ALGEBRAIC CONSTRUCTION AND PROPERTIES OF HERMITIAN ANALOGS OF K-THEORY OVER RINGS WITH INVOLUTION FROM THE VIEWPOINT OF HAMILTONIAN FORMALISM. APPLICATIONS TO DIFFERENTIAL TOPOLOGY AND THE THEORY OF CHARACTERISTIC CLASSES. II

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Abstract. The present paper is an immediate continuation of the author's paper [22], Except in the last section, it is implicitly assumed here, as in [22], that the underlying ring contains 1/2 and all the theorems relate to the theory $U \otimes Z[1/2]$ without further comment.

7. Proof of the theorems stated in § 6

We begin with a proof of the technically most difficult theorem. Theorem 6.3, which says $B^1_U \bar{B}^1_U = \pm 1$. The first stage of the proof consists of clarifying the projective class of the lagrangian plane which represents an element from $B^1_U \bar{B}^1_U(\alpha)$, where $\alpha \in U^1_j(A)$ is the lagrangian plane $p_L = L = (aX + bP)$ in the hamiltonian space $H_n = (x_1, \dots, x_n, p_1, \dots, p_n)$ and where $L^* = x_L$ is the lagrangian plane $L^* = (cX + dP)$, written in terms of the basis (for the free case), and is considered to be chosen explicitly. We recall that $\bar{B}^1_U(\alpha)$ is a form $\phi = z^{-1}\phi_{-1} + \phi_0 + z\phi_1$ with basis $E, \tilde{X}, \tilde{P}$ and with the relations

$$(z-1)E = a\tilde{X} + b\tilde{P}, \quad \langle \tilde{X}_i, \tilde{P}_j \rangle = (z-1)\delta_{ij} \quad (\tilde{X} = \tilde{P}^*), \quad \langle \tilde{X}, \tilde{X} \rangle = \langle \tilde{P}, \tilde{P} \rangle = 0.$$

For definiteness we will write all the formulas in the hermitian case so that ϕ becomes skew-hermitian. In the other case they will be the same, apart from the necessary change of sign. Recall that the projective class of the required lagrangian plane $B^1_U(\phi)$ is given as $B(\det \phi)$ — see § 6. Here we will make our calculations in terms of the basis (e) of module E and $(v) = c\tilde{X} + d\tilde{P} \cong L^*_z = x_L$ generating the module $\bar{B}^1_U(\alpha) = F$.

We have the relations

$$\tilde{x} = \bar{d}(z-1)e + \bar{b}v, \quad \tilde{p} = \bar{c}(z-1)e + \bar{a}v.$$

The module F is free. Let us denote by v^* and e^* submodules in F such that $\langle v^*, e \rangle = \langle e^*, v \rangle = 0$ in the sense of the form ϕ; thus e^* is dual to $E[z, z^{-1}]$, and v^* is dual to $V^* \cong L[z, z^{-1}]$. If E and V ($\cong L_z, L^*_z$) are free modules then it is convenient to choose canonical dual bases in (e^*) and (v^*). In all cases we choose "bases" in the modules (e^*) and (v^*) over $A[z, z^{-1}]$ which are dual to the bases of E and V in the sense that decomposition into the sums $\sum (e^*)z^i$ and $\sum (v^*)z^i$ agrees

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with E and V, i.e. detaching the variable z, where $(e^*) \cong_A L^*$ and $(v^*) \cong_A L$ over A, and calling the bases (e^*) and (v^*). The definition of $B(\det\phi)$ is

$$B(\det\phi) = F/z^s(e,v) \cup z^{-t}(e^*,v^*), \quad s \geq 1, \quad t \leq -2.$$

After the factorization $F/z^s(e,v)$, $s \geq 1$, we have the relations (in the A-module)

$$a\tilde{X} + b\tilde{P} = -e$$

(in terms of the basis, since $ze = 0$)

$$c\tilde{X} + d\tilde{P} = v, \quad a(z^{-k}\tilde{X}) + b(z^{-k}\tilde{P}) = z^{-k}e - z^{-k+1}e,$$
$$c(z^{-k}\tilde{X}) + d(z^{-k}\tilde{P}) = z^{-k}v.$$

We will denote $z^{-k}\tilde{X}, z^{-k}\tilde{P}, z^{-k}e, z^{-k}v$ by $X^{(-k)}, P^{(-k)}, e^{(-k)}, v^{(-k)}$ respectively since $F/z^s(e,v)$ is only an A-module, $s \geq 1$.

From the given relations we obtain

$$-e^{(-k)} = a\left(\sum_{i=-k}^{0} X^{(i)}\right) + b\left(\sum_{i=-k}^{0} P^{(i)}\right),$$
$$v^{(-k)} = cX^{(-k)} + dP^{(-k)}.$$

It is understood that all this makes sense for projective modules without bases. Since $z^{-k}v^* = v^{*(-k)}$ and $z^{-k}e^* = e^{*(-k)}$, $k \geq 2$, are trivial in $B(\det\phi)$ and moreover we had

$$-e^* = cz^{-1}\tilde{X} + d\tilde{P}, \quad -(z^{-1}-1)v^* = az^{-1}\tilde{X} + b\tilde{P},$$

in the $A[z,z^{-1}]$-module, it follows after factorization that $e^{*(-k)} = 0$ and $v^{*(-k)} = 0$, $k \geq 2$, and we obtain

$$cX^{(-3)} + dP^{(-2)} = 0, \quad -v^{*(-3)} = v^{*(-2)} + aX^{(-3)} + bP^{(-2)} = 0,$$
$$-v^{*(-2)} = v^{*(-1)} + aX^{(-2)} + bP^{(-1)} = 0, \quad -v^{*(-1)} = v^{*(0)} + aX^{(-1)} + bP^{(0)},$$
$$-v^{*(0)} = v^{*(1)} + aX^{(0)} + bP^{(1)},$$

where $P^{(1)} = 0$ and $v^{*(1)} = 0$. Hence we have

$$X^{(-k)} = 0, \quad k \geq 3, \quad P^{(-k)} = 0, \quad k \geq 2,$$
$$-v^{*(-1)} + aX^{(-2)} + bP^{(-1)} = 0, \quad -v^{*(-1)} = a(X^{(-1)} + X^{(0)}) + bP^{(0)}.$$

Thus for $B(\det\phi)$ in terms of the basis $X^{(0)}, X^{(-1)}, P^{(0)}$, $u = X^{(0)} + X^{(-1)} + X^{(-2)}$, $u' = P^{(0)} + P^{(-1)}$ we obtain the natural relation

$$au + bu' = 0$$

or we obtain the basis $X^{(0)}, X^{(-1)}, P^{(0)}$, $cu + du'$ where $X^{(0)}, X^{(-1)}$ and $P^{(0)}$ are free submodules and the projective class $cu + du'$ coincides, by definition, with L^*. Consequently $B^1_U \bar{B}^1_U = -1$.

Let us now turn to the module $B(\det\phi)$, which we can think of, like the element $B^1_U\bar{B}^1_U(\alpha)$, as a lagrangian plane in a hamiltonian space.

First of all we should note that if the matrix depends only on z, z^{-1} and ϕ: $Q \to Q^*$, $Q = Q_0[z,z^{-1}]$, then the "impulse space" P in the module $B^1_U(\phi)$ is distinguished in a canonical way: $P = (Q^*_0, z^{-1}Q_0)$, since

$$\langle Q^*_0, z(Q_0 + \phi(Q_0))\rangle_0 = 0 = \langle z^{-1}Q_0, z^{-2}(\phi^{-1}(Q^*_0) + Q^*_0)\rangle_0$$

and the module $B^1_U(\phi)$ is defined in terms of the two bases $L' = Q_0 + \phi(Q_0)$ and $L'' = \phi^{-1}(Q_0^*) + Q_0^*$, where $\phi\colon Q \to Q^*$, $Q = Q_0[z, z^{-1}]$, $Q^* = Q_0^*[z, z^{-1}]$; and

$$B^1_U(\phi) = E_{1,-2}(\phi)/(z^{-2}L''_- + zL'_+)$$

with the distinguished lagrangian plane of all elements of the type $(x+\phi(x), x \in Q)$.

In the case when $\phi = \bar{B}^1_U(L)$ and $L = ax + bp$ we have, in the hamiltonian space $H_n \supset L$, submodules $E \cong L_z$ and $V \cong L^*[z, z^{-1}]$ of the module Q carrying the form $\phi = \bar{B}^1_U(L)$, $Q = E = E + V$. Since

$$e = \frac{1}{z-1}(a\tilde{X} + b\tilde{P}), \quad v = c\tilde{X} + d\tilde{P},$$
$$-e^* = cz^{-1}\tilde{X} + d\tilde{P}, \quad -v^* = \frac{1}{z^{-1}-1}(az^{-1}\tilde{X} + b\tilde{P}),$$

where $\langle e^*, e\rangle_\phi = \langle v^*, v\rangle_\phi = 1$ and $\langle e^*, v\rangle_\phi = \langle v^*, e\rangle_\phi = 0$, it follows that we can distinguish the "impulse space" in $B^1_U(\phi)$ in the following way:

$$P = (\phi(e^*), \phi(v^*), z^{-1}e, z^{-1}v).$$

Let us consider the elements $X^{(-k)} = z^{-k}\tilde{X}$, $P^{(-k)} = z^{-k}\tilde{P}$ and the elements $\phi(X^{(-k)}) = X'^{(-k)}$, $\phi(P^{(-k)}) = P'^{(-k)}$. By definition the sums $X^{(-k)} + X'^{(-k)}$ and $P^{(-k)} + P'^{(-k)}$ lie in a lagrangian plane. Also, by the above calculations we see that $X^{(k)} + X'^{(k)} = P^{(k)} + P'^{(k)} = 0$ for $k \geq 1$ or $k \leq -3$, and $P^{(-2)} + P'^{(-2)} = 0$ after passing to $B^1_U(\phi)$, since we obtain the module $B(\det\phi)$ from the lagrangian plane $Q + \phi(Q)$.

By an immediate calculation of the scalar products it follows that the elements $P^{(0)}, P^{(-1)}$ and $X^{(-1)}$ (and hence $P'^{(0)}, P'^{(-1)}$ and $X'^{(-1)}$) are orthogonal to $(zL'_+ + z^{-2}L''_-)$ and therefore lie in $B^1_U(\phi)$. We have the following matrix of scalar products:

	$\phi(e^*)$	$\phi(v^*)$	$z^{-1}e$	$z^{-1}v$
$P^{(0)}$	$\bar{c}$	$\bar{a}$	0	0
$P'^{(0)}$	0	0	0	$-\bar{c}$
$P^{(-1)}$	$-\bar{c}$	0	0	0
$P'^{(-1)}$	0	0	$\bar{a}$	$\bar{c}$
$X^{(-1)}$	$-\bar{d}$	0	0	0
$X'^{(-1)}$	0	0	0	$-\bar{d}$
$X^{(0)} + X'^{(0)}$	$\bar{d}$	$\bar{b}$	0	0
$X^{-2} + X'^{(-2)}$	0	0	$\bar{b}$	$\bar{d}$

here the scalar products taken are of the type $\langle P^{(0)}, \phi(e^*)\rangle_0 = \bar{c}$ and so on.

