psi = \exp\left[\sum_{i=1}^{g} k^i(x_i - x_{i0})\right](1 + \sum_{s=1}^{\infty} \xi_s(x)k^{-s})$ has the asymptotic behavior $k \to \infty$ as $P \to P_0$ (see [6, 10]).

We introduce a matrix (noncommutative) analog of this function. Consider first of all an $l \times l$ matrix function $\Psi_0(\boldsymbol{x}, k; \boldsymbol{x}_0)$ where $\boldsymbol{x} = (x_1, \dots, x_s)$, which satisfies:

1) $\Psi_0(\boldsymbol{x}_0, k; \boldsymbol{x}_0) \equiv 1$;

2) the matrix functions $A_i = \frac{\partial \Psi_0}{\partial x_i}\Psi_0^{-1} = A_i(\boldsymbol{x}, k)$ are independent of $\boldsymbol{x}_0$, depend polynomially on k, and satisfy

$$\frac{\partial A_j}{\partial x_j} - \frac{\partial A_i}{\partial x_j} = [A_j, A_j]. \tag{3}$$

This much is obvious: if the $A_i(\boldsymbol{x}, k)$, are given, subject to (3), then there is a unique matrix function $\Psi_0(\boldsymbol{x}, k; \boldsymbol{x}_0)$ such that $\Psi_0 \equiv 1$ for $\boldsymbol{x} = \boldsymbol{x}_0$ with $A_i = \Psi_{0x_i}\Psi_0^{-1}$. Below we will always put $\boldsymbol{x}_0 = (0, \dots, 0)$ and $\Psi_0(\boldsymbol{x}, k; 0) = \Psi_0(\boldsymbol{x}, k)$. Now let there be given an arbitrary (nonsingular) Riemann surface Γ of genus g with distinguished point P_0, which we will often denote by $\infty = P_0$. The local parameter on Γ near P_0 is written $z = k^{-1}$. We pick an unordered collection (γ) of distinct points $(\gamma_1, \dots, \gamma_{lg})$ on Γ, and a collection (α) of complex $(l-1)$-vectors $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{lg}$ where $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \dots, \alpha_{i,l-1})$.

Remark. There is a connection between such parameters and the theory of holomorphic bundles. The complete set of parameters will be called "A. N. Tyurin's parameters." According to [11], they define a stable (in the sense of Mumford) l-dimensional holomorphic vector bundle of degree lg over Γ, together with an "equipment," i.e., a collection of holomorphic sections $\eta_1, \dots, \eta_l$ defined up to multiplication by a constant matrix A $(\eta_1, \dots, \eta_l) \to (\eta_1, \dots, \eta_l)A$. The points $\gamma_1, \dots, \gamma_{lg}$ are the points at which the sections η_j are linearly dependent; at each γ_i we have

$$\eta_l(\gamma_i) = \sum_{j=1}^{l-1} \alpha_{i,j}\eta_j(\gamma_i). \tag{4}$$

For $l = 1$, these parameters lead to the g-tuple $(\gamma_1, \dots, \gamma_g) \in S^g\Gamma \approx J(\Gamma)$.

We next pose the following problem: to find a vector function (of dimension l) $\psi(\boldsymbol{x}, P)$ on the Riemann surface Γ, meromorphic except at $P_0 = \infty$, with these properties:

1. The poles of ψ have order 1, do not depend on $\boldsymbol{x}$ and lie at $\gamma_1, \dots, \gamma_{lg}$. It is required that the residues $\varphi_{i,j}(\boldsymbol{x})$ of the functions ψ_j, $\boldsymbol{\psi} = (\psi_1, \dots, \psi_l)$, at γ_i be related by γ_i

$$\varphi_{i,j}(\boldsymbol{x}) = \alpha_{i,j}\varphi_{i,l}(\boldsymbol{x}), \tag{5}$$

where the $\alpha_{i,j}$ are constants, independent of $\boldsymbol{x}$.

2. In the neighborhood of $P_0 = \infty$, the vector function $\boldsymbol{\psi}(\boldsymbol{x}, P)$ has a representation

$$\boldsymbol{\psi}(\boldsymbol{x}, P) = \left(\sum_{s=0}^{\infty} \xi_s(x)k^{-s}\right)\Psi_0(\boldsymbol{x}, k), \tag{6}$$

where $\xi_0 \equiv (1, 0, \dots, 0)$, $k = k(P)$.

Following the ideas of [9], which in turn relies on the method of Koppelman [13] (see also [14]), we can show the following: 1) a vector function ψ with the desired properties (from now on we call it the Baker–Akhiezer vector function) always exists, and is uniquely determined by $\Psi_0(\boldsymbol{x}, k)$ and the Tyurin parameters

(γ,α); 2) the determination of $\boldsymbol{\psi}$ is effected by a Muskhelishvili-type [15] singular integral equation on the circle S^1 (a small circle on Γ, viz., the boundary of a neighborhood of P_0) with a Cauchy-type kernel. The kernel may be computed explicitly from the surface Γ and the point P_0. In the hyperelliptic case, the formulas become considerably simpler. The integral equation is solved separately for each x; condition 1) on the poles and residues of $\boldsymbol{\psi}$ uniquely selects a solution and determines the dependence of $\boldsymbol{\psi}$ on x.

Remark. One can construct a whole matrix $\hat\Psi(\boldsymbol{x},P)$ with $\boldsymbol{\psi}$ being its first column $\hat\Psi_1=\boldsymbol{\psi}$. The other columns are obtained in the same way as $\boldsymbol{\psi}$, except that the vector $\xi_0=(1,0,\dots,0)=e_1$ is replaced by $e_i=(0,\dots,1,\dots,0)$ to get $\hat\Psi_i$ (in formula (6)). As $P\to P_0$, we have

$$\hat\Psi=\left(\hat 1+\sum_{s=0}^{\infty}\hat\xi_s(\boldsymbol{x})k^{-s}\right)\Psi_0(\boldsymbol{x},k).$$

Aside from the Tyurin parameters (γ,α), there is arbitrariness of our construction also in the choice of Ψ_0, or equivalently in the choice of the matrices $A_i(\boldsymbol{x},k)$, which depend polynomially on k and satisfy the compatibility conditions (3). Let us look at some interesting cases involving three parameters $x_1=x$, $x_2=y$, $x_3=t$. The following examples will be important for us.

