PROPERTIES OF COSMOLOGICAL MODELS

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ABSTRACT. It is shown that in homogeneous cosmological models the Einstein equations can be reduced, on the basis of scale invariance, to systems with friction. The formalism involving friction permits one to investigate the problem of isotropization of the solutions in the Bianchi model IX at late development stages. The possibility of a statistical description of the properties of the model is discussed.

A number of investigations of the last decade have been devoted to the question (first explicitly formulated by Landau) of the singularities of the solutions of Einstein's equations in cosmological models more general than the classical Friedmann model. Special mention should be made (in addition to the ideologically useful but ineffective theorems of Penrose et al.) of the papers by Belinskii, Lifshitz, and Khalatnikov (BLKh)[4], devoted, in particular, to the asymptotic solution of the so-called Bianchi model IX (and also VIII), which they were the first to investigate, when the singularity is approached. This asymptotic behavior turned out to be quite interesting. Recently, the question of the properties of this model near the singularity was further developed intensively by a number of workers, for example Misner, I. D. Novikov and Doroshkevich, and others.

As is well known, the Bianchi models VIII and IX represent, respectively, metrics on the groups $SL(2, R)$ and SU_2; these metrics are invariant against right-hand shifts and depend on the time by virtue of Einstein's equations (for example, in a synchronous reference frame). In the present paper we use a union of the very simple scale invariance of Einstein's equations with a Hamiltonian structure (properties of this type were apparently first suggested and used by Misner[5], but he did not take notice of the friction and was interested primarily in the behavior of the model near the singularity). Our primary purpose is not to investigate the behavior of the model near the singularity (which by now is clear in the main outlines). The first questions which we believe should be clarified is the following: does this model have the property of forward isotropization in time (i.e., to the present time)? Is a probabilistic formulation of this question possible within the framework of the model? In the compact Bianchi model IX, the situation here is quite interesting[1], as is also the question of the compatibility of the compactness of the universe with an average density lower than in the closed Friedman model. We refer the reader to subsequent sections for detailed information. We note here only that the Bianchi model VIII has no isotropization whatever (not a single solution), and should therefore be discarded.

In the author's opinion, the most interesting circumstance revealed in the present paper, from the formal point of view, is the appearance of a dynamic system with

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friction in cosmological models. The formal energy U determined in the paper decreases as the universe expands, and tends to zero at the instant of maximum expansion. This formal energy has no bearing on the primary Hamiltonian of the system: as shown in the Appendix, the function that plays the role of this energy is determined uniquely by the scale invariance. We present also an invariant proof of the need for the appearance of a system with friction. This formalism uncovers a direct possibility of constructing the probability distribution over a set of solutions in the cosmological Bianchi model IX and the choice of the most probable solutions, from the point of view of an observer living near the instant of the maximum expansion (see Sec. 3; this will be completed in a succeeding paper).

1. Bianchi Models VIII and IX. Variational Principles

As is well known, the cosmological Bianchi models VIII and IX admit, by definition, the motion groups $SL(2,R)$ and SU_2, respectively, and the group orbits are space-like. The metric of each orbit is Riemannian and is determined by the time-dependent functions $g_{ij}(t)$; $(i,j=1,2,3)$ satisfying Einstein's equations in the synchronous reference frame (see [3], Appendix C; in the notation of that reference, where only g_{ii} differ from zero, we have $g_{11}=a^2$, $g_{22}=b^2, g_{33}=c^2$). Let $R_{\alpha\beta}$ denote, as usual, the Ricci tensor of four-dimensional space-time, and $T_{\alpha\beta}$ the energy-momentum tensor $(\alpha,\beta=0,1,2,3)$. For an energy-momentum tensor such that $T_{0i}=0$, the Ricci curvature components are $R_{0i}=0$, $i=1,2,3$. It follows readily, as is well known, that the matrices g_{ij}^{-1} and $\dot g_{ij}=dg_{ij}/dt$ commute. We can therefore assume, without loss of generality, that the matrix $g_{ij}(t)$ is diagonal at all instants of time. This we shall do; we put $g_{ii}=q_i^2$, $g^{ii}=q_i^{-2}$, $i=1,2,3$. We consider only the following forms of the energy-momentum tensor (without rotation): a) empty space: $T_{\alpha\beta}=0$; b) dust-like matter: $p=0$, $\varepsilon=T_{00}$; c) $p=\varepsilon/3$, where $\varepsilon=T_{00}=-T_0^0$, $p=T_i^i$, $i=1,2,3$; d) the sum of the tensors (b) and (c). The scalar curvature R of four-dimensional space is given by (see [3])

$$R=-\frac{V(q^2)}{2q_1^2q_2^2q_3^2}+\frac{(q_1q_2q_3)^{\cdot\cdot}}{q_1q_2q_3}+\ln(q_1q_2q_3)^{\cdot\cdot}+\dot q_1^2/q_1^2+\dot q_2^2/q_2^2+\dot q_3^2/q_3^2; \tag{1.1}$$

$$V(q^2)=\sum_{i=1}^{3}q_i^4-2\sum_{i\neq j}q_i^2q_j^2 \tag{1.2}$$

for Bianchi IX and

$$V(q^2)=\sum_{i=1}^{3}q_i^4-2q_1^2q_2^2+2q_3^2(q_1^2+q_2^2)$$

for Bianchi VIII.