From the form of the matrix we conclude that the elements in a column give the complete set of the linear forms on the impulse space in $B^1_U(\phi)$ composed from $\phi(e^*)$, $\phi(v^*)$, $z^{-1}e$ and $z^{-1}v$. It is sufficient moreover to take $X^{(0)} + X'^{(0)}$, $X^{(-2)} + X'^{(-2)}$, $P^{(-1)} + P'^{(-1)}$, $P^{(0)}$ for a complete set of linear forms. Since $\langle P^{(0)}, X'^{(0)}\rangle_0 = \pm 1$, to obtain the "$X$-spaces" it is necessary to replace $P^{(0)}$ by $P^{(0)} + \gamma$, where γ is an element of the P-space $(\phi(e^*), \phi(v^*), z^{-1}e, z^{-1}v)$ dual to $X^{(0)} + X'^{(0)}$. After this operation we obtain a hamiltonian basis in $B^1_U(\phi)$ where the first three of the four basis modules are free and have the forms $X^{(0)} + X'^{(0)}$, $X^{(-2)} + X'^{(-2)}$ and $P^{(-1)} + P'^{(-1)}$, i.e. lie on the lagrangian plane $Q + \phi(Q)$. As was shown above, the missing module in the lagrangian plane is $cu + du' \cong L^*$, where

$$u = X^{(0)} + X'^{(0)} + X^{-1} + X'^{(-1)} + X^{(-2)} + X'^{(-2)},$$
$$u' = P^{(0)} + P'^{(0)} + P^{(-1)} + P'^{(-1)}.$$

In this way the basis of the lagrangian plane in $B^1_U(\phi) = B^1_U \bar{B}^1_U(\alpha)$ is $X^{(0)} + X'^{(0)}$, $P^{(-1)} + P'^{(-1)}$, $X^{(-2)} + X'^{(-2)}$, $cu + du'$, where only the last module $cu + du'$ is not a submodule of the X-space in the indicated hamiltonian basis—the change to this basis is an obvious equivalence in U^*-theory, since we did not change the P-space (but chose a more convenient X-space).

Now we can carry out a "contraction" of the lagrangian plane in $B^1_U \bar{B}^1_U(L)$ over the free submodule $X^{(0)} + X'^{(0)}$, $P^{(-1)} + P'^{(-1)}$, $X^{(-2)} + X'^{(-2)}$ lying in the X-plane and after a direct calculation ascertain that the result coincides with $L^* \subset H_n$, whence follows Theorem 6.3.

We now turn to Theorem 6.4. The anticommutativity of the "Bass operators" $\bar{B}_U(z_1)\bar{B}_U(z_2) + \bar{B}_U(z_2)\bar{B}_U(z_1) = 0$ and so on follows immediately from the formulas written down for them. To check the fact that the superposition of operators $B_U(z_1) \circ B_U(z_2)$ depends only on the element of trivial degree it is obviously sufficient to verify the invariance of this superposition with respect to the substitution $z_1 \to z_1 z_2$, $z_2 \to z_2$, which is immediate. Hence follows Theorem 6.4.

We turn to Theorem 6.5. As was already shown, this theorem follows immediately from Theorem 5.4 and the anticommutativity of the Bass operators on introducing a new variable z_2 (for the second Laurent ring extension), jumping to the dimension where Theorem 5.4 applies and then applying the operator $\bar{B}_U(z_2)$. In this way this theorem follows from what has gone before.

8. A discussion of the operators $\bar{B}^k_U$, $k = 0, 1$, as processes of motion in time

We have constructed the operators $\bar{B}_U : U^k_i(A) \to U^{k+1}_i(A_z)$, $i = 1, 2$, in a purely algebraic way. At the same time the group Z is char S^1 and is generated in a natural way by the functions $\{e^{in\tau}\}$, where τ is a numerical parameter which we will call the time. The ring $A_z = A[z, z^{-1}]$ is realized naturally as a subring of the ring of functions of τ (trigonometrical polynomials) with values in $A \otimes C$, where C is the complex numbers, and having a natural involution. From this point of view the operator $\bar{B}^0_U : U^0_i(A) \to U^1_i(A_z)$ is the construction with respect to a quadratic form of a certain lagrangian plane depending on time, and $\bar{B}^1_U : U^1_i(A) \to U^2_i(A_z)$ constructs, with respect to a lagrangian plane $L \subset H_n$ in a hamiltonian space over A, a certain quadratic form with the opposite symmetry sign on the same space $(\tilde{\tilde{x}}_L, \tilde{p}_L) \cong H = (e, v)$ but depending on the time τ. In other words, $\bar{B}^1_U$ constructs with respect to $L \subset H_n$ a lagrangian plane in the doubled space H_{2n}, where the X_{2n}-plane is $\{\tilde{\tilde{x}}_L, \tilde{p}_L\} = (e, v)$ and depends on time, such that at each moment of time it projects, in H_{2n}, isomorphically onto X_{2n} and P_{2n}—this is the way in which nondegenerate forms with the opposite symmetry sign are interpreted in all of our constructions.

Quite naturally, to understand the general idea of the operators $\bar{B}_U(\alpha)$ it is useful to indicate the equations which describe this motion in time—how this equation of motion is defined by the original element α. I should remark that the formulas for $\bar{B}^1_U$ as distinct from the operators $\bar{B}^0_U$, B^0_U and B^1_U which are easy to conjecture algebraically (in the context of hamiltonian formalism), were in fact conjectured by the author (although they are not complicated) only from the algebraic meaning of the differentio-topological operation $M \to M \times S^1$ on the obstructions to surgery, bur on the other hand the annihilation operators B_U^0 and especially B_U^1 were difficult to conjecture from topological considerations, and we conjectured them precisely from our formalism. Therefore, although we have constructed the operator $\bar{B}_U^1$, its algebraic meaning has not yet been ascertained.

We first spend some time on the operator $\bar{B}_U^0\colon U_i^0(A) \to U_i^1(A[e^{i\tau}, e^{-i\tau}])$. We recall that the augmentation $A_z \xrightarrow{\epsilon} A$, where $z \to 1$ is simply represented by the boundary $\tau = 0$, i.e. the initial conditions for any process in time. When constructing the operator $\bar{B}_U^0(\phi)$ for a quadratic form ϕ on, for example, a free module F with basis $y_1, \dots, y_n$, we considered the "double" $F \oplus F'$ with basis y' in F', $\langle y', y' \rangle = -\langle y, y \rangle$, and the lagrangian plane $P = (y_1 \oplus y_1', \dots, y_n \oplus y_n')$, to which we adjoin the X-plane $X = (x_1, \dots, x_n) = \psi_1 y \oplus \pm\bar{\psi}_1 y'$ in matrix form, where $\phi^{-1} = \psi = \psi_1 \pm \bar{\psi}_1$ and $\phi = \phi_1 \pm \bar{\phi}_1$. The required distinguished lagrangian plane $L(\tau) \subset H_n(e^{i\tau})$ was obtained as $(y \oplus e^{i\tau} y') = y \oplus zy'$. Obviously $L|_{\tau=0} = P$. In addition it is useful to note that the original quadratic form ϕ on the module F was distinguished when $\tau = 0$ in the hamiltonian space $H_n|_{\tau=0}$ by its basis vectors $y = \phi(\bar{\psi}_1 P \pm X)$, the dual basis ψy with matrix ψ has the form $\psi_1 P \mp X = F \subset H_n|_{\tau=0}$. The module F' with basis $y' \subset H_n|_{\tau=0}$ has the form $\phi(\psi_1 P - X)$.

As before we denote the basis of $L(\tau)$, which depends on time, by $P_L(\tau)$ and introduce the dual basis $x_L(\tau) = \psi_1 y \oplus \pm e^{i\tau}\bar{\psi}_1 y'$, where $x_L(0) = x$, $p_L(0) = p$.

It is easy to verify that in terms of the bases $x_L(\tau), p_L(\tau)$ the equations defining B_U^0 have the form (here $i = \sqrt{-1}$)

$$\dot{x}_L = \pm i\bar{\psi}_1\phi[\psi_1 p_L(\tau) - x_L(\tau)] = \pm\frac{\partial H}{\partial \bar{p}_L},$$
$$\dot{p}_L = i\phi[\psi_1 p_L(\tau) - x_L(\tau)] = \frac{\partial H}{\partial \bar{x}_L},$$
$$x_L(0) = x, \quad p_L(0) = p,$$

where on the basis $x_L, p_L, \bar{x}_L, \bar{p}_L$, we have

$$-iH(x_L, p_L, \bar{x}_L, \bar{p}_L) = \langle \bar{p}_L\bar{\psi}_1 - \bar{x}_L|\phi|\psi_1 p_L - x_L\rangle,$$

where $H(\xi, \bar{\xi})$ is a bilinear scalar (with values in A) function of the vectors ξ and the covector $\bar{\xi}$ varying independently, the operators ψ_1 and $\bar{\psi}_1$ are considered as linear transformations $(p) \to (x)$ and $(p) \to (x)$ respectively and $\langle\bar{\xi}|\phi|\xi\rangle$ denotes a pairing between the original and the dual spaces, the bases of which are coordinated with the help of the transformation defining the quadratic form. The fact that the operator ϕ (and therefore H) is hermitian (skew-hermitian) springs from the requirement that the equations of motion written above for vectors, and the equations obtained, in an analogous way from $\partial H/\partial x_L$ and $\partial H/\partial p_L$ for the covectors dual to them, preserve this relation of duality with the course of time, i.e. that the motion of covectors is consistent with the motion of the vectors.

If we put $x_L(0) = x$ and $p_L(0) = p$, then we obtain, on solving the equation, the image of the operator B_U^0; that is, the lagrangian plane $p_L(\tau)$ (or $L(\tau)$). Thus we see that the hamiltonian H can be written simply and immediately in terms of the form ϕ or $\psi = \phi^{-1}$ (in the dual basis), or, more precisely, in terms of the plane $y' = \psi_1 p - x$ defining the form.

Example 1. Let $A = C$, R or Z and let ϕ be a hermitian form over A in $U_1^0(A)$. In this case we have the usual number space and periodic functions in $H_n(\tau)$; moreover, the lagrangian plane $L = p_L(\tau)$ projects on X with singularities at isolated moments of time. We note that on the basis of Theorem 5.4 $U_1^1(A_z) = \bar{B}_U^0 U_1^0(A) = Z$, since $U_1^1(A) = 0$ where $z = e^{i\tau}$. How can one define analytically the invariant $\sigma B_U^0 \colon U_1^1(A_z) \to Z$, where σ is the signature, for lagrangian planes $L(\tau) \subset H_n(\tau)$? It is possible to define "the cycle of singularities" W^n for the projection $\pi \colon L(\tau) \to X$ (or after making a certain change in the direction of the projection, to bring it into general position), which is an element of the "open homology" group on $L(\tau) = L(0) \times S^1$; and, since τ is a periodic variable, $W^n \in H_n^0(L(\tau))$. There is a periodic trajectory $\gamma \in H_1(L(\tau))$, and its intersection number $\gamma \circ W^n$ with the singular cycle is the "Maslov index" (see Example 1 in § 4).

Theorem 8.1. *For the ring $A = C$, R or Z and the groups $U_i^1(A[z, z^{-1}])$ the homomorphism $\sigma \circ B_U^1 \colon U_1^1(A[z, z^{-1}]) \to Z$ (where B_U^1 is the Bass annihilation operator and a is the signature of the standard hermitian form over $A = C$, R or Z) coincides with the "Maslov index" $\gamma \circ W^n$ of a periodic trajectory γ (or of a basis element of a group $H_1(L(\tau))$) on the lagrangian plane $L(\tau) \subset H_n$, $L(\tau) \approx R^n \times S^1$, $A = R$, representing the element $\alpha \in U_1^1(A[z, z^{-1}])$, $z = e^{i\tau}$. In particular, for $A = R$ or Z and $j = 2$ (skew-symmetric case) this index $j \circ W^n$ is always trivial and $U_2^1(A_z)_{1/2} = 0$, but for $A = C$ it can be nontrivial: for $A = C$ the Morse index may be nontrivial in both of the cases $j = 1, 2$, which for $A = C$ are equivalent.*

The proof of this theorem is not complicated, but we omit it since the theorem was introduced by us only to illustrate the meaning of our ideas. For the proof it is necessary to turn to the example in § 4 and to Maslov's definition of this "Morse index".

Thus for the group ring of the cyclic group Z the invariant σB_U^1 on the group $U_i^1(A[\pi])$ [and consequently on the group of obstructions to surgery in differential topology $L_{2k+1}(\pi) = V_i^1(A[\pi])$ which coincides for $\pi = Z$ with $U_i^1(A[\pi])$] can naturally be interpreted analytically as the Morse index on lagrangian planes depending periodically on time.