Example 1. Let $l=2$, and seek the matrices $A_i(\boldsymbol{x},k)$ in the form

$$\begin{gathered}A_1=\begin{pmatrix}0&1\\k+u&0\end{pmatrix}=\hat\varkappa+\begin{pmatrix}0&0\\u&0\end{pmatrix},\quad u=u(x,y,t),\\ A_2=\begin{pmatrix}k&0\\0&k\end{pmatrix}+\hat\gamma=\hat\varkappa^2+\hat\gamma,\\ A_3=\begin{pmatrix}0&k\\k^2&0\end{pmatrix}+k\hat p+\hat q=\hat\varkappa^3+k\hat p+\hat q,\end{gathered}\tag{7}$$

where $u,\hat\gamma,\hat p,\hat q$ depend a priori on x,y,t, and $\hat p,\hat q,y$ are 2×2 matrices. From (3) one finds:

$$\begin{gathered}\gamma_{12}=\gamma_{21}=p_{12}=0,\quad \gamma_{11}=\gamma_{22},\quad p_{11}=p_{22},\\ u=u(x,t),\quad p_{11}=p_{11}(t),\quad p_{21}=p_{21}(x,t),\quad q_{12}=q_{12}(x,t)\\ q_{11,y}=q_{22,y}=\gamma_{11,t},\quad \gamma_{11}=\gamma_{11}(y,t),\quad p_{21,x}=q_{22}-q_{11}=q_{12,x}.\\ (q_{11}+q_{22})=0,\quad -q_{11,x}=q_{21}-uq_{12},\\ u_t-q_{21,x}=u(q_{11}-q_{22}),\quad p_{21}=q_{12}+u.\end{gathered}\tag{8}$$

These relations easily imply:

$$\begin{gathered}\frac{\partial^2}{\partial x^2}q_{11}=\frac{1}{2}\frac{\partial^3}{\partial x^3}q_{12},\quad q_{11}=a(x,t)+b(y,t)+c(t),\\ q_{22}=-a(x,t)+b(y,t),\quad 2q_{12,x}=-u_x.\end{gathered}$$

By imposing some supplementary conditions on Ψ_0, we can remove the "inessential" functional parameters: one may assume that $\hat\gamma\equiv 0$, $p_{11}\equiv 0$.

In all cases, one gets from (8):

$$\begin{gathered}-q_{12}=\frac{u}{2}+\varphi(t),\\ u_t=q_{21,x}+uq_{12,x}=(uq_{12}-q_{11,x})_x+uq_{12,x}=-\frac{1}{4}(u_{xxx}+6uu_x+\varphi(t)u_x).\end{gathered}\tag{9}$$

In the special case $\varphi(t) \equiv 0$ we have an important corollary: if $\hat{\gamma} = p_{11} = 0$ the matrix $\Psi_0(\boldsymbol{x}, k)$ is determined by one function $u(x,t)$ which satisfies the KdV (Korteweg–de Vries) equation

$$u_t = -\frac{1}{4}(6uu_+x + u_{xxx}). \tag{10}$$

The function $u_0(x) = u(x,0)$ determines $u(x,t)$. It is this special case which we will use to construct solutions of the KP equation, so we take $\varphi(t) \equiv 0$ in the future.

Example 2. Let $l = 3$ and seek the $A_i(\boldsymbol{x}, k)$ in the form

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k+u & v & 0 \end{pmatrix} = \hat{\varkappa} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u & v & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ k & 0 & 0 \\ 0 & k & 0 \end{pmatrix} + \hat{d} = \hat{\varkappa}^2 + \hat{d}, \quad A_3 = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} = \varkappa^3,$$

where $\hat{d}(\boldsymbol{x})$ is a 3×3 matrix, $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ are functions $\boldsymbol{x} = (x, y, t)$. From (3) one finds the set of equations

$$\begin{gathered} d_{12} = d_{13} = d_{23} = 0, \quad d_{11,x} = u - d_{21}, \quad d_{12,x} = u - d_{22} + d_{11} = 0, \\ -d_{21,x} = d_{31}, \quad d_{22,x} = d_{21} - d_{32}, \quad d_{23,x} = d_{33} - d_{22} = 0, \quad u_y - d_{31,x} = \\ = u(d_{11} - d_{33}) + vd_{21}, \quad v_y - d_{32,x} = v(d_{22} - d_{33}) - d_{31} = -d_{31}, \\ -d_{33,x} = u - d_{32}. \end{gathered} \tag{11}$$

These equations lead to

$$\begin{gathered} d_{32} = u + \frac{v_x}{3}, \quad d_{33} - d_{11} = v, \quad d_{31} = -\left(u + \frac{2}{3}v_x\right)_x = -d_{21,x}, \\ d_{11,x} = -\frac{2}{3}v_x, \quad d_{21} = u + \frac{2}{3}v_x, \quad \operatorname{Tr}\hat{d} = 3d_{11} + 2v = \varphi(y). \end{gathered}$$

Thus, we find

$$u_y = -u_{xx} - \frac{2}{3}v_{xxx} + \frac{3}{2}vv_x, \quad v_y = 2u_x + v_{xx} = \frac{\partial}{\partial x}(2u + v_x). \tag{12}$$

Introduce $w(x,y)$, where $w_x = v$, $w_y = 2u + v_x$. From (12) we obtain

$$3w_{yy} = \frac{\partial}{\partial x}(-w_{xxx} + 2w_x^2). \tag{13}$$

For $v = w_x$, this is the Boussinesq equation:

$$3v_{yy} = \frac{\partial}{\partial x}(-v_{xxx} + 4vv_x). \tag{14}$$

Equation (14) can be integrated completely by inverse scattering, and is known to have a large number of explicit exact solutions. This equation has order two in y, and y is a time-like parameter. There are then two arbitrary functions $w(x,0), w_y(x,0)$, which completely determine Ψ_0, if $\varphi = \operatorname{Tr}\hat{d} = 0$. Thus, in the present case the matrix Ψ_0 is defined by solutions of Eq. (13).