The variational principle consists, as is well known, in the following:

$$\delta S=\delta\int[R(-g)^{1/2}+\Lambda(-g)^{1/2}]\,dG\,dt=0, \tag{1.3}$$

[1] The question of stability of the properties of the closed Friedmann model during the expansion state was investigated by E. M. Lifshitz[6] back in the 40s. Serious investigation of the dynamics of the Bianchi model IX during the later stages of development was started only very recently[7]. The author agrees with a remark by E. M. Lifshitz, that the definition used by us for isotropization is insufficient for application to a real universe. In the author's opinion, the question whether strong isotropization does or does not exist in the Bianchi model IX cannot be answered without a probabilistic analysis of this model during the later stage of development.

where $(-g)^{1/2}\,dG\,dt$ is an element of four-dimensional volume, dG is a standard volume element on the group $SL(2,R)$ or SU_2, and Λ is the action term responsible for the matter. It would be incorrect, however, to substitute directly into this action our metric components, where $g_{00}=-1$, and to vary only q_i. The point is that this variational principle is four-dimensional; we must vary S, knowing R, $(-g)^{1/2}$, and Λ as functions of g_{00}, but substitute at the end $g_{00}=-1$. We now turn to the form of Λ. For both energy-momentum tensors $p=0$ and $p=\varepsilon/3$ we introduce the parameter $E=\varepsilon(-g)^{1/2}$ (the total energy of the space-like section apart from multiplication by the normalization constant $\int dG=\text{const}$). According to the well known definition of Λ, we have

$$-\frac{1}{2}(-g)^{1/2}T_{ij}=\frac{\partial}{\partial g^{ij}}[(-g)^{1/2}\Lambda], \tag{1.4}$$

and we assume here that E is independent of g^{ij}.

From (1.4) and from the form of T_{ij} we have

$$\begin{gathered}(-g)^{1/2}\Lambda=-\frac{1}{2}E\ln g_{00}=\frac{1}{2}E\ln g^{00}\quad(p=0),\\ (-g)^{1/2}\Lambda=+\frac{1}{2}E\left(\ln g^{00}+\sum_{i=1}^{3}\frac{1}{3}\ln g^{ii}\right)\quad\left(p=\frac{\varepsilon}{3}\right),\\ g_{ii}=q_i^2,\quad g^{ii}=q_i^{-2},\quad T_i^i=p.\end{gathered} \tag{1.5}$$

Further, varying S with respect to g_{00} and then putting $g_{00}=-1$, we obtain from (1.1) and (1.5)

$$H^0(q,\dot q)=-E=\dot q\frac{\partial L^0}{\partial \dot q}-L, \tag{1.6}$$

$$\begin{gathered}L^0=-(\dot q_1\dot q_2q_3+\dot q_1q_2\dot q_3+q_1\dot q_2\dot q_3)-V(q^2)/4q_1q_2q_3,\\ P_i=\partial L^0/\partial\dot q_i=(q_jq_k)^{\cdot},\quad i\neq(j,k).\end{gathered} \tag{1.7}$$

We carry out the canonical transformation

$$p_i'=q_iP_i,\quad q_i'=\ln q_i,$$

and obtain ultimately

$$\begin{gathered}H^0(p',q)=\frac{1}{4q_1q_2q_3}(P_2(p')+V(q^2)),\\ P_2(p')=\sum_{i=1}^{3}{p_i'}^2-2\sum_{i\neq j}p_i'p_j'\end{gathered} \tag{1.8}$$

and the system of equations

$$\dot p_i'=-q_i\frac{\partial H^0}{\partial q^i},\quad \dot q_i=q_i\frac{\partial H^0}{\partial p_i'}\quad(p=0), \tag{1.9}$$

$$\dot p_i'=-q_i\frac{\partial H^0}{\partial q^i}+\frac{E}{3},\quad \dot q_i=-q_i\frac{\partial H^0}{\partial p_i'}\quad\left(p=\frac{\varepsilon}{3}\right) \tag{1.10}$$

(the parameter E was not differentiated when S was varied). $E=0$ corresponds in both cases to empty space.