Now we consider the operator $\bar{B}_U^1 \colon U_i^1(A) \to U_i^2(A[z, z^{-1}])$ which was defined algebraically in § 6. Here a hermitian (skew-hermitian) form depending on the time τ is constructed on the $2n$-dimensional module F, isomorphic to H_n, from a lagrangian plane $L = (p_L) \subset H_n$. This form $\phi = \bar{B}_U^1(L)$ has been considered in a natural way in terms of the basis $X_1 = E = (\tilde{\bar{p}}_L)$, $X_2 = (\tilde{X}_L)$ on $F \approx H_n$ and interpreted in the hamiltonian space H_{2n} with the basis $X = (X_1, X_2)$ and $P = (P_1, P_2)$ as a lagrangian plane $\tilde{L}$ with basis $P - \phi X$, depending on the time τ, which projects isomorphically for all τ onto X and P. Thus if $L = p_L = (ax + bp)$ the lagrangian plane $\tilde{L}$ or the form ϕ for $\tau = 0$ had the form $\begin{pmatrix} b\bar{a} & 1 \\ \pm 1 & 0 \end{pmatrix}$ in terms of the basis X_1, X_2, where $b\bar{a}$ was the "action hessian" on L in the coordinates x, p; in this way the initial conditions $\tilde{L}|_{\tau=0}$ were trivially defined by this "action hessian". It only remained to write down the equations of motion in H_{2n} . They have the form

$$\dot{X}_1 = \dot{X}_2 = 0,$$
$$\dot{P}_1 = ie^{-i\tau} b\bar{c} X_2 = \frac{\partial H}{\partial \bar{X}_1},$$
$$\dot{P}_2 = \pm i[e^{i\tau}(c\bar{b}X_1 \pm d\bar{c}X_2) \mp e^{-i\tau} c\bar{d} X_2] = \pm \frac{\partial H}{\partial \bar{X}_2},$$

The plane $\tilde{L}(\tau)$ is defined by these equations and the initial conditions $\tilde{L}|_{\tau=0} = P - \phi(0)X$, where $\phi(0) = \begin{pmatrix} b\bar{a} & 1 \\ \pm 1 & 0 \end{pmatrix}$. The hamiltonian $H(\xi, \bar{\xi})$ is here free (does not depend on P) and may be expressed in terms of the basis as $iH = \bar{V}W \pm \bar{W}V$, where $V = e^{i\tau}\bar{c}X_2$, and $W = \bar{b}X_1 + \bar{d}X_2$. We recall here that the change of basis from (x, p) to $(L^* = x_L,\, L = p_L)$ is given by the matrix $\begin{pmatrix} \pm c & \pm d \\ a & b \end{pmatrix}$, but the change from (x_L, p_L) to x, p is given by the matrix $\begin{pmatrix} \bar{b} & \bar{d} \\ \bar{a} & \bar{c} \end{pmatrix}$ and $p = \bar{a}x_L + \bar{c}p_L \to \bar{a}X_1 + \bar{c}X_2$ and $x = \bar{b}x_L + \bar{d}p_L \to \bar{b}X_1 + \bar{d}X_2$ and $\bar{c}p_L \to \bar{c}X_2$ is simply the image $\pi_L(p)$ (the projection of the p-plane onto p_L along x_L) which is involved in the definition of the action hessian on (p) in terms of the coordinates (x_L, p_L). Consequently to obtain the hamiltonian $H(X_1, X_2, \bar{X}_1, \bar{X}_2, \tau)$ we proceed as follows: we shift $\pi_L(p)$ by an amount z in the coordinates (x_L, p_L) (multiply by $e^{i\tau}$) and thereby obtain V; then we take X (the plane $\bar{b}x_L + \bar{d}p_L$), we obtain W and from them we construct the form $\bar{V}W \pm \bar{W}V = iH$, where $x_L \to X_1$, $p_L \to X_2$. Further we consider the initial "phase space" $H_n(x, p)$ as a configuration and construct its double—the phase space $H_{2n}(X_1, X_2, P_1, P_2)$—in which we consider this hamiltonian $iH = \bar{V}W \pm \bar{W}V$ to be free: $H = H(X, \bar{X}, \tau)$. Further, we solve the hamiltonian equations with initial condition $\tilde{L}|_{\tau=0} = X - \phi(0)P$, where $\phi(0) = \begin{pmatrix} b\bar{a} & 1 \\ \pm 1 & 0 \end{pmatrix}$, and obtain as a result the plane $\tilde{L}(\tau) \subset H_{2n}$ which is projected isomorphically onto X and P for all τ; that is, $\bar{B}^1_U(L)$.

Let us note that the inverse matrix $\phi^{-1}(\tau)$ or the basis $X - \phi^{-1}P$ is obtained from the original by a substitution such as (see § 6)

$$\tau \to -\tau, \quad P \to X,$$
$$x_L \to p_L, \quad X \to \pm P,$$
$$p_L \to \pm x_L.$$

Thus $X - \phi P \to P - \phi^{-1}X$ and $\phi \to \phi^{-1}$.

One can give another explanation of the structure of $\bar{B}^1_U$. Namely, first of all, without introducing $L^* = x_L$, we define $\bar{B}^1_U(L)$ in terms of the basis $E, \tilde{X}, \tilde{P}$, where

$$(z-1)E = L = a\tilde{X} + b\tilde{P}, \quad \langle \tilde{X}, \tilde{X} \rangle = 0, \quad \langle \tilde{P}, \tilde{P} \rangle = 0, \quad \langle \tilde{X}, \tilde{P} \rangle = (z-1)\delta_{ij} = e^{i\tau}.$$

Then $\tilde{X} = \tilde{X}(\tau)$, $\tilde{P} = \tilde{P}(\tau)$ and the spaces $E = \tilde{p}_L$, and $\tilde{x}_L = c\tilde{X} + d\tilde{P}$ are regarded as not depending on the time τ. In this case $\tilde{X}(\tau)$ and $\tilde{P}(\tau)$ at any moment of time $\tau \neq 0 \pmod{2\pi}$ give a basis for the space $F \cong H_n$ on which the form $\phi(\tau)$ is defined, and moreover $\tilde{X}(\tau)$ and $\tilde{P}(\tau)$ is a hamiltonian basis in $F(\tau)$ when $\tau \neq 0$. However, when $\tau = 0$ the space $(\tilde{X}(\tau), \tilde{P}(\tau))$ is degenerate, in view of the relation $a\tilde{X}(0) + b\tilde{P}(0) = 0$, and the whole of the space $\tilde{X}(\tau), \tilde{P}(\tau)$ for small $\tau \neq 0$ is "born from" the space $L^* = c\tilde{X}(0) + d\tilde{P}(0)$. The differential equation $\tilde{X}(\tau)$ and $\tilde{P}(\tau)$ has a singularity when $\tau = 0 \pmod{2\pi}$.

9. The interrelation of U^*- and V^*-theory over $Z[\pi]$ and the Bass operators B and $\bar{B}$ with the theory of obstructions to surgery in differential topology $L_n(\pi_1) \otimes Z[1/2]$

We will denote collectively by $L_n(\pi_1)$ all the possible obstructions to surgery in all the situations which in the simply connected case correspond to various theorems due to the author and Browder ([4], [11]). Here it turns out that the various theorems for the simply connected case correspond in fact to different kinds of groups of the type $L_n(\pi)$ which have a definite relation to the various U^* and V^* of our theory, but the relation is not completely precise. This occurs essentially when we talk about projections of the Bass type and decomposition theorems of the type $L_n(\pi \times Z) = L_n(\pi) + L_{n-1}(\pi)$; it is incorrect, apart from isolated special cases of the group π (where, however, it is also necessary to revise the arguments in [16]) to relate these assertions to vague notions of $L_n(\pi)$ by making use of geometrical arguments from different geometrical situations. C.T.C. Wall in his papers [19] and [20] defines much more clearly the class of objects with which the definition of $L_n(\pi)$ is related: he considers all the Poincaré complexes to be finite, which excludes K^0 from the definition of $L_n(\pi)$ and permits all quadratic forms on free modules to be considered in $L_{2k}(\pi)$. As for the definition of $L_{2k+1}(\pi)$, this permits lagrangian planes L to be considered as free, not projective, submodules in H_n, reducing them to automorphisms carrying the X-plane into L (see [20], § 6), and allows a basis on L to be distinguished (these are elements of the group $V_j^1(\pi) = L_{2k+1}(\pi)$). Obviously Wall does not consider in [20] questions about Bass projections, but his L_n is correctly defined for each n individually, reflecting the solution of a single well-determined geometrical problem. Besides, it is well known, for example, that the problem of the simple homotopy type of $K^1(\pi \times Z)$ for $M \times S^1$ becomes under the Bass projection $B\colon K^1(\pi \times Z) \to K^0(\pi)$ the Wall obstruction [18] to the homotopy finiteness of complexes, and in certain special cases it becomes the obstruction of the author and Siebenmann to a smooth PL decomposition of $M = V \times R$ (see [13], Proposition 2). These examples indicate the impossibility of restricting oneself to homotopically finite Poincaré complexes while at the same time preserving projections of the Bass type. However, this causes another difficulty in purely topological approaches: the quadratic forms on free modules in $L_{2k}(\pi)$ are easily realized geometrically as "obstructions to uniqueness", but all quadratic forms on projected modules are difficult to realize. There is an analogous problem for $n = 2k + 1$: projective lagrangian planes are also difficult to realize geometrically. This explains the restrictions imposed in [20]. We prefer to construct and systematize a separate algebraically correct homology theory and to explain the interrelations with topology later. We have the following problems:

1. "The existence problem": this is the question of the possibility of modifying a map $f\colon M_1^n \to M_2^n$ of degree 1, where $f^*(\xi)$ is tangent to M_1^n, $\xi \in KO(M_2^n)$ and M_2^n is a Poincaré complex. We may suppose that $[f] \in \pi_{N+n}(-\xi)$, where $M_N(-\xi)$ is the Thom complex.

1′. "The unimodular existence problem": this is the same question, but M_2^n is a finite complex and we wish to modify f to give a simply homotopy equivalence.

2. "The uniqueness problem": this is the question of the possibility of modifying a manifold W^n to give an h-cobordism, when W^n has two boundaries $\partial W = M_1^{n-1} \cup M_2^{n-1}$ and the tangent map of $W \xrightarrow{r} M^{n-1}$ is of degree 1 on the boundaries (we may suppose that $r|M_1 = 1$).

2′. "The unimodular uniqueness problem": this is the same question, but we wish to modify r to give a diffeomorphism $W = M \times I$.

3. "The strengthened uniqueness problem": here we assume that $r|M_2^n$ is a simple homotopic equivalence, and we are required to modify the connecting manifold W and in addition deform the map r so that $r|M_2^n$ is a diffeomorphism and $\tilde{r}\colon W \to M \times I$ is a pseudo-isotopy between $r|M_1 = 1$ and $r|M_2$, where $\tilde{r}$ is homotopic to r.