Example 3. Let $l > 3$. The matrices $A_i(\boldsymbol{x}, k)$, where $\boldsymbol{x} = (x, y, t)$, which are of interest to us will be sought in the form

$$A_1 = \hat{\kappa} + \begin{pmatrix} 0 & \dots\dots\dots\dots & & 0 \\ \vdots & & 0 & \vdots \\ 0 & & & 0 \\ u_0 & u_1 & \dots & u_{l-2} & 0 \end{pmatrix}, \quad \hat{\varkappa} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ k & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{15}$$

$$A_2 = \hat{\varkappa}^2 + \hat{\gamma}, \quad A_3 = \hat{\varkappa}^3 + \hat{p},$$

where $\hat{\gamma}, \hat{p}$ are $(l \times l)$ matrices, and the functions $u_0, \dots, u_{l-2}$ depend on x, y, t. Note that the matrix $\hat{\varkappa}$ has the property $\hat{\varkappa}^l = k \cdot \hat{1}$. For $l > 3$ neither of the matrices $\hat{\varkappa}^2$ and $\hat{\varkappa}^3$ are scalars (the cases $l = 2, 3$ are singular in this respect). To construct Ψ_0 it is necessary to find a class of solutions of the compatibility equations (3). We will look at this in more detail in the next paper. Here we note only that the "trivial" case $u_\alpha = \hat{p} = \hat{\gamma} = 0$ leads us to nontrivial solutions of the KP equation, which depend on a finite number of parameters: the Riemann surface Γ, the point $P_0 \in \Gamma$ and the Tyurin parameters (γ, α), defining the holomorphic bundle over Γ.

Remark. For $l = 1$ we have $\hat{\varkappa} = k$ and the functional parameters are inessential. In this case we recover the scalar Baker–Akhiezer function; for the corresponding solutions of the Kadomtsev–Petviashvili equation see [6, 7].

2. Solutions of the KP Equation

We will be especially interested in the case where the Baker–Akhiezer vector function $\boldsymbol{\psi}(\boldsymbol{x}, P)$ is annihilated by a linear partial differential operator with coefficients that do not depend on the point of the Riemann surface Γ. It turns out that this property depends only on the choice of the matrix $\Psi_0(\boldsymbol{x}, k)$ but not on Γ, P_0 or the parameters (γ, α). Apparently, our construction, by permitting a choice of different classes of matrices $A_i = \frac{\partial \Psi_0}{\partial x_i} \Psi_0^{-1}$, makes it possible to find a broad class of such matrices Ψ_0; these lead, in general, to matrix linear differential operators T_q, $q = 1, \dots, s$, such that $T_q \boldsymbol{\psi} = 0$ (or $T_q \boldsymbol{\psi} = \lambda_q(P) \boldsymbol{\psi}$), $T_q = \sum_{k,\alpha} v_{kq}^\alpha \frac{\partial^\alpha}{\partial x_k^\alpha}$, with $v_{kq}^\alpha(\boldsymbol{x})$ being $l \times l$ matrices, and $\lambda_q(P)$ an algebraic function of $P \in \Gamma$.

The problem of finding solutions of the KP equation requires isolation of the case where for $\boldsymbol{x} = (x, y, t)$ one has two scalar operators T_1, T_2 of a form independent of l and Ψ_0,

$$\begin{aligned} T_1 &= \frac{\partial}{\partial t} - A = \frac{\partial}{\partial t} - \frac{\partial^3}{\partial x^3} - \frac{3}{2} U(x, y, t) \frac{\partial}{\partial x} - W(x, y, t), \\ T_2 &= \frac{\partial}{\partial y} - L = \frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2} - U(x, y, t) \end{aligned} \tag{16}$$

such that $T_1 \boldsymbol{\psi} = T_2 \boldsymbol{\psi} = 0$. In this situation, the equation $[T_1, T_2] \boldsymbol{\psi} = 0$ for all $P \in \Gamma$ implies that the coefficients of T_1, T_2 satisfy the KP equation,

$$[T_1, T_2] = 0$$

or, after elimination of W,

$$\frac{3}{4} \frac{\partial^2 U}{\partial y^2} + \frac{\partial}{\partial x} \left(\frac{\partial U}{\partial t} + \frac{1}{4} \left(6U \frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3} \right) \right) = 0.$$

We now have the following result.

Theorem 1. *Let* $\boldsymbol{x} = (x, y, t)$ *and let the matrices* $A_i(\boldsymbol{x}, k)$*, depending polynomially on* k *and satisfying* (3) *be chosen in the forms exhibited in Examples* 1–3 *of Sec.* 1. *Then the Baker–Akhiezer vector function* $\boldsymbol{\psi}(\boldsymbol{x}, P)$*, is determined by the "inverse problem data": the matrix* $\Psi_0(\boldsymbol{x}, k)$*, an algebraic curve* Γ*, a point* $P_0 \in \Gamma$ *and the parameters* $(\gamma_1, \ldots, \gamma_{lg}, \alpha_{i,j})$ $(i = 1, \ldots, lg,\ j = 1, \ldots, l-1)$*, and it satisfies the equation*

$$T_1\boldsymbol{\psi} = \left[\frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2} - U(x,y,t)\right]\boldsymbol{\psi} = 0,$$
$$T_2\boldsymbol{\psi} = \left[\frac{\partial}{\partial t} - \frac{\partial^3}{\partial x^3} - \frac{3}{2}U\frac{\partial}{\partial x} - W(x,y,t)\right]\boldsymbol{\psi} = 0,$$

where $U(x, y, t)$ *is some scalar function of* x, y, t*. Consequently, this function solves the KP equation*

$$\frac{3}{4}\frac{\partial^2 U}{\partial y^2} + \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial t} + \frac{1}{4}\left(6U\frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3}\right)\right) = 0.$$