Since $H^0(p', q) = -E$, we can, for $p = \varepsilon/3$, divide the right-hand side by H^0 (change the time) and obtain from (1.9) a system with Hamiltonian H', from which we get the integral

$$I = e^{H'} = |H^0(p', q)|\,(q_1 q_2 q_3)^{1/3} = A > 0. \tag{1.11}$$

On the other hand, if we have an energy-momentum tensor in the form of matter + radiation, then it is necessary to introduce the densities ε_1 and ε_2 of the matter and of the radiation, and two parameters $E_1 = \varepsilon_1(-g)^{1/2}$ and $E_2 = \varepsilon_2(-g)^{1/2}$, with E_1 regarded as a constant. By similar reasoning, we obtain

$$\begin{gathered} H^0(p', q) = -(E_1 + E_2), \\ \dot{p}'_i = -q_i \frac{\partial H^0}{\partial q_i} + \frac{E_2}{3}, \quad \dot{q}_i = q_i \frac{\partial H^0}{\partial p'_i} \end{gathered} \tag{1.12}$$

and, replacing dt by the factor $H^0 + E_1 = -E_2$, we obtain the Hamiltonian $H'(p', q)$ and the integral

$$I = e^{H'} = |H^0 + E_1|(q_1 q_2 q_3)^{1/3} = A > 0. \tag{1.13}$$

2. Friction in Cosmological Models

In the preceding section we have pointed out Hamiltonian forms that follow more or less in standard fashion from the universally known four-dimensional variational principle (for the case of homogeneous metrics). We proceed now to use the scale invariance of Einstein's equations; this scale invariance was first used (since 1969) by Misner (see[5, 8]). Using scale invariance, we can always decrease the number of degrees of freedom by unity; the Lagrangian of the system, however, then becomes time dependent. It is convenient to use as the time, following Misner, the logarithm of the volume of the spatial metric. This is indeed done in Misner's papers, but he did not take notice of a simple transformation that recasts (in the case of empty space) the system in a form in which the energy does not depend on the time, but pure friction appears in addition. Of course, not all systems with time-dependent Lagrangians reduce to such a form. We can therefore state that the reduction just to a system with friction constitutes a final and most convenient utilization of the scale invariance of the primary Einstein equation. Moreover, the formalism connected with friction operates well also in filled space. This formalism, as will be shown below, is very convenient in the investigation of cosmological models during the later stages of development.

In the Appendix we present a general analysis of scale invariance. It is shown that it leads uniquely to friction and it is indicated that the choice of the function that plays the role of the energy is unique. We now present the required derivation. For an arbitrary number α, the transformation

$$p'_i \to \alpha^2 p'_i, \quad q_i \to \alpha q_i, \quad t \to \alpha t \tag{2.1}$$

leaves Einstein's equation unchanged, since $H^0 \to \alpha H^0$. We assume, as always, confining the action of the scale group to one variable,

$$p'_i = \lambda^2 b_i, \quad q_i = \lambda \gamma_i; \tag{2.2}$$

the variables b and γ are connected by some constraint equation $F(\gamma, b) = 1$, where

$$F(\lambda\gamma, \lambda^2 b) = \lambda^m F(\gamma, b). \tag{2.3}$$

Misner operated only with the constraint $F = \gamma_1\gamma_2\gamma_3 = 1$, which we shall also use. Substituting (2.2) in Einstein's equation at the level $H^0(p', q) = 0$, we obtain from (1.9) and (2.1)

$$\lambda^2 \dot{b}_i = -\lambda\gamma_i \frac{\partial H^0(\gamma, b)}{\partial \gamma_i} - 2\lambda\dot{\lambda} b_i, \tag{2.4}$$
$$\lambda\dot{\gamma}_i = \gamma_i \frac{\partial H^0(\gamma, b)}{\partial b_i} - \dot{\lambda}\gamma_i;$$
$$m\dot{\lambda} = 3\dot{\lambda} = \sum_{i=1}^{3} \frac{\partial F}{\partial \gamma_i}\frac{\partial H^0}{\partial b_i}\gamma_i = \frac{1}{2}\sum_{i=1}^{3} b_i.$$

We multiply dt by $1/6\,\lambda(\sum_{i=1}^{3} b_i)$, so that in terms of the new time τ the equations for the variables (γ, b) and λ are separated. We introduce the following canonical coordinates on the constraint surface and on the energy level H^0:

$$p_k = b_k - b_3, \quad y_k = \ln x_i = \ln \gamma_i, \quad k = 1, 2. \tag{2.5}$$

We consider the function

$$|b| = \left| \sum_{i=1}^{3} b_i \right| = H(y, p),$$

where $b/2 = 3\, d\lambda/dt$. In the new coordinates (2.5), the earlier equation for λ, γ, and b takes the following form:

$$\frac{dp_k}{d\tau} = -\frac{\partial H}{\partial y_k} - 2p_k = -x_k \frac{\partial H}{\partial x_k} - 2p_k,$$
$$\frac{y_k}{d\tau} = \frac{\partial H}{\partial p_k}, \quad k = 1, 2, \tag{2.6}$$

where

$$H(y, p) = |b| = [(4p_1^2 - 4p_1p_2 + 4p_2^2) + 3V(\gamma^2)]^{1/2}. \tag{2.7}$$