We show where the obstruction to the solution of these problems lies:

Problem 1:

$$\alpha(f) \in V_j^{(0)}(A) \quad \text{for } n = 4k,\ 4k+2,$$
$$\alpha(f) \in W_j^{(0)}(A) \quad \text{for } n = 4k+1,\ 4k+3.$$

Problem 1′:

$$\alpha(f) \in W_j^1(A) \quad \text{for } n = 4k,\ 4k+2,$$
$$\alpha(f) \in V_j^1(A) \quad \text{for } n = 4k+1,\ 4k+3.$$

Problem 2:

$$\alpha(f) \in V_j^0(A) \quad \text{for } n = 4k,\ 4k+2,$$
$$\alpha(f) \in W_j^0(A) \quad \text{for } n = 4k+1,\ 4k+3.$$

Problem 2′:

$$\alpha(f) \in W_j^1(A) \quad \text{for } n = 4k,\ 4k+2;$$
$$\alpha(f) \in V_j^1(A) \quad \text{for } n = 4k+1,\ 4k+3.$$

Problem 3:

$$\alpha(f) \in W_j^2(A) \quad \text{for } n = 4k,\ 4k+2,$$
$$\alpha(f) \in V_j^2(A) \quad \text{for } n = 4k+1,\ 4k+3.$$

Here, of course, we have neglected the Arf-invariant and certain differences between the Whitehead group Wh, K^1 and the analogies of K^0. Thus we see that the obstructions in problem 1 do not really lie in our theories but have homomorphisms $V_j^0 \to U_j^0$ and $W_j^0 \to U_j^1$, where the kernels of these homomorphisms are generated by the images of $\tilde{K}^0(A) \to V_j^0(A)$ and $\tilde{K}^0(A) \to W_j^0(A)$.

Conclusion. *If the Arf-invariant is neglected then the obstructions to surgery in problem* 1 *coincide with the theory* $U_{1/2}^*$ *when* $\tilde{K}^0 = 0$.

All the other obstructions in problems 1′, 2, 2′ and 3 lie in our theories V^* and W^*, but a theorem on the realization of the groups W and V in these problems is generally speaking not true—it is only in problem 2′ for W^1 and V^1, and, apparently, in problem 3. We will not explain all these questions in more detail, since the aim of the present work is purely algebraic; but the way to obtain relations between obstructions and these or other geometric problems is well known in contemporary differential topology (for n odd we will show this below), and it was useful just to give a systematic statement of them from the point of view of the spaces in our algebraic "hermitian K-theories" U^*, V^* and W^*.

We will indicate further the geometrical meaning of the Bass operators B_U and $\bar{B}_U$ and the formula for them which explains the algebraically-involved processes in the proofs in §§ 5, 6 and 7. Before this we indicate the geometrical path by which we associate an invariant in the groups U^1_j and V^1_j for odd $n = 2k+1$ and $j = 1, 2$ with a map $M^n_1 \xrightarrow{f} M^n_2$ of degree 1, where $f^*(\xi)$ is tangent to M^n_1.

This was first given in [20], but the complexity of the procedure in [20] is explained by the artificality of the algebraic approach and the calculation of the pair.

I. If $f_*\colon \pi_i(M^n_1) \to \pi_i(M^n_2)$ is an isomorphism for $i < k$ and $n = 2k+1$, it is possible to choose a sufficiently large number of spheres $\{S^k_q\} \subset M^n_1$, $q = 1, \dots, N$, such that they give a complete basis for $\operatorname{Ker} f^{(\pi_k)}_*$ as an A-module and, further, give all the cells of dimension k which are superfluous in M^n_1 in comparison with M^n_2. We assume that these spheres do not intersect and that $f(S^k_q)$ is a point.

We cut out a tubular neighborhood T of the spheres S^k_q (of their connected sum) from M_1. Put $Q = M^n_1 \setminus \bigcup_q T(S^k_q)$ and $\partial Q = \bigcup_q S^k_q \times \tilde{S}^k_q$, where $\tilde{S}^k_q$ is linked with S^k_q. Introduce the notation

$$x_q = [S^k_q] \in \pi_k(\partial\hat{Q}),$$
$$p_q = [\tilde{S}^k_q] \in \pi_k(\partial\hat{Q}),$$

where $\hat{Q}$ is the universal covering space and S^k_q and $\tilde{S}^k_q$ are the natural cycles lifted to $\hat{Q}$. By calculating the intersection number we find that $H_N = \pi_k(\partial\hat{Q})$ is a hamiltonian A-module with basis $x_1, \dots, x_N, p_1, \dots, p_N$.

The obstruction to surgery is a lagrangian plane $L \subset H_N$, where L is by definition the kernel of the inclusion $\pi_k(\partial\hat{Q}) \to \pi_k(\hat{Q})$. Generally speaking, L is a projective module, but if M^n_2 is a finite Poincaré complex then L is a free module in which a basis can be naturally distinguished. This will also give an element in $V^1_j(A)$; generally speaking we have an element in $W^0_j(A) \to U^1_j(A)$.

To establish these facts we ought to indicate the geometric meaning of the elementary operations which give the equivalence between planes $L \subset H_N$.[1]

1) Replacement of the X-plane by X' which projects isomorphically onto X along the same P. This corresponds to a change of the spheres $\{S^k_q\}$ by a homotopy of the system where each of the S^k_q is deformed regularly, intersecting other spheres or even itself along the way.

2) Stabilization. This is obviously adding an extra "small" sphere S^k_{N+1} to the collection $\{S^k_q\}$.

3) A hamiltonian operation. This corresponds to a Morse modification. In fact, let the sphere with respect to which we make the modification be chosen on M in a neighborhood of the boundary $\partial\hat{Q}$ and let it represent on this boundary an element $\sum \gamma_i x_i + \sum \mu_i p_i \in \pi_k(\partial\hat{Q}) = H_N$. This sphere is given together with a field of frames which on the boundary of its tubular neighborhood determines the coordinates $(p_{N+1}, x_{N+1}) = (H, t)$, where H is the displacement of this sphere to the boundary and t is linked to the initial sphere S^k_{N+1}. Let B be a connecting manifold in $\hat{M} \setminus \hat{Q}$ which realizes a cobordism of the sphere H with an element $\gamma x + \mu p$ from $\pi_k(\partial\hat{Q}) = H_N$, and let $\phi \in A$ be the "intersection number" $\langle B, S^k_{N+1}\rangle = \phi \in A = Z[\pi_1]$. After adjoining the sphere S^k_{N+1} to the collection $\{S^k_1, \dots, S^k_N\}$ we obtain a basis $p_1, \dots, p_N, H, x_1, \dots, x_N, t$, but instead of L we will have $L \oplus H$ as the new

[1] It is relevant to note the obvious fact that die cokernel of the projection of L on X coincides with the homological kernel in dimension k on the covering spaces but the kernel of the projection is in dimension $k+1$, $n = 2k+1$.

kernel of the map $\pi_k(\partial\hat{Q}') \to \pi_k(\hat{Q}')$, and instead of the P-plane we will have on the other hand $p'_i = p_i \pm \gamma_i t$, $H' = H - \gamma x - \mu p - \phi t$ as the kernel of the inclusion of the new boundary $\partial Q'$, since we also removed a neighborhood of the cycle S^k_{N+1} from the join $T(S^k_q)$ (more precisely, from its total preimage on $\hat{M}_1$).

It is convenient to represent the elements of $U^1_j(A)$ and $V^1_j(A)$ as obstructions to "uniqueness problems": if $M^{2k}_1 \xrightarrow{f} M^{2k}_2$ is a map such that $\operatorname{Ker} f^{(\pi_i)}_* = 0$ for $i < k$ ($\neq 0$ only when $i = k$) and $Q = \operatorname{Ker} f^{(\pi_k)}_*$ is a free module with a form ϕ which can be reduced to hamiltonian type with the help of a lagrangian plane $P \subset Q$ which gives a basis $(P, X) \subset Q$ and can be reduced to zero with the help of another lagrangian plane $L \subset Q$ with even action hessian in the coordinates (P, X), then both processes of reduction (P) and (L) can be realized by handles attached to $M^{2k}_1 \times I(0,1)$ or to $M^{2k} \times 0$ and $M^{2k} \times 1$ respectively; we obtain a manifold W^{2k+1} with a tangential map $r\colon W^{2k+1} \to M^{2k}_2$, where $r|M^{2k}_1 \times 1/2 = f$ and $f|\partial W^{2k+1}$ are homotopy equivalences (or simple homotopy equivalences). This interprets naturally and geometrically the elements of U^1_j, V^1_j and W^0_j; moreover, it does so exactly in accordance with the point of view of this paper on "the reduction processes" in the construction of a K-theory.

II. Let us give now the geometrical interpretation of the initiation operators $\bar{B}^0_U, \bar{B}^1_U$, although of course the geometrical situation does not correspond absolutely precisely to the theory U^* and not all the elements can be realized geometrically.

1. *The operator* $\bar{B}^0_U$. We have a map $M^{2k}_1 \xrightarrow{f} M^{2k}_2$ with a unique nontrivial kernel $(Q, \phi) = \operatorname{Ker} f^{(\pi_k)}_*$ with form ϕ. Let us consider $f \times 1\colon M^{2k}_1 \times S^1 \to M^{2k}_2 \times S^1$ and an immersion of spheres $\lambda_q\colon S^k_q \to M^{2k}_1$ which realize the basis cycles of Q. Let us order the elements σ of the group $\pi = \pi_1(M^{2k}_1) = \pi_1(M^{2k}_2)$ in any way such that for any pair (σ, σ^{-1}) only one of them is positive.

For simplicity we assume that the group π has no 2-torsion, i.e. $\sigma^2 = 1 \to \sigma = 1$. Consider the maps $\hat{\lambda}_q\colon S^k_q \to \hat{M}^{2k}_1 \to \hat{M}^{2k}_1 \times R$ into the universal covering spaces. We "perturb" the map $\hat{\lambda}_1\colon S^k_1 \to \hat{M}^{2k}_1 \subset \hat{M}^{2k}_1 \times R$ so that it becomes an embedding $\tilde{\lambda}_1\colon S^k_1 \subset \hat{M}^{2k}_1 \times R$; we do this according to the following rule: at the outset we suppose that $\hat{\lambda}_q\colon S^k_q \to \hat{M}^{2k}_1$ is an embedding but that there is an intersection of the type $\langle S^k_{q_1}, \sigma S^k_{q_2}\rangle$ for various q_1, q_2 and σ, in particular when $q_1 = q_2 = 1$. If $\sigma > 1$ we displace one of the intersecting pieces of σS^k_1 "up" a little along the R-axis; if $\sigma_2 > \sigma_1 > 1$ then the "displacement" along the R-axis is higher for σ_2 than it is for σ_1. Having done this we obtain an embedding $\tilde{\lambda}_1\colon S^k_1 \subset \hat{M}^{2k}_1 \times R$ depending on the above-mentioned introduction of an order between the elements $\sigma \in \pi$. For the sphere $S^k_2 \to M^{2k}_1$ we begin by moving it a little, parallel to the R-axis, above the image $\tilde{\lambda}_1(S^k_1)$ (but always below $\hat{M}^{2k}_1 \times 1$) where initially $\hat{M}^{2k}_1$ and all the images of $\tilde{\lambda}_q$ lie on $\hat{M}^{2k} \times 0$ and the displacement z along the R-axis transforms α into $\alpha + 1$. Then we do the same as for $\hat{\lambda}_1$. We continue thus until we have done it for all q. After this we remove from $\hat{M}^{2k}_1 \times R$ all the images of the spheres $\sigma\hat{\lambda}_q(S^k_q)$ and all their displacements under elements $\sigma \in \pi \times Z$, and we obtain a manifold $\hat{Q}$ with boundary $\partial\hat{Q}$, where there is a natural basis x_i, p_j, $\langle x_i, p_j\rangle = \delta_{ij}$ in $\pi_k(\partial\hat{Q})$, where the p_j are linked with $\tilde{\lambda}_q(S^k_q)$ and the x_q are displacements of $\tilde{\lambda}_q(S^k_q)$ onto the boundary of a tubular neighborhood and x_q and p_q are in natural one-to-one correspondence with the spheres $\tilde{\lambda}_q(S_q^k)$. Which lagrangian plane L in this hamiltonian module is the kernel of the embedding $\pi_k(\partial\hat{Q}) \to \pi_k(\hat{Q})$?