Corollary 1. a) *For* $l = 2$ *every stable holomorphic bundle, i.e., a selection of Tyurin parameters* (γ, α) *over the curve* Γ *with distinguished point* $P_0 = \infty$*, together with an arbitrary solution* $u(x, t)$ *of the KdV equation generates a solution of the KP equation* (see Example 1 of Sec. 1). b) *For* $l = 3$*, every set of Tyurin parameters* (γ, α) *over the curve* Γ *with distinguished point* $P_0 = \infty$*, together with an arbitrary solution* $w(x, y)$ *of the Boussinesq equation* (14) *generate a solution of the KP equation* (see Example 2 of Sec. 1).

Proof of the Theorem. We study the Baker–Akhiezer vector function $\boldsymbol{\psi}(x, y, t, P)$ and find operators T_1 and T_2 which annihilate $\boldsymbol{\psi}$. By definition, near P_0, where $z = k^{-1}(P)$ is the local parameter, $\boldsymbol{\psi}$ has the form

$$\boldsymbol{\psi}(\boldsymbol{x}, k) = \left(\sum_{s=0}^{\infty} \boldsymbol{\xi}_s(\boldsymbol{x})k^{-1}\right)\Psi_0(\boldsymbol{x}, k),$$

where $\boldsymbol{\xi}_0 = (1, 0, \ldots, 0)$. For $\frac{\partial\boldsymbol{\psi}}{\partial x}, \frac{\partial^2\boldsymbol{\psi}}{\partial x^2}, \frac{\partial^3\boldsymbol{\psi}}{\partial x^3}, \frac{\partial\boldsymbol{\psi}}{\partial t}, \frac{\partial\boldsymbol{\psi}}{\partial y}$ we have the expansions

$$\begin{aligned}
\boldsymbol{\psi}_x\Psi_0^{-1} &= (\boldsymbol{\xi}_0 + \boldsymbol{\xi}_1 k^{-1})A_1 + O(k^{-1}),\\
\boldsymbol{\psi}_y\Psi_0^{-1} &= (\boldsymbol{\xi}_0 + \boldsymbol{\xi}_1 k^{-1})A_2 + O(k^{-1}),\\
\boldsymbol{\psi}_t\Psi_0^{-1} &= (\boldsymbol{\xi}_0 + \boldsymbol{\xi}_1 k^{-1} + \boldsymbol{\xi}_2 k^{-2})A_3 + O(k^{-1}),\\
\boldsymbol{\psi}_{xx}\Psi_0^{-1} &= (\boldsymbol{\xi}_0 + \boldsymbol{\xi}_1 k^{-1})(A_{1x} + A_1^2) + 2\boldsymbol{\xi}_{1x}k^{-1}A_1 + O(k^{-1}),\\
\boldsymbol{\psi}_{xxx}\Psi_0^{-1} &= (\boldsymbol{\xi}_0 + \boldsymbol{\xi}_1 k^{-1} + \boldsymbol{\xi}_2 k^{-2})(A_1^3 + 2A_{1x}A_1 + A_1A_{1x} + A_{1xx}) +\\
&\quad + 3\boldsymbol{\xi}_{1x}A_1^2k^{-1} + 3\boldsymbol{\xi}_{1xx}k^{-1}A_1 + O(k^{-1}).
\end{aligned} \tag{17}$$

Formulas (17) and the explicit form of the matrices A_i, $i = 1, 2, 3$, show, after a little calculation, that

$$\left(\frac{\partial\boldsymbol{\psi}}{\partial y} - \frac{\partial^2\boldsymbol{\psi}}{\partial x^2}\right)\Psi_0^{-1}, \quad \left(\frac{\partial\boldsymbol{\psi}}{\partial t} - \frac{\partial^3\boldsymbol{\psi}}{\partial x^3}\right)\Psi_0^{-1}$$

have the representations

$$\left(\frac{\partial \boldsymbol{\psi}}{\partial y} - \frac{\partial^2 \boldsymbol{\psi}}{\partial x^2}\right)\Psi_0^{-1} = U\boldsymbol{\psi}\Psi_0^{-1} + O(k^{-1}),$$
$$\left(\frac{\partial \boldsymbol{\psi}}{\partial t} - \frac{\partial^3 \boldsymbol{\psi}}{\partial x^3}\right)\Psi_0^{-1} = \left(\frac{3}{2}U\frac{\partial \boldsymbol{\psi}}{\partial x} + W\boldsymbol{\psi}\right)\Psi_0^{-1} + O(k^{-1}),$$

where $U = U(x,y,t)$, $W = W(x,y,t)$ are scalar functions.

The functions

$$\varphi_1(\boldsymbol{x}, P) = \left(\frac{\partial}{\partial y} + \frac{\partial^2}{\partial x^2} - U\right)\boldsymbol{\psi}, \quad \varphi_2(\boldsymbol{x}, P) = \left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3} - \frac{3}{2}U\frac{\partial}{\partial x} - W\right)\boldsymbol{\psi}$$

have the same poles $\gamma_1, \dots, \gamma_{lg}$, as $\boldsymbol{\psi}$; the residues of their components φ_j at these poles satisfy (5) with the same constants $\alpha_{i,j}$. Asymptotically, $\boldsymbol{\varphi}_q$ $(q = 1, 2)$ behave as $k \to \infty$ just as in (6), but with $\xi_0 = 0$. From this it follows that $\varphi_1 \equiv 0$ and $\varphi_2 \equiv 0$, by analogy with [7]. This proves the theorem. □

For the potential $U(x,y,t)$ one has the formulas:

$$\begin{aligned} l = 2&: \quad U(x,y,t) = -(u + \xi_{1x}^{(2)}), \quad \boldsymbol{\xi}_1 = (\xi_1^{(1)}, \xi_1^{(2)}), \\ l = 3&: \quad U(x,y,t) = -2\xi_{1x}^{(3)}, \quad \boldsymbol{\xi}_1 = (\xi_1^{(1)}, \xi_1^{(2)}, \xi_1^{(3)}), \\ l \geq 3&: \quad U(x,y,t) = -2\xi_{1x}^{(l)}, \quad \boldsymbol{\xi}_1 = (\xi_1^{(1)}, \dots, \xi_1^{(l)}). \end{aligned} \tag{18}$$

By no means all the parameters—the arbitrary functions which enter into $\Psi_0(\boldsymbol{x}, k)$, —are "essential" in the sense that changing them will change the potential $U(x,y,t)$. It is trivial to see, at least, that all the parameters listed in Corollary 1a and b, are "essential."