This can be verified by direct substitution. Thus, we have a system with friction. We denote the function $H^2 = b^2$ by $U(p, x)$. We write down the equation in terms of the variables p_1, p_2, x_1, and x_2. The theoretical justification for the appearance of a system with friction on the surface $F(\gamma, b) = 1$ and of the choices of the time, of the canonical coordinates, and of the energy can be found in the Appendix. Multiplying the time by $U^{-1/2}$, we obtain the time ds and the equations

$$\frac{dp_k}{ds} = -\frac{1}{2}x_k \frac{\partial U}{\partial x_k} - 2p_k U^{1/2},$$
$$\frac{dx_k}{ds} = +\frac{1}{2}x_k \frac{\partial U}{\partial p_k}, \quad \frac{d\lambda}{ds} = \lambda U^{1/2}. \tag{2.8}$$

We emphasize that Eq. (2.6) or (2.8) is valid so long as the function $U(p, x)$ is strictly positive. Equation (2.8), in which we put formally $E = 0$, will be called the *transverse drift* of the model about the instant of the maximum expansion. This is a Hamiltonian system on the surface $U = 0$ with a Hamiltonian $\tilde{H}(p, x) = U^{\text{empty}}$. In a filled space (see (2.12) and (2.15)) we also obtain a transverse drift with the same Hamiltonian, but in the region $\tilde{H}(p, x) = U^{\text{empty}} < 0$. The sign of the friction is such that the formal energy U decreases with expanding universe, and, conversely, increases when the universe contracts to a point. It follows from (2.6) that

$$0 \geq \frac{dU}{d\tau} = -2\sum_{i=1}^{2} p_i \frac{\partial U}{\partial p_i} = -16(p_1^2 - p_1 p_2 + p_2^2). \tag{2.9}$$

We shall naturally call $K = 4(p_1^2 - p_1 p_2 + p_2^2)$ the kinetic energy, and $3V(\gamma^2)$ the potential (on the surface $\gamma_1\gamma_2\gamma_3 = 1$).

We have obtained the function $U = b^2$ from the obviously positive function b^2; we are therefore permitted to be only in the phase space $U(p, x) \geq 0$. This seemingly simple circumstance determines the possibility of fully classifying all the possible states at the instant of the maximum expansion, and the subsequent possibility of constructing the statistics on the set of trajectories. The point is that in terms of the primary coordinates p', q or λ, γ, b the function $b = \sum b_i$ was one of the coordinates and its vanishing yielded nothing. We shall show later that the condition $b^2 = U = 0$ gives a much better classification of the states of the system at the instant of maximum expansion in a filled space with both energy-momentum tensors ($p = 0$ and $p = \varepsilon/3$). This is one of the advantages of the method in which systems with friction are considered. We present now an equation for filled space in both cases ($p = 0$ and $p = \varepsilon/3$).

1. The case $p = 0$. From (1.9) and (2.1) we get

$$H^0(\gamma, b) = -E\lambda^{-1} = -Ee^{-\tau}. \tag{2.10}$$

Solving the equation

$$P_2(b) = -V(\gamma^2) = -4e^{-\tau} \tag{2.11}$$

with respect to the function $U = b^2$, we obtain in terms of the earlier coordinates (2.5)

$$U = U^{\text{empty}} + 12Ee^{-\tau} \tag{2.12}$$

and equations that are formally the same as in empty space, but with a new function $U(p, x, \tau)$.

2. The case $p = \varepsilon/3$. From (1.11) and (2.1) we obtain

$$H^0(\gamma, b) = -A\lambda^{-2} = -Ae^{-2\tau}. \tag{2.13}$$

Solving the equation

$$P_2(b) = -V - 4Ae^{-2\tau}, \tag{2.14}$$

we obtain

$$U(p, x, \tau) = U^{\text{empty}} + 12Ae^{-2\tau} \tag{2.15}$$

and the equations are again the same as in empty space, but with a different function U where $H = U^{1/2}$ is the energy.

3. Certain Deductions and Consequences. Isotropization and Average Density of the Universe at the Instant of Maximum Expansion

The time transformations employed by us involved multiplication by the obviously positive function λ, and also multiplication by a function b which can reverse sign. Our equations are therefore valid only in a region where b has only one sign. This region corresponds to $U > 0$ or $U^{1/2} > 0$ (we note that $b = U^{1/2}$ in the case of expansion and $b = -U^{1/2}$ in the case of contraction).

The derivative $dU/d\tau$ has always the same sign (negative in expansion and positive in contraction). This derivative can turn out to be equal to zero only at those points where the kinetic energy K is equal to zero—in the case of empty space; on the basis of (2.9), (2.12), and (2.15), it is never equal to zero in either case of filled space.