Let us note that the division of the group π into two sets $(\pi)^+ \cup (\pi)^- \cup 1$ and the condition that the scalar product ϕ on Q be even makes it possible to decompose $\phi = \phi_1 \pm \bar{\phi}_1$ canonically in such a way that the matrix of ϕ_1 is triangular and $\phi_1 = 0$ is below the diagonal. On the diagonal we have

$$\phi(x_q, x_q) = \sum_{\sigma\in\pi} (S_q^k \cdot \sigma S_q^k)\sigma.$$

Let us put

$$\phi_1(x_q, x_q) = \begin{cases} \sum_{\sigma>1}(S_q^k \circ \sigma S_q^k)\sigma, & k = 2l+1, \\ \sum_{\sigma>1}(S_q^k \circ \sigma S_q^k)\sigma + \frac{1}{2}(S_q^k \circ S_q^k), & k = 2l, \end{cases}$$

where $a \circ b$ is the usual intersection number.

A direct geometrical calculation of the kernel of the embedding $\pi_k(\partial Q) \to \pi_k(\hat{Q})$, where $\pi_k(\partial\hat{Q}) = H_m$ is a hamiltonian module with basis (x, p), leads to a formula of this kind for the basis of the lagrangian plane L which is equal to this kernel

$$\psi L = (z-1)X + (z\phi_1 \pm \bar{\phi}_1)P,$$

where $\psi = \phi^{-1}$ does not depend on z.

Turning to Remark 2.6, we obtain an interpretation of the operator $\bar{B}_U^0$ in terms of a basis.

2. *The operator* $\bar{B}_U^1$. We have here a map $f\colon M_1^{2k+1} \to M_2^{2k+1}$ of degree 1 which is tangential and such that $\operatorname{Ker} f_*^{(\pi_s)} = 0$, $s < k$. In dimension k spheres $(S_1^k, \dots, S_m^k)$ are chosen such that $f(S_q^k)$ is a point and the connected sum of the tubular neighborhoods of these spheres is removed from M_1^{2k+1}, where $N = M_1 \setminus \#_q T(S_q^k)$ and $\hat{M} \supset \hat{N}$ are their covering spaces. The module $\pi_k(\partial\hat{N})$ is a hamiltonian module with basis $x_1, \dots, x_m, p_1, \dots, p_m$ as above; L is the kernel of the embedding $\pi_k(\partial\hat{N}) \to \pi_k(\hat{N})$.

Let us turn to $f \times 1\colon M_1^{2k+1} \times S^1 \to M_2^{2k+1} \times S^1$ and $\hat{M} \times R \supset \hat{M} \times 0 \supset \hat{N} \times 0$.

We carry out Morse modifications with respect to the cycles $x_q \times 0$ which are realized by $S_q^k \times 0$, and consider the effect of the modifications on the covering spaces. After the modification we obtain a map (the modification of $f \times 1$) $g\colon \tilde{M}^{2k+2} \to M_1^{2k+1} \times S^1$ the kernel of which will be nontrivial only in dimension $k+1$ and which is a nondegenerate form (even) of the opposite symmetry sign. Let us calculate this form. First of all it is necessary to indicate a basis for the cycles on $\tilde{M}^{2k+2}$. When we carried out modifications on the spheres $S_q^k \subset \hat{M}_1^{2k+1} \times 0 \subset \hat{M}_2^{2k+1} \times R$ we performed the following operations:

a) We removed from $\hat{M}_1 \times R$ the tubular neighborhoods T_q^{2k+2} of these spheres, where $T_q^{2k+1} \cap (\hat{M}_1 \times 0)$ are neighborhoods of these spheres in $\hat{M}_1^{2k+1}$ and on the boundary we have the "old" hamiltonian basis x_q, p_q in $M_1^{2k+1} \setminus \#_q(T_q \cap M_1^{2k+1})$, which we will then consider on the covering spaces. The lagrangian plane L may be represented by cells e_{k+1} of dimension $k+1$ lying in

$$\hat{N} = (M_1^{2k+1} \setminus \#_q(M_1^{2k+1} \times 0 \cap T_q))^\wedge,$$

where the boundaries $\partial e_{k+1} \in H_m(x, p)$ form L. It is possible to suppose that in terms of the basis on L we have $\partial(e_{k+1}) = aX + bP$.

b) We adjoined new cells β_q to $\tilde{M}^{2k+2}$ such that $\partial\beta_q = x_q \in \partial\hat{N}$. In addition there are obvious pairs of connecting manifolds $\gamma_{q,1}, \gamma_{q,2}$, which lie in $\hat{M}_1 \times R \setminus \#_q T_q^{2k+2}$ which have each side on $\hat{N} \times 0 \subset \hat{N} \times R$ in $(M_1^{2k+1} \times R \setminus \#_q T_q^{2k+2})^\wedge$, which have boundaries p_q, and where $\gamma_{q,1}$ is above $\hat{M}_1 \times 0$ and $\gamma_{q,2}$ is below $\hat{M}_1 \times 0$ with respect to R.

We have the following cycles in $\tilde{M}^{2k+2}$:

$$\gamma_{q,1} - \gamma_{q,2} = \tilde{P}_q,$$
$$x_q \times I(0,1) + z\beta_q - \beta_q = \tilde{X}_q,$$
$$e_{k+1}^{(\alpha)} + a\{\beta_q\} - b\{\gamma_{q,1}\} = e^{(\alpha)},$$

where a runs over the basis of the lagrangian plane L.

We have the scalar products

$$\langle \tilde{X}_q, \tilde{P}_s \rangle = (z-1)\delta_{qs},$$
$$\langle \tilde{X}, \tilde{X} \rangle = 0,$$
$$\langle \tilde{P}, \tilde{P} \rangle = 0,$$

and also the relation (in matrix form)

$$(z-1)e = a\tilde{X} + b\tilde{P}.$$

This gives us the description of $\bar{B}_U^1$ (see § 6).

III. We turn now to a more difficult problem, namely the geometric interpretation of B_U^0 and B_U^1.

1. *The operator* B_U^0. The general geometrical situation with which we have to deal in the given case consists of the following: we have a map $f\colon M^{2k+1} \to M^{2k} \times S^1$ of degree 1, and so on, and we consider the t-regular preimage $f^{-1}(M^{2k} \times 0) = M_2^{2k}$ together with the map $g = f|M_2^{2k} \to M^{2k}$. We have an invariant $\alpha(f) \in U_j^1(A_z)$ and wish to find $\alpha(g) \in U_j^0(A)$, where $A = A[z, z^{-1}] = Z[\pi \times Z]$ and $A = Z[\pi]$.

There will be analogous situations for other problems where $\alpha(f) \in V_j^1$. It goes without saying that is is impossible to calculate a precise formula for $\alpha(g)$ in such a general form. We introduce certain hypotheses (supposing from the outset that $\operatorname{Ker} f_*^{(\pi_i)} = 0$ for $i < k$).

a) Let us assume that the total preimage of M_2^{2k} is such that $\operatorname{Ker} g_*^{(\pi_i)} = 0$, $i < k$.

b) Suppose that M_2^{2k} does not intersect the spheres $\{S_q^k\} \subset M^{2k+1}$ containing those tubular neighborhoods which have to be cut out of M^{2k+1} in order to obtain a hamiltonian basis $(x_q, p_q) \in \pi_k(\partial\hat{N})$ together with a lagrangian plane L

$$0 \to L \to \pi_k(\partial\hat{N}) \to \pi_k(\hat{N}), \quad N = M^{2k+1} \setminus \bigcup_q T(S_q^k);$$

it is convenient for us here to cut out a disconnected sum.

c) We suppose that a basis $L^* = x_L$, $L = p_L$ is chosen in the module $H_m(x,p) = \pi_k(\partial\hat{N})$; of course we do not exclude the possibility that L^* and L are projective; here the elements of the basis x_L are realized by spheres which do not intersect M_2^{2k} in the interior of $N \supset M_2^{2k}$.

d) We have $\hat{M}_2^{2k} \subset \hat{N}$ on the covering space $\hat{N}$ where the displacements $z^j \hat{M}_2^{2k}$ do not intersect and $\hat{M}_2^{2k}$ separates $\hat{N}$ into $N^+ \cup N^- = \hat{N}$, $N^+ \cap N^- = \hat{M}_2^{2k}$. We suppose that the coverings of the basis spheres which realize $x_L \in \pi_k(\hat{N})$ lie in N^- and do not intersect the $z^j \hat{M}_2^{2k}$ for $-\infty < j < \infty$.

These hypotheses can be fulfilled by various geometrical operations. If they are fulfilled, then we can find representatives of an element $B_U^0 \alpha(f) = \alpha(g)$. For definiteness we suppose that the spheres $\{S_q^k\}$, $q = 1, \dots, m$, lie in the domain N^+ between $\hat{M}_2^{2k} \times 0$ and $\hat{M}_2^{2k} \times 1 = z(\hat{M}_2^{2k} \times 0)$, and that the x_L lie in N^-. We fix connecting manifolds $B^\alpha \subset \hat{N}$ which have boundaries lying in $\partial \hat{N}$ and define a basis for the lagrangian plane L in the coordinates $x, p \in \pi_k(\partial \hat{N})$. Further we fix connecting manifolds C^β which represent a homotopy of the line $x_L^{(\beta)}$ into the boundary $\partial \hat{N}$. The manifold M_2^{2k} can be chosen so that the intersections $z^j B^\alpha \cap \hat{M}_2^{2k} \times 0$ and $z^j C \cap M_2^{2k} \times 0$, which we denote by κ_j^α and κ_j^β respectively, generate the group $\operatorname{Ker} g_*^{(\pi_k)}$ on $\hat{M}_2^{2k} \times 0$. It is easy to see that these intersections are trivial if $z^j B^\alpha$ or $z^j C^\beta$ lie in N^+ or in N^- respectively: for this it is sufficient that $z^j L \geq 0$ or $z^j L^* > 0$, $z^j L < 0$, $z^j L^* < 0$ in the sense of the notation of § 5. The parts of these connecting manifolds lying in N^+ represent a homotopy of these cycles relative to N^+ into the boundary $\partial \hat{N} \cap N^+$, where $\pi_k(\partial \hat{N} \cap N^+) = H_m^+$ in the sense of § 5. The corresponding cycles on the boundary $\partial \hat{N} \cap N^+$ are, as is easily seen, $(z^j L)_+$ and $(z^j L^*)_+$ where $-N_1 < j < N$ and moreover N and N_1 are numbers such that $(z^{-N_1} L^-) < 0$ and $(z^N L^+) \geq 0$; we have $L = L^- + L^+$ as an A-module. The intersection numbers of these cycles on $\partial \hat{N} \cap N^+$ are the same as on $\hat{M}_2^{2k} \times 0$ in view of the connecting manifolds $z^j B^\alpha \cap N^+$ and $z^j C^\beta \cap N^+$. This implies that the geometric interpretation of the projection $B_U^0 \colon U_j^1(A_z) \to U_j^0(A)$ corresponds in fact to the algebraic definition in § 5.

2. *The operator* B_U^1. The geometric realization of the operator $\bar{B}_U^1$ is more complicated than the three cases just investigated. For simplicity we will assume that the quadratic form $\phi \in U_j^2(A_z)$ contains only z and z^{-1} and does not contain $z^{\pm k}$ for $k \geq 2$, i.e. it is a trigonometrical polynomial of the first degree. It is only forms of this sort that appear as images of the operator $\bar{B}_U^1$. The form, as before, is given on a free module with basis (or on a projective module $Q = Q_0[z, z^{-1}]$ with a basis Q_0).