We postpone to the next paper the general question about the analytical form of our solutions for $l > 1$. Here we consider only the simplest case, in which Γ degenerates into a rational curve with singularities, and everything may be computed through to the final formulas.

Example 1. Rational curve with double points and parameter Γ, $z = k^{-1}$ near $P_0 = \infty$.

Let the points $\gamma_1, \dots, \gamma_{Nl}$ be given, and look for Ψ_0 in a form independent of functional parameters, $A_1 = \Psi_{0x}\Psi_0^{-1} = \hat{\varkappa}$, $A_2 = \Psi_{0y}\Psi_0^{-1} = \hat{\varkappa}^2$, $A_3 = \Psi_{0t}\Psi_0^{-1} = \hat{\varkappa}^3$. The Baker–Akhiezer vector function is sought in the form

$$\boldsymbol{\psi} = \left(\xi_0 + \sum_{q=1}^{Nl} \boldsymbol{a}_q(x,y,t)(k - \gamma_q)^{-1}\right)\Psi_0,$$

where $\boldsymbol{a}_q = (a_{q1}, \dots, a_{ql})$, $\boldsymbol{\xi}_0 = (1, 0, \dots, 0)$. This function is completely determined by conditions (5) and by the following requirements at the "double" points:

$$\begin{aligned} &1. \quad \sum_{i=1}^{l} a_{si}\Psi_0^{ij} = \alpha_{sj}\left(\sum_{i=1}^{l} a_{si}\Psi_0^{il}\right)\Big|_{k=\gamma_s}, \\ &2. \quad \boldsymbol{\psi}(x,y,t,\varkappa_{r1}) = \boldsymbol{\psi}(x,y,t,\varkappa_{r2}) \end{aligned} \tag{19}$$

for all points $\varkappa_{11}, \varkappa_{12}, \varkappa_{21}, \varkappa_{22}, \dots, \varkappa_{N1}, \varkappa_{N2}$ (the points $\varkappa_{r1} \sim \varkappa_{r2}$ are "double").

The collection of parameters (γ,α) and the double points determine the vector $\boldsymbol{\psi}(x,y,t,P)$. Using (18), we find for the potential (k has been changed to $-k$ in the matrix $\hat{\varkappa}$)

$$U(x,y,t)=-2\frac{\partial}{\partial x}\left(\sum_{j=1}^{Nl}a_{jl}\right),$$

In the case $l=2$, we obtain

$$\Psi_0(x,y,t,k)=e^{-ky}\begin{pmatrix}\cos\theta & \frac{1}{\sqrt{k}}\sin\theta\\ -\sqrt{k}\sin\theta & \cos\theta\end{pmatrix},\quad \theta=\sqrt{k}(x+kt).$$

For real $\gamma,\alpha,\varkappa$, we get real solutions of the KP equation, with $U(x,y,t)$ expressed rationally in terms of the entries of the matrices $\Psi_0^{ij}(x,y,t,k)$ at the points $k_m=\{\gamma_q,\varkappa_{\tau\varepsilon}\}$ that is to say in terms of the exponentials $e^{k_m y}$, $\cos\sqrt{k_m}(x+k_m t))$, $\sin(\sqrt{k_m}(x+k_m t))$ for all these points k_m.

The simplest case $N=1$, $l=2$ gives

$$a_{s1}=-a_{s2}\frac{\Psi_0^{11}-\alpha_s\Psi_0^{12}}{\Psi_0^{21}-\alpha_s\Psi_0^{22}}=\frac{-a_{s2}(\cos\theta_s-\alpha_s/\sqrt{\gamma_s}\sin\theta_s)}{-\sqrt{\gamma_s}\sin\theta_s-\alpha_s\cos\theta_s}=-a_{s2}\lambda_s(x+\gamma_s t),$$
$$s=1,2,\quad \theta_s=\sqrt{\gamma_s}(x+\gamma_s t).$$

If one puts $\alpha_s^2=-\gamma_s$, then $\lambda_s=-\alpha_s^{-1}=\text{const}$.

Let $\varkappa_{11}=\varkappa_1$ and $\varkappa_{12}=\varkappa_2$. Let us solve the equation $\boldsymbol{\psi}(\boldsymbol{x},\varkappa_1)=\boldsymbol{\psi}(\boldsymbol{x},\varkappa_2)$. With $\theta_3=\sqrt{\varkappa_1}(x+\varkappa_1 t)$, $\theta_4=\sqrt{\varkappa_2}(x+\varkappa_2 t)$,

$$D=(d_{ij})=\begin{pmatrix}\cos\theta_3 & \frac{1}{\sqrt{\varkappa_1}}\sin\theta_3\\ -\sqrt{\varkappa_1}\sin\theta_3 & \cos\theta_3\end{pmatrix}\cdot\begin{pmatrix}\cos\theta_4 & -\frac{1}{\sqrt{\varkappa_2}}\sin\theta_4\\ \sqrt{\varkappa_2}\sin\theta_4 & \cos\theta_4\end{pmatrix}.$$