In the case of empty space, it should be noted in addition that the gradient of the potential $3V(\gamma^2)$ is never equal to zero on the surface $\gamma_1\gamma_2\gamma_3 = 1$ (for Bianchi VIII) and is equal to zero only at the point $\gamma_1 = \gamma_2 = \gamma_3 = 1$ (in the case of Bianchi IX) where the potential is equal to -9. Further, we have, (see (1.2)):

$$3V(\gamma^2) \geq -9 \quad \text{for Bianchi IX,} \tag{3.1}$$

$$3V(\gamma^2) > 0 \quad \text{for Bianchi VIII.} \tag{3.2}$$

From this we readily deduce that in the region of applicability of our time τ and at $U > 0$ the function U decreases monotonically along the trajectories in the case of expansion (and increases to ∞ upon contraction).

In the case of the Bianchi model VIII the function U never reaches $U = 0$, since $V > 0$, so that expansion does not give way to contraction in this model (our time covers the entire region). Further, to determine the character of the solution as $\tau \to \infty$ (towards expansion), we should consider for Bianchi VIII the regions

$$V_\delta = \{V(\gamma^2) < \delta\}, \quad \delta \to 0, \tag{3.3}$$

since all the solutions enter in these regions V_δ as $\tau \to \infty$. On the surface $\gamma_1\gamma_2\gamma_3 = 1$ we see that the condition $V < \delta$ yields (as $\delta \to 0$)

$$\gamma_1 \approx \gamma_2, \quad \gamma_3 \to 0. \tag{3.4}$$

This model therefore does not have the isotropization property as $\tau \to \infty$, and will henceforth be disregarded (the direction γ_3 is singled out for Bianchi VIII because of the anisotropy in the commutation relations of the group $SL(2, R)$, see Sec. 1).

We now turn to the Bianchi model IX.

1. Empty space. It follows from (2.9) that the function U decreases monotonically in expansion, reaching $U = 0$. To change over to the contraction stage, it is necessary to make the substitutions

$$p \to -p, \quad d\tau \to -d\tau \tag{3.5}$$

and to consider this system as it contracts, when the function U increases from 0 to ∞. Thus, in this model, expansion gives way to contraction once (and only once). This fact was apparently first established by Matzner et al.[7][1] At the instant of maximum expansion we have

$$U(p, x) = 0.3\, V(\gamma^2) \leq 0. \tag{3.6}$$

The region in γ space, on going through the point of maximum expansion, takes the form $V(\gamma^2) \leq 0$, where $\gamma_1\gamma_2\gamma_3 = 1$. This region is not compact: it goes off to ∞ in three directions: $\gamma_1 \to 0$, $\gamma_j = \gamma_k$, $i \neq j \neq k$. Thus, even in empty space there is a relative isotropization in the sense that at the instant of maximum expansion it is permitted to be only in the region $V(\gamma^2) \leq 0$ which surrounds the point $(1, 1, 1)$ but which, unfortunately, is not compact.

[1]The derivation of this fact in[7] is also based on the monotonicity of a function of the type b^2, but there this function is not connected with the energy and with friction.

2. Filled space. Let $p = 0$. From (2.12) we have

$$U = 4(p_1^2 - p_1 p_2 + p_2^2) + 3V + 12Ee^{-\tau}. \tag{3.7}$$

$U = 0$ at the instant of maximum expansion, and we therefore obtain

$$4(p_1^2 - p_1 p_2 + p_2^2) + 3V = U^{\text{empty}} = -12Ee^{-\tau_0} < 0, \tag{3.8}$$

where τ_0 is finite, depends on the trajectory and, as a function of the trajectory, is an integral of the system. Since $K \geq 0$, we get from (3.8)

$$3V(\gamma^2) \leq -12Ee^{-\tau_0} < 0. \tag{3.9}$$

Since $V(\gamma^2) \geq -3$, it follows that

$$e^{-\tau_0} \leq 3/4E. \tag{3.10}$$

This function reaches the maximum value 3/4 at $\gamma_1 \equiv \gamma_2 \equiv \gamma_3 \equiv 1$, which corresponds Friedmann's classical solution $K \equiv 0$ and $V \equiv -3$. This is only one trajectory at the given total energy E. For all the remaining trajectories of the Bianchi model IX, obviously, $\exp(-\tau_0) < 3/4E$. To the contrary, we can impose the following initial conditions at $E = 0$: we choose any τ_0, where $0 < \exp(\tau_0) \leq 3/4E$, and any initial point on the surface $E = 0$; trajectories pass through all these points, with only one trajectory through each point. Thus, it is possible to parametrize mutually and uniquely the set of trajectories by means of the parameter τ_0, $\exp(-\tau_0) \leq 3/4E$ and by the points of the levels

$$\tilde{H}(p, x) = U^{\text{empty}} = -12Ee^{-\tau_0} < 0. \tag{3.11}$$

All the negative level surfaces of the function $H(p, x)$ are compact; the lowest of them is a point (Friedmann solution), and the remainder are three-dimensional spheres that depend on the parameter τ_0. On each of these spheres there is a circle specified by the additional equation $\gamma_1 = \gamma_2 = \gamma_3 = 1$ at the instant of maximum expansion.