In a geometrical realization it is convenient to consider only free modules when only such forms can arise from $\bar{B}_U^1$. As in the algebraic construction of the operator B_U^1, it is necessary to begin the geometrical realization with an interpretation of the form ϕ as a lagrangian plane $L = P + \phi X$, where $\phi(P) = \phi X$ and (P) is a space carrying the form

$$\phi \colon (P) \to (X), \quad \langle x_i, p_j \rangle = \delta_{ij}, \quad H^+ + H^- = H(x, p),$$
$$L'_+ = \sum_{i \geq 0} z^i L', \quad L''_- = \sum_{i \leq 0} z^i L'',$$
$$L' = P + \phi X, \quad L'' = \psi P + X,$$
$$\psi = \phi^{-1}, \quad \phi = z^{-1}\phi_{-1} + \phi_0 + z\phi_1, \quad \psi = z^{-1}\psi_{-1} + \psi_0 + z\psi_1.$$

The space $E_{1,-2}(L)$ is taken orthogonal to $zL'_+ \cup z^{-2} L''$ in the sense $(\ ,\ )_0$, and is assumed to contain the lagrangian plane $L \subset E_{1,-2}(L)$. The quotient $E_{1,-2}(L)/(z^{-2} L''_- +$

$zL'_+)$ is $B^1_U(\phi)$ with the lagrangian plane $B(L) = L/(z^{-2}L''_- + zL'_+)$ of projective class $B(\det\phi)$ and with hamiltonian basis $\tilde{X}^{(1)} = W_2$, $\tilde{X}^{(2)} = V_2$, $\tilde{P}^{(1)} = W_1$, $\tilde{P}^{(2)} = V_1$, where $W_1 = X$, $W_2 = (P + \phi X)_+$, $V_1 = z^{-1}P$ and $V_2 = (z^{-1}X + z^{-1}\psi P)_-$; the suffix signs $\pm$ denote the projection $H \xrightarrow{\pi_\pm} H_\pm$, where $H_+ = \sum_{i\geq 0} z^i(X,P)$, $H_- = \sum_{i<0} z^i(X,P)$, $\psi(X) = \psi P$, $\phi(P) = \phi X = \sum \phi_{ij}x_j$, ϕ and ψ being matrices over the bases in X and P.

To indicate a complete system of elements which generate the module $B(L) = L/(z^{-2}L''_- + zL'_+)$ it is sufficient to take the elements

$$A_1 = P + \phi X, \quad z^{-1}(P + \phi X) = A_2, \quad B_1 = z^{-1}(X + \psi P), \quad B_2 = X + \psi P.$$

After some simple calculations we obtain these elements in matrix form:

$$\begin{aligned} A_1 &= W_2 - \phi_{-1}(V_2 - \psi_0 V_1), \\ A_2 &= \phi_0 V_2 - \phi_1(W_1 + \psi_{-1}V_1), \\ B_1 &= V_2 - \psi_1(W_2 - \phi_0 W_1), \\ B_2 &= \psi_0 W_2 \pm \psi_1(V_1 + \phi_{-1}W_1), \end{aligned}$$

It is necessary to take into account the relations $\psi\phi = 1$, $z^{-2}L''_- = 0$ and $zL'_+ = 0$ for these calculations. The signs indicated here are for a skew-hermitian form ϕ (for hermitian forms everything is the same except that the opposite signs are taken). Note that the triple (A_1, A_2, B_1) or (A_1, B_1, B_2) already generates $B(L)$.

Let us turn now to a geometrical situation. Interpreting, as before, the form ϕ as a "process of converting a trivial lagrangian plane into an isomorphically projected plane", we arrive at the following geometrical situation on the covering space. We start with an odd-dimensional $(2k+1)$-manifold $M = V \times R$ where z acts as a "displacement" along a line and $V \times 0$ separates $M = V \times R$. We choose a collection of spheres $\{S^k_q, \dots, S^k_m\}$ with fields of frames for the Morse modifications and cut out tubular neighborhoods $T(S^k_j)$ of the spheres; we examine the whole picture on the universal covering spaces $\hat{M} = \hat{V} \times R$. We will denote by $P_1, \dots, P_m$ these same spheres S^k_j copied onto the boundaries of the tubular neighborhoods ∂T_j and denote the "linked" cycles on the boundary of the tubes by $X_1, \dots, X_m$, where $m = \operatorname{rk}\phi$. We select the linking matrix (and the fields of frames) for the spheres S^k_j so that it coincides with the form ϕ; this means that in $\hat{M}_1 = (M \setminus \bigcup_j T_j)^\wedge$ we will have a homotopy relation $P + \phi X = 0$ between elements on the boundary $\partial\hat{M}_1$. Performing a Morse modification on M for each of the cycles S^k_j in the given setting, we pass from M to $M \setminus \bigcup_j T_j$, thereupon adding the relations $\{P_j = 0\}$ to the final manifold M_2. As a result, in view of the nondegeneracy of ϕ, we have $P_j = 0$, $X_j = 0$ and M_2 is homotopically equivalent to M. The process of performing $M \to M_2$ has the form ϕ as its "action hessian".

What happens to $V \subset M$ under this transformation? We choose the spheres S^k_j so that the intersections $z^s S^k_j \cap (\hat{V} \times 0)$ in the manifold $\hat{M} = \hat{V} \times R$ are empty if and only if the number s is such that $z^s L'_+ \subset H_+$ or $z^s L'_- \subset H_-$, i.e. when $s \neq 0, -1$. If the intersection $z^s S^k_j \cap \hat{V}$ is nonempty then it is a sphere $S^{k-1}_{j,k} \cap \hat{V}$ which is homotopic to zero in $\hat{V}$. Such a choice is possible and natural when $\phi = \phi_{-1}z^{-1} + \phi_0 + z\phi_1$ and $\psi = \psi_{-1}z^{-1} + \psi_0 + z\psi_1$.

Further, in performing $\hat{M} \to \hat{M}_1$, neighborhoods of these spheres $S^{k-1}_{j,s}$, when $s = 0, -1$, are "cut out" from $\hat{V}$. After performing $\hat{M}_1 \to \hat{M}_2$, connecting manifolds are adjoined to the spheres $S^{k-1}_{j,s}$ displaced to the boundaries of the tubes $z^s(\partial T_j)\cap\hat V$. As a result of performing $\hat M\to\hat M_2$ we obtain $\hat V\to\hat V_2\subset\hat M_2$, which has the form of a Morse modification on the cycles $S^{k-1}_{j,s}\subset V$. Therefore the manifold $\hat V_2\to\hat M_2$ will have as the homotopy kernel of the inclusion $\hat V_2\to\hat M_2$ a free module (hamiltonian) with coordinates $\tilde{\tilde X}_{j,s}$ = cycles linked with $S^{k-1}_{j,s}$ and $\tilde{\tilde P}_{j,s}$ = cycles obtained by the union of connecting manifolds and representing homotopies in the interior of $\hat V$ from the cycles $S^{k-1}_{j,s}$ to zero and to the connecting manifold which is added in the process of modification. Thus $\langle\tilde{\tilde X}_{j,s},\tilde{\tilde X}_{k,s_1}\rangle=0$, $\langle\tilde{\tilde P}_{j,s},\tilde{\tilde P}_{k,s_1}\rangle=0$ and $\langle\tilde{\tilde X}_{j,s},\tilde{\tilde P}_{k,s_1}\rangle=\delta_{(j,s)(k,s_1)}$.

We will denote by A, $\partial A=P+\phi X$, a connecting manifold in $\hat M_1$ which realizes the homotopy relation $P+\phi X=0$, and we will denote by B_j, where $\partial B_j=P_j$, $z^sB_j\cap\hat V_2=\beta_{s,j}$, $\partial B_{0,j}=S^{k-1}_{0,j}$ and $\partial B_{-1,j}=S^{k-j}_{-1,j}$, a connecting manifold for P_j in $\hat M_2$. The cycles $\tilde{\tilde X}_{j,s}$ can be identified in a natural way with the cycles z^sX_j in $\hat M_1$ when $s=0,-1$:

$$\tilde{\tilde P}_{j,s}=\beta_{j,s}=\partial_V^{-1}(S^{k-1}_{j,s}),\quad s=0,-1.$$

As $\hat V_2$ separates $\hat M_2$ into 2 parts (the upper and the lower with respect to the R-coordinates, or with respect to powers of z), to calculate the lagrangian plane $B^1_U(\phi)$ in the hamiltonian module $(\tilde{\tilde X}_{j,s},\tilde{\tilde P}_{j,s})$ it is necessary to calculate the (homotopy) kernel of the inclusion of the manifold $\hat V_2$ into the "lower half": $M_2^-\subset\hat M_2$, $\hat M_2=M_2^-\cup M_2^+$, $M_2^-\cap M_2^+=\hat V_2$. We utilize for this calculation the connecting manifold A, $\partial A=P+\phi X=P+\phi(P)$. We suppose here that the connecting manifold A intersects $\hat V\times0$ and $\hat V\times1=z(\hat V\times0)$ only in the interior of $\hat V\times R=\hat M$; this means that the connecting manifold A lies between $\hat V\times(-1)$ and $\hat V\times(2)$. Dividing A into three parts by means of $\hat V\times0$ and $\hat V\times1$, we obtain three connecting manifolds $A=A_{-1}+A_0+A_1$, where A_{-1} lies below $\hat V\times0$, A_0 lies between $\hat V\times0$ and $\hat V\times1$ and A_1 lies above $\hat V\times1$.

Let us introduce the notation

$$\begin{gathered}\partial A_{-1}=q_0^{(1)}+\lambda_{-1}X,\\ \partial(A_0+A_1)=-q_0^{(2)}+(\lambda_0+\lambda_1)X,\\ \partial A_1=-zq_{-1}^{(2)}+\lambda_1X,\\ \partial(A_0+A_{-1})=zq_{-1}^{(1)}+(\lambda_0+\lambda_{-1})X,\end{gathered}$$

here everything is written in matrix form, $\lambda_{-1}+\lambda_0+\lambda_1=\phi=z^{-1}\phi_{-1}+\phi_0+z\phi_1$ and the $q_s^{(j)}$ (where $s=0,-1$ and $j=1,2$) are the cycles consisting of the piece of the cycle z^sP below $\hat V\times0$ (or above when $j=2$) and the piece of boundary of the connecting manifold A_{-1},A_0+A_1 (or $z^{-1}A_1,z^{-1}(A_0+A_1)$) on the manifold $\hat V\times0$. Note moreover that $\lambda_1=\lambda_{1,1}z+\lambda_{1,0}$, $\lambda_0=\lambda_{-1,0}z^{-1}+\lambda_{0,0}+\lambda_{0,1}z$ and $\lambda_{-1}=\lambda_{-1,-1}z^{-1}+\lambda_{-1,0}$. As long as we do not know the matrix λ_{ij} we cannot complete the calculation of $B^1_U(\phi)$. Of course in changing the connecting manifold A we change the λ_{ij}; then sum is equal to ϕ. (It is possible to choose the connecting manifold A such that $\lambda_{-1}=0$, $\lambda_0=z^{-1}\phi_{-1}$ and $\lambda_1=\phi_0+z\phi_1$.) Further, let us note that in view of the connecting manifold B we have a homotopy $q_0^{(j)}=P_0$,

$q_{-1}^{(j)} = P_{-1}$, where the homotopy lies below $\hat{V}_2 \times 0$ when $j = 1$ (or above when $j = 2$).

Which connecting manifolds (relations on $\tilde{\tilde{X}}, \tilde{\tilde{P}}$) lie below $\hat{V}_2 \times 0$? These connecting manifolds give the lagrangian plane $B_U^1(\phi)$ in the hamiltonian module $(\tilde{\tilde{X}}, \tilde{\tilde{P}})$.

One such connecting manifold is A_{-1}, and $\partial A_{-1} = q_0^{(1)} + \lambda_1 X$ gives an element $\tilde{\tilde{P}}_0$ in the required lagrangian plane. The connecting manifold $z^{-1}(A_0 + A_{-1})$ does not give yet another such element since $(\lambda_0 + \lambda_{-1})X$ contains $(\lambda_{0,-1} + \lambda_{-1,-1})z^{-2}X = \phi_{-1}z^{-2}X$, where $z^{-2}X$ cannot be displaced onto $\hat{V}_2 \times 0$ along $\hat{M}_2$. Recall that only $z^{-1}X$ and X have a natural displacement onto $\hat{V}_2$ (and they give the elements $\tilde{\tilde{X}}_{-1}$ and $\tilde{\tilde{X}}_0$ there).