Set $\varkappa=\varkappa_1-\varkappa_2$, $\delta_{ij}=1/(\varkappa_i-\gamma_j)$, $i,j=1,2$. For the potential $U(x,y,t)=-2\frac{\partial}{\partial x}(a_{12}+a_{22})$ we find from the equation $\boldsymbol{\psi}(\boldsymbol{x},\varkappa_1)=\boldsymbol{\psi}(\boldsymbol{x},\varkappa_2)$ that

$$U(x,y,t)=-2\frac{\partial}{\partial x}\frac{a_1e^{-\varkappa y}+c_1e^{\varkappa y}+b_1(x,t)}{a_2e^{-\varkappa y}+c_2e^{\varkappa y}+b_2(x,t)},\tag{20}$$

where

$$a_1=\delta_{11}-\delta_{12},\quad c_1=\delta_{21}-\delta_{22},\quad a_2=(\lambda_2-\lambda_1)\delta_{11}\delta_{22},\quad c_2=(\lambda_2-\lambda_1)\delta_{21}\delta_{22},$$
$$b_1(x,t)=d_{11}(\delta_{12}-\delta_{11})+d_{12}(\delta_{21}\lambda_1-\delta_{11}\lambda_1+\delta_{12}\lambda_2-\delta_{22}\lambda_2)+d_{22}(\delta_{22}-\delta_{21});$$
$$b_2(x,t)=d_{11}(\delta_{12}\delta_{21}\lambda_1-\delta_{22}\delta_{11}\lambda_2)+(d_{21}-\lambda_1\lambda_2d_{12})(\delta_{11}\delta_{22}-\delta_{21}\delta_{22})+$$
$$+d_{22}(\delta_{22}\delta_{11}\lambda_1-\delta_{21}\delta_{12}\lambda_2).$$

The numerator and denominator under the derivative symbol in (20) are, for these special solutions of the KP equation, linear combinations, with constant coefficients, of $e^{-\varkappa y}$, $e^{\varkappa y}$, $\cos\theta_3\cos\theta_4$, $\sin\theta_3\cos\theta_4$, $\cos\theta_3\sin\theta_4$, $\sin\theta_3\sin\theta_4$, if $\lambda_s=-\alpha_s^{-1}$ and $\alpha_s^2=-\gamma_s$. If all $\lambda_s,\gamma_s<0$ and $\varkappa_s$ are real, then we have a real solution for which $U(x,y,t)\to 0$, $y\to\pm\infty$. Both trigonometric ($\varkappa_s>0$) and hyperbolic ($\varkappa_s<0$) forms are possible.

If $\prod_{i,j=1}^2\delta_{ij}<0$, then for fixed x_0,t_0 the solution (20) invariably has a singularity, and at exactly one point $y^*(x_0,t_0)$. In the trigonometric case ($\varkappa_1>0$, $\varkappa_2>0$), we always have $\prod_{i,j=1}^2\delta_{ij}>0$. The presence of a singularity for given x,t depends on the solvability of $4a_2c_2<b_2^2(x,t)$ under the constraint, where $b_2(x,t)/c_2<0$, where $a_2c_2=(\lambda_1-\lambda_2)^2\prod_{i,j=1}^2\delta_{ij}>0$. It can be shown that in the trigonometric case these special solutions will always develop a singularity; it is confined to a region $|y| <$ const, bounded uniformly for all x and t.

Example 2. A rational curve Γ with more complicated degeneracies.

Again, let there be given points $\gamma_1, \dots, \gamma_{Nl}$, and let some of the pairs of points end $\varkappa_{i2}$ coalesce: $\varkappa_{i1} \to \varkappa_{i2}$, $i = i_1, \dots, i_p$. We look for $\boldsymbol{\psi}$ in the form

$$\boldsymbol{\psi}(x,y,t,k) = \left(\boldsymbol{\xi}_0 + \sum_{q=1}^{Nl} a_q(x,y,t)(k-\gamma_q)^{-1} \right) \Psi_0(x,y,t,k).$$

$1 = 1'$.

$$\sum_{i=1}^{l} a_{si} \Psi_0^{ij} = \alpha_{sj} \left(\sum_{i=1}^{l} a_{si} \Psi_0^{il} \right) \Bigg|_{k=\gamma_q}, \quad 1 \le j \le l-1,\ 1 \le q \le Nl.$$

$2 \to 2'$.

$$\frac{\partial \boldsymbol{\psi}}{\partial k}\Big|_{k=\varkappa_{i1}=\varkappa_{i2}} = 0, \quad i = i_1, \dots, i_p,$$

$$\boldsymbol{\psi}(x,y,t,\varkappa_{i1}) = \boldsymbol{\psi}(x,y,t,\varkappa_{i2}), \quad i \ne i_1, \dots, i_p.$$

Again we find a solution $U(x,y,t) = -2\frac{\partial}{\partial x}\left(\sum_{j=1}^{Nl} a_{jl}\right)$.

In the simplest case $N = 1$, $l = 2$, $p = 1$, $\varkappa_{11} = \varkappa_{12} = \varkappa$ we obtain solutions of the KP equation which are rational in x, y, t, if $\varkappa = 0$, $\alpha_s^2 = -\gamma_s$, $\alpha_{s1} = -a_{s2}\lambda_s$, where $\lambda_s = -\alpha_s^{-1}$, just as in Example 1. Equation ($2'$) takes the form

$$-\frac{\boldsymbol{a}_1}{\gamma_1^2} - \frac{\boldsymbol{a}_2}{\gamma_2^2} = \left(\boldsymbol{\xi}_0 - \frac{\boldsymbol{a}_1}{\gamma_1} - \frac{\boldsymbol{a}_2}{\gamma_2}\right)\left(\frac{\partial \Psi_0}{\partial_k}\Psi_0^{-1}\right)_{k=0} =$$

$$= \left(\boldsymbol{\xi}_0 - \frac{\boldsymbol{a}_1}{\gamma_1} - \frac{\boldsymbol{a}_2}{\gamma_2}\right)\begin{pmatrix} y - \frac{x^2}{2} & t - \frac{x^2}{2} - \frac{5x^3}{12} \\ -\frac{1+x}{2} & y - \frac{x}{2} - x^2 \end{pmatrix},$$

where $-\gamma_s = \lambda_s^{-2}$, $\boldsymbol{a}_s = (-\lambda_s a_{s1}, a_{s2})$, $\boldsymbol{\xi}_0 = (1, 0)$,

$$U(x,y,t) = -2\frac{\partial}{\partial x}(a_{12} - a_{22}).$$

These rational solutions do not decay as $x \to \infty$, $y = y_0$, $t = t_0$, and are therefore not contained among the known rational solutions (see [8, 10]).