We thus draw the following conclusions:

1) At fixed E there is a two-dimensional set of trajectories (filling a three-dimensional manifold in a phase space of $5 = 6 - 1$ dimensions) which are isotropic at the instant of maximum expansion $U = 0$ ($\gamma_1 = \gamma_2 = \gamma_3 = 1$); these trajectories are present on each level of the function $H(p, x)$, which lies between (-9) and 0. The remaining trajectories pass through a compact region in γ-space

$$V(\gamma^2) \leq -4Ee^{-\tau_0}, \quad \gamma_1\gamma_2\gamma_3 = 1, \tag{3.12}$$

that is, the scatter around the isotropizing trajectories increases as $\exp(-\tau_0) \to 0$.

2) Since we had $\tau = \ln \lambda$, it follows that $\lambda_{\max} = \exp(\tau_0)$. For the minimum value we have $\lambda^0_{\max} = 4E/3$, and for the remaining trajectories $\infty > \lambda_{\max} > 4E/3$. Since $\lambda^3_{\max} = (-g_{\max})^{1/2}$, larger values of $\lambda_{\max}$ correspond to a lower density at constant E (at the instant of maximum expansion or at a zero Hubble constant).

In order for the density to be lower by one order of magnitude than in the Friedmann solution, it is necessary that $\lambda_{\max}$ be approximately 2–3 times larger than $\lambda^0_{\max}$. Since for any $\lambda > \lambda^0_{\max}$ there is a bundle of isotropic (at the instant of maximum expansion) solutions, the conclusion is that a low density does not contradict compactness of the universe in the Bianchi model IX (together with isotropization during the later stage of expansion). It is easily seen that the conclusions at $p = \varepsilon/3$ are perfectly analogous (instead of $12E\exp(-\tau_0)$ we have $12A\exp(-2\tau_0)$).

However, a density lower than in the Friedmann solution causes a larger scatter around the isotropizing solutions, and also a shorter stay in the isotropic state. Therefore the final program of the research should be as follows: find the probability distribution on the set of trajectories parametrized by our section at the instant of maximum expansion (the parameters constitute the region $\tilde{H}(p,x) < 0$). This probability distribution depends only on two factors: 1) the average rate of change of the functions U and $U^{\text{empty}} = \tilde{H}$

$$\left|\frac{dU^{1/2}}{ds}\right| = 2K + 6Ee^{-\tau_0} \tag{3.13}$$

which is the longitudinal component of the system relative to the variable U (the time s); 2) the transverse drift of the system along the region $U = 0$, which is given by the Hamiltonian system with Hamiltonian $\tilde{H}(P,x) = K + 3V$, if we take the time s (2.8). On the levels $\tilde{H}(p,x) < 0$ we have a measure which should determine, together with (3.13), the sought probability distribution. If this distribution were to be determined only by the transverse drift, then it would coincide approximately with a Gibbs distribution with Hamiltonian $\tilde{H}(p,x)$ in the region $\tilde{H}(p,x) < 0$. However, the probability distribution has so far not been determined exactly, and therefore the probable trajectories and densities have not yet been calculated; all that is clear is that the probable density can be somewhat lower than in Friedmann's solution.

By virtue of equations (2.6), (2.12), and (2.15) we have in filled space ($p = k\varepsilon$)

$$\begin{aligned} &\frac{dU}{d\tau} = -16(p_1^2 - p_1p_2 + p_2^2) - Ce^{-\alpha\tau} < 0, \\ &\frac{d\tilde{H}}{d\tau} = -16(p_1^2 - p_1p_2 + p_2^2) \le 0, \\ &C > 0, \quad \alpha > 0, \quad \alpha = 3k + 1. \end{aligned} \tag{3.14}$$

At the instant of maximum expansion we have $U = 0$. Therefore any trajectory falls sooner or later into this region $\tilde{H}(p,x) < 0$. By virtue of (3.4), this region is a trap: it does not let the trajectories out up to the instant of the maximum expansion $U = 0$. The height of the level $H(p,x)$ at the instant of maximum expansion depends on how long ago the universe fell into this trap $\tilde{H}(p,x) < 0$[2].

It apparently fell into this trap long ago, and by now it succeeded in dropping over the levels of the function $\tilde{H}$ towards the Friedmann solution. This lifetime in the trap with slow dropping over the levels of $\tilde{H}$ can be described statistically, as indicated above. In the earlier period, the BLKh buildup took place[1]–[4].

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APPENDIX

Scale Invariance and Hamiltonian Structure. Friction

The function $H^0(p', q)$ (see (1.8), which enters in all the equations under consideration (and is the Hamiltonian of the system for ($p = 0$)) has definite homogeneity

[2] An important problem is to calculate the most probable time of falling into this trap.

properties: if $q_i \to \alpha q_i$, $p'_i \to \alpha^2 p'_i$, then $H^0 \to \alpha H^0$ for $(p = 0)$. Thus, transformations of the type

$$q \to \alpha q, \quad p' \to \alpha^2 p', \quad t \to \alpha t + \beta \tag{A.1}$$

leave the Einstein equations unchanged. We consider first the case of dust-like matter $p = 0$. On the basis of this scale invariance we can introduce the convenient homogeneous coordinates

$$q_i = \lambda \gamma_i, \quad \lambda^2 b_i = p'_i. \tag{A.2}$$

Here γ and b are connected by some constraint equation $F(\gamma, b) = 1$

$$F(\lambda\gamma, \lambda^2 b) = \lambda^m F(\gamma, b). \tag{A.3}$$

Substituting in the equations, we get

$$\begin{aligned} 2\lambda\dot{\lambda} b_i + \lambda^2 \dot{b}_i &= -\lambda\gamma_i \frac{\partial H^0(\gamma, b)}{\partial \gamma_i}, \\ \dot{\lambda}\gamma_i + \lambda\dot{\gamma}_i &= \gamma_i \frac{\partial H^0(\gamma, b)}{\partial b_i}. \end{aligned} \tag{A.4}$$

Since

$$\sum_i \dot{\gamma}_i \frac{\partial F}{\partial \gamma_i} + \dot{b}_i \frac{\partial F}{\partial b_i} = 0, \quad \sum_i \gamma_i \frac{\partial F}{\partial \gamma_i} + 2b_i \frac{\partial F}{\partial b_i} = mF. \tag{A.5}$$

It follows from (A.4) that

$$m\dot{\lambda} = \sum_i \gamma_i \left(\frac{\partial H^0}{\partial b_i} \frac{\partial F}{\partial \gamma_i} - \frac{\partial H^0}{\partial \gamma_i} \frac{\partial F}{\partial b_i} \right) = \{H^0, F\}, \tag{A.6}$$

where $\{H^0, F\}$ are Poisson brackets. Further, we replace $\dot{\lambda}$ by

$$\psi(\gamma, b) = m^{-1}\{H^0, F\} \tag{A.7}$$

and then divide dt by λ/ψ. This choice of time is determined uniquely by the requirement that the equation be separable for the variables (γ, b) and λ. In the new time, we get from (A.4)–(A.7)

$$\begin{aligned} \frac{db_i}{d\tau} &= -\frac{1}{\psi} \left(\gamma_i \frac{\partial H^0}{\partial \gamma_i} \right) - 2b_i, \\ \frac{d\gamma_i}{d\tau} &= \frac{1}{\psi} \left(\gamma_i \frac{\partial H^0}{\partial b_i} \right) - \gamma_i. \end{aligned} \tag{A.8}$$

In all cases, λ drops out from the closed system of the first two equations and enters only in the definition of the time, $\tau = \ln \lambda$.

We now consider the differential form

$$\Omega = \sum_i p_1 \wedge dq_i / q_i \tag{A.9}$$

(the symbol $\wedge$ denotes the alternated tensor product on going to the usual notation in which the tensors are skew-symmetrical), which determines the Hamiltonian structure in the primary equation. In the case of empty space, we move on the level $H^0 = 0$ and therefore any changes of the time do not change the Hamiltonian properties of the system. Therefore, as before, the total derivative of Ω along the trajectory, with respect to the time τ, is equal to zero (on the level $H^0 = 0$, $F = 1$)

$$\dot{\Omega} = 0. \tag{A.10}$$

We have (from (A.2) and (A.9)):

$$\Omega = \lambda^2\Omega^0 + d\lambda^2 \wedge \omega, \tag{A.11}$$

where

$$\Omega^0 = \sum db_i \wedge \frac{d\gamma_i}{\gamma_i},$$

$$\omega = \frac{1}{2}\sum_i -db_i + \sum_i b_i\frac{d\gamma_i}{\gamma_i}, \quad d\omega = \Omega^0 \tag{A.12}$$

($d\omega$ is the differential of the form ω). By virtue of equations (A.8) we have $\dot\lambda = \lambda$ (the dot denotes here $d/d\tau$).

We recall that the differentiation of forms along trajectories commutes with the differential d, and for the product (or alternation) of forms it satisfies the Leibnitz formula

$$(\Omega_1 \wedge \Omega_2)^{\cdot} = \dot\Omega_1 \wedge \Omega_2 + \Omega_1 \wedge \dot\Omega_2. \tag{A.13}$$

Using these fact and Eq. (A.10) (at the level $H^0 = 0$), we obtain

$$\dot\omega = 2\omega, \quad \dot\Omega^0 = 2\Omega^0. \tag{A.14}$$

We already have the following important conclusion: if we choose a time τ that is directly connected with the scale invariance (since $\tau = \ln\lambda$), then we get Eq. (A.14) which expresses in invariant language the fact that on the section $F(\gamma, b) = 1$ and on the level $H^0 = 0$ we obtain a system with friction, where the friction coefficient is equal to 2.

Further, it is easily seen that the six-dimensional vector field $Y = (db_i/d\tau, d\gamma_i/d\tau)$ is orthogonal to ω:

$$\omega Y = 0. \tag{A.15}$$

The simplest equation with the foregoing properties (A.14) and (A.15) is as follows: we denote by J the operator that transforms vectors into covectors (forms) in accordance with a skew-symmetrical scalar product defined by the form Ω_0: then the properties (A.14) and (A.15) are possessed by the equation

$$(\dot\gamma, \dot b) = J^{-1}(2\omega). \tag{A.16}$$

Direct calculation shows that, indeed, (A.8) is precisely of this type on the constraint surface $F = 1$ and on the level $H^0 = 0$.

We have already called attention to the fact that in the usual Hamiltonian phase space systems of this kind constitute systems with a Hamiltonian

$$H = b = \sum_{i=1}^{3} b_i$$

plus friction, which is accounted for by the second term in the form

$$2\omega = dH + 2\sum_{i=1}^{3} b_i\frac{d\gamma_i}{\gamma_i}. \tag{A.17}$$

In which cases is it obligatory to choose precisely the function

$$b = \sum_{i=1}^{3} b_i$$

as the energy? We consider the case when the constraint surface (section) $F = F(\gamma, b)$ is chosen in the form $F = F(\gamma)$. What is the function b? By virtue of (A.2) we obtain from (1.7)

$$\lambda^2 b = \sum_{i=1}^{3} q_i (q_j q_k)^{\cdot} = 2\dot{q}, \quad q = q_1 q_2 q_3. \tag{A.18}$$

Since $F = F(\gamma)$, we get

$$\begin{gathered} 2\omega = -db + 2(p_1\, d\ln x_1 + p_2\, d\ln x_2), \\ x_1 = \gamma_1, \quad x_2 = \gamma_2, \\ p_k = b_k - b_3 \frac{\gamma_k\, \partial F/\partial\gamma_k}{\gamma_3\, \partial F/\partial\gamma_3}, \quad k = 1, 2. \end{gathered} \tag{A.19}$$

We have the simplest canonical coordinates $p_1, p_2, \ln x_1, \ln x_2$ and the form (see A.12)

$$\Omega^0 = dp_1 \wedge \frac{dx_1}{x_1} + dp_2 \wedge \frac{dx_2}{x_2}, \tag{A.20}$$

which defines the operator J. We should express the function b in the form $b = H(p_1, p_2, x_1, x_2)$ by virtue of (A.16) and (A.19), with the constraint $F = F(\gamma) = 1$. Equations (A.16) take, by virtue of (A.20), the form

$$\dot{p}_k = -x_k \frac{\partial H}{\partial x_k} - 2p_k, \quad \dot{x}_k = x_k \frac{\partial H}{\partial p_k}. \tag{A.21}$$

Our problem consists only of calculating the function $b = H(p, x)$. If $F = \gamma_i \gamma_2 \gamma_3 = 1$, then we get from (A.19)

$$p_1 = b_1 - b_3, \quad p_2 = b_2 - b_3, \quad x_1 = \gamma_1, \quad x_2 = \gamma_2. \tag{A.22}$$

Further, from (1.8), (A.6), and (A.7) we obtain

$$\psi = -b/6 = d\lambda/dt. \tag{A.23}$$

We now calculate the function b. Let $U = b^2 = H^2$.

Since $H^0(\gamma, b) = 0$, we should find $U^{\text{empty}}(p, x)$ from the equation

$$P_2(b) = -V(\gamma^2). \tag{A.24}$$

The following important algebraic fact can be simply and directly verified: given the equality

$$P_2(b) = -D, \tag{A.25}$$

then

$$\left(\sum b_i\right)^2 = 4(p_1^2 - p_1 p_2 + p_2^2) + 3D, \tag{A.26}$$

where $p_1 = b_1 - b_3$, $p_2 = b_2 - b_3$. We obtain from (A.26)

$$U = U^{\text{empty}}(p, x) = 4(p_1^2 - p_1 p_2 + p_2^2) + 3V(\gamma^2). \tag{A.27}$$

The final equations (in empty space) are

$$\frac{dp_k}{d\tau} = -x_k \frac{\partial U^{1/2}}{\partial x_k} - 2p_k, \quad \frac{dx_k}{d\tau} = x_k \frac{\partial U^{1/2}}{\partial p_k}. \tag{A.28}$$

It is easy to verify that in the case of filled space Eqs. (A.28) remain the same, but the function $U = b^2$ is altered: Eq. (A.24) is replaced, by virtue of (1.6), (1.11), and (A.2), by Eq. (A.25) where

$$D = -4E\lambda^{-1} \quad (p = 0), \quad D = -4A\lambda^{-2} \quad (p = \varepsilon/3). \tag{A.29}$$

From (A.26) and (A.27) we obtain ultimately

$$\begin{aligned} U &= U^{\text{empty}} + 12Ee^{-\tau} \quad (p = 0), \\ U &= U^{\text{empty}} + 12Ae^{-2\tau} \quad (p = \varepsilon/3), \end{aligned} \tag{A.30}$$

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