Since $\partial A = P + \phi X$ and $\partial B = P$, we have $\partial(z^{-2}\psi A - z^{-2}\psi B) = z^{-2}X$ in $\hat{M}_2$, where only $z^{-1}\psi_1 A_1$ and a bit of B lies above $\hat{V}_2 \times 0$. Therefore "below" $\hat{V}_2 \times 0$ we will have

$$0 = z^{-2}X + \psi(-\tilde{\tilde{P}}_{-1} + \lambda_{1,0}z^{-1}X + \lambda_{1,1}X),$$

which is obtained from the boundary $\partial(z^{-1}\psi_1 A_1)$. Replacing $\psi_1(\tilde{\tilde{P}}_{-1} - \lambda_{1,0}z^{-1}X - \lambda_{1,1}X)$ by $z^{-2}X$ in the formula for the boundary $\partial(z^{-1}(A_0 + A_{-1}))$, we obtain other elements in the lagrangian plane:

$$\tilde{\tilde{P}}_{-1} + \lambda_{0,1}X + (\lambda_{0,0} + \lambda_{-1,0})z^{-1}X + (\lambda_{0,-1} + \lambda_{-1,-1})z^{-2}X,$$

where $z^{-1}X = \tilde{\tilde{X}}_{-1}$ and $X = \tilde{\tilde{X}}_0$.

Further, we have $\partial(z^{-1}\psi A + \text{a bit of } B) = z^{-1}X$ in $\hat{M}_2$. Separating the part of the connecting manifold "above" $\hat{V}_2 \times 0$, we obtain

$$z^{-1}X = \tilde{\tilde{X}}_{-1} = \psi_1(\tilde{\tilde{P}}_0 + (\lambda_0 + \lambda_1)X) + \psi_0(-\tilde{\tilde{P}}_{-1} + \lambda_1 z^{-1}X).$$

Noting that $\psi_1\phi_1 = 0$ and $\lambda_{0,1} + \lambda_{1,1} = \phi_1$, after substituting $z^{-1}X = \tilde{\tilde{X}}_{-1}$, $X = \tilde{\tilde{X}}_0$ we find yet another relation "below".

To compare the formulas obtained for the "geometric" lagrangian plane in the coordinates $\tilde{\tilde{P}}_0, \tilde{\tilde{P}}_{-1}, \tilde{\tilde{X}}_0, \tilde{\tilde{X}}_{-1}$ with the "algebraic" lagrangian plane obtained previously in the coordinates W_1, W_2, V_1, V_2 it is necessary to complete the following hamiltonian transformation:

$$V_2' = V_2 - \psi_1(W_2 - \phi_0 W_1),$$
$$V_1' = V_1, \quad W_2' = W_2 \pm \phi_0\psi_{-1}V_1, \quad W_1' = W_1 \pm \psi_{-1}V_1$$

and then make the comparison

$$V_2' \to \tilde{\tilde{P}}_0, \quad W_2' \to \tilde{\tilde{P}}_{-1}, \quad V_1' \to \tilde{\tilde{X}}_0, \quad W_1 \to \tilde{\tilde{X}}_{-1}.$$

After some simple calculations we see that the lagrangian planes in the new coordinates coincide (more precisely, will differ by obviously inessential terms) if we take $\lambda_{-1} = 0$, $\lambda_0 = z^{-1}\phi_{-1}$ and $\lambda_1 = \phi_0 + z\phi_1$.

Thus the geometrical and algebraic definitions are equivalent, given those restrictions which enable us to relate the elements of U^2 geometrically.

We will not analyze the geometrical interpretations in more detail; in particular, we will not analyze the proofs of Theorems 5.4 and 6.3–6.5.

Let us note that a rigorous identification of the usual $K^2(A)$ for $A = Z[\pi]$ with the problem of pseudo-isotopies of diffeomorphisms, and therefore with the interpretation of the groups $V_j^2(A)$ and $W_j^2(A)$ in Problem 3 (see above), has not, as far as the author knows, been developed in the literature. However, it could be obtained by analyzing the "simply-connected" paper of Browder [4], although we will not make this analysis here.

The result of this section is, in particular,

Theorem 9.1. *If $\tilde{K}^0(\pi_1) = 0$, then the obstructions to Morse modifications in Problem 1 lie in the groups $U_1^k(Z[\pi_1]) = U_2^{k+2}(Z[\pi_1])$ (modulo $\otimes Z[1/2]$), the operation $M \to M \times S^1$ on the obstruction to modifications corresponds to the operators $\bar{B}_U$ and the inverse operation $M \times S^1 \to M$ (and passage to the total preimage M) corresponds to the operator B_U.*

The proof of this theorem was given above. It is more complicated for the operators $B_U : U^k(A[z, z^{-1}]) \to U^{k-1}(A)$.

10. Certain applications to the theory of characteristic classes. Relative formulas of the Hirzebruch type

If we have a map $f : M_1^n \to M_2^n$ of degree 1 which induces an isomorphism between the fundamental groups $\pi_1(M_1) = \pi_1(M_2)$ and is such that there is an element $\xi \in KO(M_2)$ for which $f^*(-\xi)$ is the tangent bundle to M_1, then the following question arises: since the bordism class $[M_1, f]$ in the group $\pi_{N+n}(M_N(\xi))$ of the Thom complex $M_N(\xi)$ defines an element of the group $\alpha(f) \in U_1^n(A)$, $A = Z[\pi_1]$, then which characteristic classes of the manifold M_1^n, coinciding with the characteristic classes of the fiber bundle $f^*(-\xi)$, are defined by the original manifold M_2^n and the element $\alpha(f)$? In [12], [13] and [15] the scalar products of the Hirzebruch classes $L_k(M_1^n)$ with cycles which were intersection cycles of codimension 1 were considered.

Theorem 10.1. *If $H^1(M_1^n)/\mathrm{Torsion} = \mathrm{Hom}(\pi_1, Z)$ and $\Lambda^{n-4k}H^1 \xrightarrow{\kappa} H^{n-4k}(M_1^n)$ is generated by multiplication, then for $u \in \Lambda^{n-4k}H^1(M_1^n) = \Lambda^{n-4k}\pi_1^*$ the scalar product $(L_k(M_1^n) - f^*L_k(M_2^n), D\kappa(u))$ is completely defined by the element $\alpha(f) \in U_1^n(A)$ according to the following "Hirzebruch formula":*

$$(L_k(M_1^n) - f^*L_k(M_2^n), D(z_1^* \wedge \cdots \wedge z_{n-4k}^*)) = \sigma B_U(z_1^* \wedge \cdots \wedge z_{n-4k}^*)\tilde{\alpha}(f),$$

where D is the Poincaré duality operator, $z_i^* \in H^1(M_1^n)$, $B_U(z_1^* \wedge \cdots \wedge z_{n-4k}^*)$ is the iterated Bass operator $B_U(z_1^*) \circ \cdots \circ B_U(z_{n-4k}^*)$ depending only on the element $z_1^* \wedge \cdots \wedge z_{n-4k}^* \in \Delta^{n-4k}\pi_i^*$ (see § 6), $\sigma : U_1^{4k}(A) \to Z$ is the usual signature homomorphism on the Z-module $M \otimes_A Z = M_0$, $M \in U_1^{4k}$, $\tilde{\alpha}(f)$ is the image of the element $\alpha(f)$ under the homomorphism $U_1^*(A) \to U_1^*(A')$, $A' = Z[\pi_1^{**}]$, $A = Z[\pi_1]$, and the group π_1^{**} is free abelian.

We will not cite the proof of this formula—it could have been written down, rather ineffectively, before this present paper was written (see for example [13]–[16]), but it would not have been an algebraic formula, by which we mean that the algebraic definition of the operators B_U and $\bar{B}_U$ was not known, although for free abelian groups $\pi = \pi^{**}$ a rather ineffective "theorem on the existence of Bass operators" was considered for differential topology (see [16]) (at least for geometrically realizable elements (and it is not clear in exactly what theory of homotopy type)). In view of § 9 our algebraic operators B_U and $\bar{B}_U$ coincide with the geometric operators on elements such as these. Therefore such a formula is true and is now purely algebraic.

Comparing the above with the papers [14] and [16], we can derive

Corollary 10.2. *The non-simplyconnected Hirzebruch formula gives a complete collection of algebraic relations on the realization of elements in* $U_1^n(Z \times \cdots \times Z)$ *as obstructions to modifications* $\alpha(f)$ *if the maps* f*, the elements* $\alpha(f)$ *and the group* $\pi = Z \times \cdots \times Z$ *are considered modulo passing to finite coverings* $\hat{f}, \alpha(\hat{f})$ *(and modulo subgroups of finite index).*

11. Unsolved problems

1. Here we consider first of all the following general question: what is a "general non-simplyconnected Hirzebruch formula"?

This question could be answered in the following way: there should exist a certain homomorphism "of generalized signature"

$$\sigma_k \colon U_1^n(A) \to H_{n-4k}(\pi_1, Q)$$

such that for any n-dimensional closed oriented manifold M^n with fundamental group π_1 and for a natural map $f \colon M^n \to K(\pi_1, 1)$ the scalar product $(L_k(M^n), Df^*(x))$ is homotopically invariant for all $x \in H^*(\pi_1, Q)$, and DL_k as a linear form on $H^*(\pi_1)$ (or an element of $H_*(\pi_1, Q)$) belongs to the image of σ_k. We have constructed explicitly such homomorphisms for one abelian group; here they even turn out to be isomorphisms over Q (this was known ineffectively in topology—see [7], [16] and [19]).

Of course this problem can be posed for finite modules p, at least for p large compared with n.

Let us note that a number of considerations suggest that, for example, for the fundamental groups of "solv-" and "nil-" manifolds such a homomorphism exists and is an epimorphism over Q, such that the allowable classes of cycles are not just the intersections of cycles of codimension 1. Here we can introduce the "non-commutative extension" of the ring A, namely augmentation by z and z^{-1} without assuming that they commute with A, to generalize the theory of operators of Bass type. However, this does not clarify the general question. It goes without saying that the question of a "relative Hirzebruch formula" is simpler. Let us note that the question of the intrinsic calculation of scalar products of L_k with cycles of the form $Df^*(x)$ is essentially more complicated even for abelian π—it cannot be solved even for $\pi = Z \times Z$ (see [13]–[15]).[2]

2. Let us see what the question on a "non-simplyconnected Hirzebruch formula" and the construction of "generalized signature" homomorphisms $\sigma \colon U_1^*(A) \to H^*(\pi; Q)$ becomes when we replace the group rings A by the ring of functions $A = C(X)$.

If we replace $H_*(\pi, Q)$ by $H^*(X)$ then we arrive at a problem about the abstract algebraic construction of the Chern character

$$\mathrm{Ch} \colon U^*(A) = K^*(X) \to H^*(X).$$

For this it is necessary to start from some purely ring theoretic formulism for constructing $H^*(X)$ from the ring $C(X)$. Let us note in this connection that for $A = Z[\pi]$ the group π is distinguished by the equation $\sigma\bar{\sigma} = 1$. In the ring $C(X)$ this equation distinguishes the functions on X with modulus equal to the identity,

[2] A. S. Miščenko has found a peculiar analog to the classical signature: an element from $U^*(\pi_1) \otimes Z[1/2]$ is associated in a homotopically invariant way with a manifold, and this defines a homomorphism from the bordism theory $SO_*(\pi_1) \to U^*(\pi_1) \otimes Z[1/2]$ into hermitian K-theory which is connected, apparently, with the L-type.

i.e. the group $(X \to S^1) = (S^1)^X$, which is a peculiar infinite-dimensional torus. It is possible that this analogy is meaningful and that one could construct an analogy to the theory of the "Bass operators" $B_U, \bar{B}_U$ and then use it to define $H^*(X)$ and Ch algebraically. Here, in view of § 6, it is impossible to think of continuous (or smooth, if smooth functions are taken) operators B_U and $\bar{B}_U$ without a thorough investigation of the relationship with the formalism or derived quantization over the ring of functions; the operators $B_U(z^*)$, in essence, depend on the linear functionals, i.e. on generalized functions. This question, however, does not seem clear.

Let us note that for the ring $R(X)$ the involution is trivial and $\sigma\bar{\sigma} = 1$ distinguishes only the identity. The bad properties of such a ring are connected with this fact. There are other reasons.

3. We have constructed analogs of K-theory (namely the theory U^*) requiring only the existence of the Bass operators B and $\bar{B}$. However, generally speaking, U^*-theory is not a real homology theory. For example, for a ring of the type $R(X)$ only $U^* \otimes Z[1/2]$ is a homology theory and coincides with $KO^*(X) \otimes Z[1/2]$. Here the incompleteness of our investigation is obvious. It is possible that it was incorrect to take $U^2(A) = U_i^0(A)$, $i \neq j$, by definition and to carry over to U_j^2 (in constructing U_j^k, $k \geq 3$) the idea of a "reduction process" which is characteristic for U_i^0. Here there are many subtle and vague questions from the point of view of pure algebra. For a clarification of this it is necessary to analyze and unite from our point of view the schemes of Karoubi–Atiyah type and the construction of bigraded theories (see [1]).

We have restricted ourselves to the Bass projections because this operation does not take us out of the class of group rings which are necessary in differential topology. In addition to this we were interested in clarifying the analogies with the hamiltonian formalism well known in other domains, which is already sufficient (in the true sense of the word) for our narrow aims. Nevertheless the questions indicated here ought undoubtedly to be clarified in the future.

4. The question of the analytic meaning of the algebraic ideas given here has been discussed repeatedly in the present paper. In particular, in Example 1 of § 4 we pointed out that this algebraic formalism over the ring R (real numbers) has already appeared in the construction of the so-called global quasi-classical when passing from quantum mechanics to classical mechanics in this situation (see [9]): there is a lagrangian submanifold L in a hamiltonian space $H_n(x, p) \supset L$ and finite functions on L which depend on a small parameter h. It is necessary to construct a correspondence (a canonical operator) between functions on L and functions on X, which is well defined up to terms in h (mod $O(h^2)$). For a motion L in the hamiltonian system (in the sense of classical mechanics) the image of the functions under the canonical operator varies as the solution of a Schrödinger equation, up to terms in h (mod $O(h^2)$).

Up to terms in h, on isomorphically projected L's such an operator K has the form $f(l) \xrightarrow{K} J^{-1/2} e^{i/hS(l)} f[\pi l]$, where $l \in L$, $x = \pi l$, $\pi \colon L \to X$ is the projection, J is the Jacobian of the projection and $S = \int_{l_0}^{l} p\,dx$ along the path in L. We pointed out in Example 1 of § 4 that the construction of such an operator on non-isomorphically projected L's requires strict analysis of the process of passing from the P-space to the X-space near the singular points of the projection into X, and is connected in the same way with $U_2^2(R)$ (see [9], part II, Chapter 2, § 2). However, we isolated in essence only the abstract algebraic definition of "the Maslov index" in [9] and the reasons for its appearance in such problems. This was not sufficient for a thorough algebraic analysis of the idea of a "canonical operator" as well as the quantization conditions. Here it would be desirable to develop the corresponding formalism on the ring of functions, and then to make all these constructions the topic of a purely algebraic theory. This, however, does not appear too easy.

12. On the role of the Arf-invariant

In every result in the present paper we have started from the idea of a bilinear form $\langle\ ,\ \rangle$ on an A-module and required in addition that the ring A contain $1/2$. At the same time we often used the expression "even form", which is not essential when a $1/2$ is present. This implied in fact that many of our constructions were suitable for integral group rings, and we indicate here the corresponding addenda for a ring A which does not contain a $1/2$. Following Wall ([20], § 5), we consider a module with "a quadratic form" $[x,x] \in A/(a-\bar{a})$ (hermitian case) or $[x,x] \in A/(a+\bar{a})$ (skew-hermitian case), where $\langle x,x\rangle = [x,x] \pm \overline{[x,x]} \in A$. It is obvious that when there is a $1/2$ in the ring the form $[x,x]$ is defined by the bilinear form $\langle\ ,\ \rangle$ but when there is no $1/2$ ambiguous distinctions arise. As an example we note that, when $A = Z$, in the skew-symmetric case we have $Z/(a+\bar{a}) = Z_2$ and we obtain a so-called Arf-invariant in the form $[x,x] \in Z_2$, since in $A/[a+\bar{a}]$ we always require the identity

$$[x+y, x+y] = [x,x] + [y,y] + \langle x,y\rangle \bmod (a \pm \bar{a}).$$

Therefore we will call the form $[\ ,\ ]$ an Arf-invariant for all A.

The idea of a "lagrangian plane" $L \subset Q$ will now include an additional requirement that $[x,x] = 0$ for $x \in L$ when we consider an A-module Q with a bilinear form $\langle\ ,\ \rangle$ together with an Arf-invariant $[\ ,\ ]$.

In the case of the integral group ring (with the usual involution) $Z[\pi] = A$ the group $A/(a+\bar{a})$ of values taken by the form $[\ ,\ ]$ has generators $-g = g^{-1}$ which are free abelian when $g^2 \neq 1$ and of order 2 when $g^2 = 1$. The free abelian part of the form $[x,x]$ in $A/(a+\bar{a})$ is determined completely by the bilinear form $\langle\ ,\ \rangle$, but the 2-torsion is associated with the elements $g^2 = 1$ in π that give the usual Arf-invariant when $g = 1$ and its analog when $g \neq 1$ but $g^2 = 1$. (The group $A/(a-\bar{a})$ for $Z[\pi]$ with the usual involution does not generally have 2-torsion, and the form $[\ ,\ ]$ is defined by the bilinear form $\langle\ ,\ \rangle$ (even). Therefore in the case $A = Z[\pi]$ with symmetric forms the Arf-invariant $[\ ,\ ]$ is not necessary.) For skew-symmetric forms it reduces to the classical Arf-function, which is associated not only with the identity element $g \in \pi$, but also with all $g^2 = 1$, $g \in \pi$. Namely if $S \subset \pi$ is the set of all $g \in S$ with $g^2 = 1$, and the group π acts on $S\colon g \to \sigma g \sigma^{-1}$, then we have the Arf-invariant $\Phi(x,g) \in Z_2$, $x \in Q$, $g \in S$, where $\Phi(\sigma x, g) = \Phi(x, \sigma g \sigma^{-1})$, $\Phi(x+y, g) = \Phi(x,g) + \Phi(y,g) + (x, gy)_2$ and $\langle x, gy\rangle = -(gy, x) = -(y, gx)$ when $g^2 = 1$, $(gx, gy) = (x,y)$. Such a situation with an integral group ring with the usual involution corresponds to orientable manifolds.

At the same time one meets a group ring in topology with an unusual involution given by the "orientation homomorphism"

$$f\colon \pi \to Z_2(\pm 1), \quad \bar{a} = \sum_{\sigma\in\pi} a_\sigma f(\sigma)\sigma^{-1} \quad \text{for} \quad a = \sum_{\sigma\in\pi} a_\sigma \sigma.$$

Moreover,

$$\langle x, y\rangle = \sum_{\sigma\in\pi}(x, \sigma y)f(\sigma)\sigma,$$

where $(x, y) = f(\sigma)(\sigma x, \sigma y)$ is the intersection number. Taking into account the new involution in $Z[\pi]$, we have $\langle x, y\rangle = \pm < \overline{y, x}\rangle$ and $\sigma\langle x, y\rangle = \langle\sigma x, y\rangle$.

Here, as is easily seen, 2-torsion can appear in both cases $A/(a \pm \bar{a})$—the symmetric and the skew-symmetric. For $\pi = Z_2$ and nontrivial $f\colon \pi \to (\pm 1)$ the Arf-invariants appear in the symmetric case $A/(a - \bar{a})$.

We note here the additions and changes entailed by the Arf-function in hamiltonian formalism and the constructions of this paper for the case of a group ring with the usual involution; the remaining cases will then be easy to analyze.

If we first construct the U^*-theory for $A = [\pi]$ with the usual involution in a skew-hermitian category, then when constructing $U_2^0(A)$ we ought to consider, in § 2, modules Q with a bilinear form $\langle x, y\rangle \in A$ and with an Arf-function $\Phi(x, g)$, $g^2 = 1$, $g \in \pi$ (or with a form $[\ ,\ ] \in A/(z + \bar{a})$). Here only those submodules $L \subset Q$ such that $\Phi/L = 0$ and $\langle L, L\rangle = 0$ will be called lagrangian. In a hamiltonian space $H = (X) + (P)$ it is necessary to require that $\Phi/P = 0$ and $\Phi/X = 0$, and, when constructing $U_2^1(A)$, to consider those $L \subset H$ such that $\Phi/L = 0$.

There is a simple lemma which explains the role of the parity of the "action hessian".

Theorem 12.1. *In a hamiltonian space $H = (X) + (P)$ such that $\Phi/X = 0$ and $\Phi/P = 0$ the lagrangian plane L has an even action hessian if and only if $\Phi/L = 0$.*

The proof is obtained in an obvious way from the additive identity for the Arf-function.

Remark 12.2. If there is distinguished an "impulse subspace" $P \subset H$ in a hamiltonian module H_n, then the requirement $\Phi/X = 0$ means that the extension to a basis $X, P \subset H_n$ can be carried out uniquely up to an equivalence; in fact X can be replaced by $X' = X + \lambda P$, where λ is an odd form. The Arf-function makes invariant the restriction of evenness for the action hessian on L and gives a unique rule for choosing X up to an equivalence provided that only $P \subset H_n$ is given. This point was not clarified in §§ 2 and 3, and is only eliminated when a 1/2 is introduced into the ring.

Thus a revised definition is that $U_2^0(A)$ is constructed from skew-hermitian forms with an Arf-function, and $U_2^1(A)$ is constructed from the "processes of reducing U^0 to zero", i.e. from lagrangian planes with even action hessian in the X, P-coordinates (or, what is the same thing, from lagrangian planes in hamiltonian modules with Arf-functions Φ for which $\Phi/X = \Phi/P = 0$). The construction of $U_2^2(A) = U_1^0(A)$ for $A = Z[\pi]$ $(\sigma^{-1} = \bar{\sigma})$ remains the same, and the Arf-function will be taken into account automatically in the construction of the Bass projections $B_U^0, \bar{B}_U^0, B_U^1, \bar{B}_U^1$.

However, when we obtain $U_2^0(A)$, like $U_1^2(A)$, in the form of the action hessian, the Arf-invariant does not arise naturally. This violates the 4-periodicity and causes difficulties, and it is only possible to leave all the results modulo $\otimes_Z Z[1/2]$.

Note that for $A = R(X) \ni 1/2$ there is also no 4-periodicity; our formalism really only gives a transition from the skew-symmetric category $U_2^0 = KO^6(X)$, $U_2^1 = KO^7(X)$ law the symmetric category $U_1^0 = U_2^3(A) = KO^8(X) = KO^0(X)$

(according to Bott). However, it is just this case which corresponds to the classical hamiltonian formalism. Conversely, for $A = R(X)$ the transition from the symmetric category where $U_1^0(A) = KO^0(X)$ and $U_1^1(A) = KO^1(X)$ into the skew-symmetric category where $U_2^0(A) = KO^6(X)$ is verified only after introducing a $1/2$, i.e. for $U^* \otimes Z[1/2]$. This shows that thinking of the "action hessian" in the hermitian category simply as a skew-hermitian form is not sufficient even for an $A = R(X)$ which contains a $1/2$ (probably it plays the role here of a complex structure). This question has already been posed in § 11.3, so that the difficulties connected with the number two are not entirely due to neglect of Arf-function.

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