Claim. Rational solutions are obtained for all $N \ge 1$, if one imposes, instead of condition $2'$, the following condition $2''$ at the point $\varkappa = 0$.

$2''$.
$$\frac{d\boldsymbol{\psi}}{dk} = 0, \dots, \frac{d^n\boldsymbol{\psi}}{dk} = 0.$$

Conjecture. 1) These are all the rational solutions of the KP equation which do not decay as $x \to \infty$. 2) For all even N there are solutions of this type which have no singularities.

3. Multiparameter Variation of the Equipped Bundle. KP Solutions of Genus $g = 1$, $l = 2$

In the case $l = 1$ there is, for the KdV equation, a well-known system of differential equations in x and t for the parameters $\gamma_1, \dots, \gamma_g$, and the potential $u(x,t)$ can be expressed very simply through these parameters (see [2, Chap. II, Secs. 3, 4]). In the case $l = 1$, however, the use of these equations may be circumvented entirely because there are explicit formulas for the scalar Baker–Akhiezer function in terms of the Riemann θ-function (see [2, 10]). In the present paper, it is clear that the situation for $l > 1$ is much more complicated: the computation of the Baker–Akhiezer vector function $\boldsymbol{\psi}(\boldsymbol{x}, P)$ has not been carried through to the end, and it leads to the solution of a system of singular integral equations on the circle, following the method of [9] (see Sec. 1). At least we do not need to know the whole vector $\boldsymbol{\psi}(\boldsymbol{x}, P)$, but rather it is sufficient to know one coefficient in the expansion of $\boldsymbol{\psi}\Psi_0^{-1}$ at P_0, which then determines $U(x, y, t)$ (see formula (18)). Hoping to get a more explicit answer for $U(x, y, t)$, we turn now to the computation of the x, y, t dynamics of the Tyurin parameters (γ, α) the moduli of the holomorphic equipped bundle, and we consider the resulting equations to generalize to the case $l > 1$ the Dubrovin equations for the parameters $\gamma_1, \dots, \gamma_g$ in the $l = 1$ KdV case.

We study the Baker–Akhiezer vector function $\boldsymbol{\psi}(\boldsymbol{x}, P)$, defined by the following data: an algebraic curve Γ of genus g, a collection of Tyurin parameters $(\gamma_1, \dots, \gamma_{lg}, \boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{lg})$, a distinguished point $P_0 = \infty \in \Gamma$ and the "input" matrix $\Psi_0(\boldsymbol{x}, k)$ (see (1), $\boldsymbol{x} = (x, y, t)$.

Let Ψ denote the Wronskian determinant of the vector ψ. The following are some elementary properties of the Wronskian matrix:

a) $\Psi_x \Psi^{-1}$ is a rational matrix function of $P \in \Gamma$, of the form

$$\Psi_x \Psi^{-1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \chi_1 & \chi_2 & \dots & \dots & \dots & \chi_l \end{pmatrix} = \hat{\chi}(\boldsymbol{x}, P)$$

(i.e., the scalar functions χ_α are rational);

b) for $\boldsymbol{x} = \boldsymbol{x}_0 = \boldsymbol{0}$ the poles of the matrix $\Psi_x \Psi^{-1} = \bar{\chi}(\boldsymbol{0}, P)$ coincide with $\gamma_1, \dots, \gamma_{lg}$, and the ratios of the residues of the χ_j at the points γ_i coincide with the parameters $\alpha_{i,j}$: $\alpha_{i,j} = \chi_j \chi_l^{-1}|_{P=\gamma_i}$.

Definition. The dependence upon $\boldsymbol{x}$ of the poles of the matrix $\hat{\chi}$ and of the ratios of the residues of the functions χ_j at these poles will be called the $\boldsymbol{x}$ dynamics of the Tyurin parameters (γ, α).

Consider the Baker–Akhiezer vector function corresponding to the choice of the A_i in the form $A_i = \varkappa^i$, $i = 1, \dots, l(g + 1) - 1 = N$. It defines a multiparameter variation of the Tyurin parameters. Thus:

Theorem 2. *There exists a commutative $l(g + 1) - 1$-dimensional group of transformations of the space of moduli of the equipped l-dimensional holomorphic vector bundle of degree lg over a nonsingular algebraic curve of genus g. Its generators are meromorphic vector fields.*

Note that the space of moduli is l^2g-dimensional. For $l = 1$ it is just the Jacobian torus $J(\Gamma)$, which in this case is itself a group. For $l > 1$ the whole moduli space is no longer a group. The group $GL(l, C)$, acts on this space by permuting the equipment. It is important to realize that the action of our $l(g+1)-1$-dimensional group does not commute with the action of $GL(l, C)$, and so is not defined on the space of moduli of bundles without equipment.

For one variable x, this dynamics was discussed in [9] (Sec. 3), and an algorithm for the computation of the right-hand sides of the equations $\gamma_{ix} = \dots$, $\alpha_{ix} = \dots$ was given. In the present paper we obtain, for genus $g = 1$ and $l = 2$, a peculiar analog of the "trace formulas," which connects γ, α with the potential $U(x, y, t)$ of primary interest to us. This makes it possible to close the system of equations for the (x, y, t) dynamics of the parameters γ, α. It should be mentioned that an explicit representation of $U(x, y, t)$ in terms $\gamma_i(\boldsymbol{x}), \boldsymbol{\alpha}_i(\boldsymbol{x})$ and $u(x, t)$ has still not been obtained.

For the Wronskian matrix $\Psi(\boldsymbol{x}, P)$, of the Baker–Akhiezer vector function $\boldsymbol{\psi}(\boldsymbol{x}, P)$, introduced for the construction of solutions of the KP equation with $l = 2$ (see Theorem 1 of Sec. 2 and Example of of Sec. 1) we find

$$
\begin{aligned}
B_1 &= \Psi_x\Psi^{-1} = \begin{pmatrix} 0 & 1 \\ k - U & 0 \end{pmatrix} + O(k^{-1}), \quad U = -u - 2\xi^{(2)}_{1x}, \\
B_2 &= \Psi_y\Psi^{-1} = \begin{pmatrix} k & 0 \\ v_1 & k \end{pmatrix} + O(k^{-1}), \quad v_1 = \xi^{(2)}_{1y}, \\
B_3 &= \Psi_t\Psi^{-1} = \begin{pmatrix} \omega_1 & k + \frac{U}{2} \\ k^2 - \frac{Uk}{2} + \omega_3 & \omega_4 \end{pmatrix} + O(k^{-1}).
\end{aligned} \tag{21}
$$

Using the technique of [9], we extract equations for $\gamma_1, \gamma_2, \alpha_1, \alpha_2$ from formula (21):

$$
\begin{cases} \gamma_{ix} = (-1)^i(\alpha_2 - \alpha_1)^{-1}, \\ \alpha_{ix} = \alpha_i^2 - U + (-1)^i(\zeta(\gamma_2 - \gamma_1) + \zeta(P_0 - \gamma_2) - \zeta(P_0 - \gamma_1)), \end{cases} \tag{22}
$$

$$
\begin{cases} \gamma_{iy} = 1, \\ -\alpha_{iy} = -v_1, \end{cases} \tag{23}
$$

$$
\begin{cases} \gamma_{it} = (-1)^i\left(\alpha_1\alpha_2 + \dfrac{U}{2}\right)(\alpha_2 - \alpha_1)^{-1}, \\ \alpha_{it} = \alpha_i(\omega_4 - \omega_1) + \dfrac{\alpha_i^2 U}{2} - \omega_3 - \wp(P_0 - \gamma_i) + \\ \qquad + (-1)^i\left(\alpha_i^2 + \dfrac{U}{2}\right)(\zeta(\gamma_1 - \gamma_2) + \zeta(P_0 - \gamma_1) - \zeta(P_0 - \gamma_2)). \end{cases} \tag{24}
$$

Here $\frac{d\zeta(z)}{dz} = -\wp(z)$, where $\wp$ is the Weierstrass function (see [12]).

In [9], with just one parameter x, it was possible to regard $U(x, 0, 0)$ as an arbitrary function of x, which then replaced the functional parameter $u(x)$ in the matrix $A_1 = \Psi_{0x}\Psi_0^{-1}$. In the present case it is necessary to compute this U as function of γ, α and of the coefficient $u(x, t)$ in the matrix $\Psi_{0x}\Psi_0^{-1} = A_1$. To this end we use the commutativity of the flows (22)–(24) with respect to x, y, t. Compatibility of (22), (23) in x, y yields

$$
B_{2x} - B_{1y} = [B_1, B_2] \implies v_1 = (\alpha_1 - \alpha_2)^{-1}(\wp(P_0 - \gamma_1) - \wp(P_0 - \gamma_2)), \tag{25}
$$

$$
U_y = -(\alpha_1^2 + \alpha_2^2)_y \quad \text{or} \quad U_y = -(\alpha_1^2 + \alpha_2^2) + u_0(x, t). \tag{26}
$$

Using compatibility of the flows in x, t and y, t, we find a relation between $u_0(x, t)$ and $u(x, t)$, where $u(x, t)$ satisfies the KdV equation

$$
u_t = -\frac{1}{4}\left(6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right)
$$

(see Example 1 of Sec. 1). The KdV-type equation for u_0 is too complicated to give here.

The remaining parameters figuring in (24) assume the form

$$\omega_4 - \omega_1 = \frac{1}{\alpha_1 - \alpha_2}[\wp(P_0 - \gamma_1) - \wp(P_0 - \gamma_2)] + \frac{U'}{2}, \tag{27}$$

$$\omega_3 = Z_2 - \frac{U^2}{2} - \frac{1}{2}\left[\frac{\alpha_2^2 - \alpha_1^2 - 2Z_1}{(\alpha_1 - \alpha_2)^2} Z_2 + \frac{U''}{2} + \right. \\ \left. + \frac{1}{(\alpha_1 - \alpha_2)^2}(2\wp'(\gamma_1 - \gamma_2) - \wp'(P_0 - \gamma_1) - \wp'(P_0 - \gamma_2))\right], \tag{28}$$

where $Z_1 = \zeta(\gamma_1 - \gamma_2) + \zeta(P_0 - \gamma_1) - \zeta(P_0 - \gamma_2)$, $Z_2 = \wp(P_0 - \gamma_1) - \wp(P_0 - \gamma_2)$. If the functional parameter $u_0(x, t)$ reduces to zero, then (22)–(24) become autonomous equations for the Tyurin parameters $\gamma_1, \gamma_2, \alpha_1, \alpha_2$. In all cases, upon substitution of (25)–(28) into (22)–(24), we obtain a set of commuting flows in the variables x, y, t, with $u_0(x, t)$ figuring explicly on the right-hand sides.

Conclusions

Every solution of (22)–(24) from which $U, v_1, \omega_1 - \omega_4, \omega_3$, have been removed by formulas (25)–(28), generates a solution of Kadomtsev–Petviashvili equation. This circumvents the use of singular integral equations for $g = 1$, $l = 2$. In principle, this procedure extends to all $l \geq 2$, $g \geq 1$, but the formulas become very complicated.

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G. M. Krzhizhanovskii Power Institute, Moscow. L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR.