ADAMS OPERATORS AND FIXED POINTS

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Abstract. The aim of this article is to calculate the Conner–Floyd invariants of the fixed points of the action of a cyclic group, by analogy with Adams operators. We shall correct the mistakes made in a previous article.

First of all, let me state that Appendices 3 and 4 in my article [2] contain fundamental errors. In particular, an internal contradiction in the formulas of Theorem 3 in Appendix 4 was pointed out to me by G. G. Kasparov, a student of the mechanico-mathematics faculty of Moscow State University. I am grateful to him.

The main aim of this article is to calculate definitively the Conner–Floyd functions $\alpha_{2n-1}(x_1, \dots, x_n)$ or fixed points by means of "Adams operators" in cobordism theory, introduced by the author in [2][1].

I. Correction of the errors in Appendix 3 in the article [2]

In Appendix 3 in the article [2], Lemma 1 and the construction of the cell complexes S_X in the general homology theory of X^* are incorrect.

The basic Theorem 1 of this appendix is correct, since it follows from the Adams spectral sequence in the cobordism theory (this was shown in a footnote on p. 945). Here, by definition we must have $U_*(X, A) = U_*(X), k_*(Z), K_*(X), H_*(X)$ for $A = \Omega_U$, $Z[x]$, $Z[x, x^{-1}]$, Z and $U^*(X, A) = U^*(X), k^*(X), K^*(X), H^*(X)$ for the same Ω-modules A. This theorem follows easily from the identities

$$U_*(MU) \otimes_\Omega A = U_*(MU, A)$$

and

$$\mathrm{Hom}_\Omega(U_*(MU), A) = U^*(MU, A),$$

which reduce the second term of the Adams spectral sequence to the form

$$\mathrm{Tor}_\Omega(U_*(X), A) \quad \text{or} \quad \mathrm{Ext}_\Omega(U_*(X), A).$$

Theorem 2 is correct if we replace k_* by $k_*(Z_p)$, since here the construction of the cell complex is correct.

In the remaining cases the construction of the cell complexes is also correct, for example $S_k(BZ_p)$ and $S_U(BZ_p)$. Therefore all the applications of the results in

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[1]Noted added in proof. Theorem 1 (see section III) was simultaneously and independently obtained by A. S. Miščenko and G. G. Kasparov (in press).

Appendix 3 to the classical lenses in Appendix 4 are correct (but other mistakes occur there).

II. Correction of the errors in Appendix 4 in the article [2]

In Appendix 4 two problems were solved:

1. A complete calculation of the bordisms and cobordisms of the space BZ_p by means of Adams operators in U-theory, and also a description of the structure of the automorphism $\lambda_x\colon BZ_p \to BZ_p$, which is generated by multiplication by an invertible residue $x \in Z_p$ (Theorems 1 and 2). Here there are no errors.

2. Calculation of the symmetric functions $\alpha_{2n-1}(x,\dots,x_n) \in U_{2n-1}(BZ_p)$ on a set of invertible residues $x_1,\dots,x_n$, defined by the action of the group Z_p on the sphere S^{2n-1}, linear in the complex space C^n and given by a diagonal matrix (a_{ij}), where $a_{jj} = e^{2\pi i x_j/p}$.

We formulated three obvious requirements that these functions must satisfy:

a) $\nu a_{2n-1}(x_1,\dots,x_n) = x_1^{-1} \circ \dots \circ x_n^{-1} a_{2n-1}$, where $a_{2n-1} \in H_{2n-1}(BZ_p)$ is a standard basis element and $\nu\colon U_* \to H_*$ is a homomorphism of the augmentation ("normalization condition").

b) $(\lambda_x)_* \alpha_{2n-1}(1,\dots,1) = \alpha_{2n-1}(x^{-1},\dots,x^{-1})$, where λ_x was described above and $(\lambda_x)_*\colon U_{2n-1} \to U_{2n-1}$ ("values on the diagonal").

c) If two relations of the following form hold:

$$\sum_{i,t} \gamma_t^{(i)} \alpha_{2(k-i)-1}(x_{1,t}^{(i)},\dots,x_{k-i,t}^{(i)}) = 0,$$
$$\sum_{j,s} \delta_s^{(j)} \alpha_{2(l-j)-1}(y_{1,s}^{(j)},\dots,y_{l-j,s}^{(j)}) = 0$$
$$(\gamma_t^{(i)} \in \Omega_U^{2i}, \quad \delta_s^{(i)} \in \Omega_U^{2j}),$$

then their "direct product" is also a relation of the form

$$\sum_{i,j,s,t} \gamma_t^{(i)} \delta_s^{(j)} \alpha_{2(k+l-i-j)-1}(x_{1,t}^{(i)},\dots,x_{k-i,t}^{(i)}, y_{1,s}^{(j)},\dots,y_{l-j,s}^{(j)}) = 0$$

("multiplicative condition").

We also showed that from these three conditions we can recursively calculate the values of the functions $\alpha_{2n-1}(x_1,\dots,x_n)$ in terms of $\alpha_{2n-1}(1,\dots,1)$, and we gave the result of the calculation for $n = 2, 3$ (Theorem 3). This theorem is not true. The error lies in the fact that on page 946 (bottom) the elements $\alpha_{2n-1}(1,\dots,1)$ are incorrectly identified with the elements v_{2n-1}, which are canonically conjugate to the basis (vu^k) in the complex $S_U(BZ_p)$. In fact $(v_{2n-1}, vu^{n-1}) = 1$ and $(v_{2n-1}, vu^k) = 0$ for $k \neq n-1$. In fact, we have the formulae

$$(\alpha_{2n-1}(1,\dots,1), vu^{n-1-k}) = [CP^k] \in \Omega_U,$$
$$\alpha_{2n-1}(1,\dots,1) = v_{2n-1} + [CP^1]v_{2n-3} + \dots + [CP^{n-1}]v_1. \qquad \text{(I)}$$

Also, on page 946 we wrote x instead of x^{-1} throughout, and in Lemma 3 we must write

$$(\lambda_x)_* v_5 = x v_5 + \frac{x^2 - x^3}{2}[CP^1]v_3,$$

changing the sign of the second term. A correct recalculation leads to the following formulae:

a) $\alpha_1(x^{-1}) = x\alpha_1(1)$;

b) $\alpha_3(x^{-1}, y^{-1}) = \sigma_2\alpha_3(1,1) + (\sigma_1 - 2\sigma_2)\frac{[CP^1]}{2}\alpha_1(1)$;

c) $\alpha_5(x^{-1}, y^{-1}, z^{-1}) = \sigma_3\alpha_5(1,1,1) + (\sigma_2 - 3\sigma_3)\frac{[CP^1]}{2}\alpha_3(1,1)$

$$+ (\sigma_1 - 2\sigma_2 + 3\sigma_3)\frac{[CP^1]^2}{4}\alpha_1(1) + \frac{\sigma_2^2 - 2\sigma_1\sigma_3 - 3\sigma_3^3}{\sigma_3}\left(\frac{[CP^2]}{3} - \frac{[CP^1]^2}{4}\right)\alpha_1(1), \tag{II}$$

where σ_1 are the elementary symmetric functions. In general, we have the following simple

Lemma 1. *The normalization, multiplicative, and value-on-the-diagonal conditions formulated above completely determine the functions $\alpha_{2n-1}(x_1, \dots, x_n)$ for all $p \geq n$.*

For the proof of the lemma we first consider primes $p > n$. By the normalization conditions we have

$$\alpha_{2n-1}(x_1^{-1}, \dots, x_n^{-1}) - \sigma_n\alpha_{2n-1}(1, \dots, 1) = \sum_{i>0}\gamma_i^{(n)}(x_1^{-1}, \dots, x_n^{-1})\alpha_{2(n-i)-1}(1, \dots, 1).$$

If we know the functions α_{2k-1} for all $k < n$, then, by virtue of the multiplicative condition, we can use the relations written above to form various relations for α_{2n-1} for all possible $x_1, \dots, x_n$. These are linear equations that are non-homogeneous, since we know $\alpha_{2n-2}(x^{-1}, \dots, x^{-1})$. It is easy to see that this system of equations reduces to triangular form. First we find $\alpha_{2n-1}(x^{-1}, 1, \dots, 1)$, and so forth. This implies Lemma 1. For the remaining p the proof is similar, but the linear equations will not be over a field.

Remark. It is possible in principle to find various relations on $\alpha_{2n-1}(x_1, \dots, x_n)$, by constructing a large number of examples of actions of Z_p on complex manifolds, and then to calculate α_{2n-1} recursively. Such a method of calculation was realized by G. G. Kasparov for $n = 2$, 3, and he obtained the formula (II) in a different manner from that used here. His method requires the essential use of certain constructions and theorems of Conner and Floyd. Kasparov first observed (from geometrical considerations) that the coefficients $\gamma_i^{(n)}$ in $\alpha_{2n-1}(x^{-1}, 1, \dots, 1) = x\alpha_{2n-1}(1, 1, \dots, 1) + \sum_{i>0}\gamma_i^{(n)} \cdot \alpha_{2(n-i)-1}(1, \dots, 1)$ depend only on i for $i < n$. This fact will be proved by algebraic methods and will be used later.

III. The complete calculation of the functions $\alpha_{2n-1}(x_1,\dots,x_n)$

We shall seek the functions $\alpha_{2n-1}(x_1,\dots,x_n)\in U_{2n-1}(BZ_p)$ in the form

$$\alpha_{2n-1}(x_1,\dots,x_n)=\frac{1}{\sigma_n}\alpha_{2n-1}(1,\dots,1)+\sum_{i>0}\gamma_i^{(n)}(x_1,\dots,x_n)\alpha_{2(n-i)-1}(1,\dots,1),$$

where $\gamma_i^{(n)}(x_1,\dots,x_n)\in\Omega_U$, assuming by definition that $\alpha_{2(n-i)-1}=0$ for $i\geq n$.

We denote by $u\in U^2(BZ_p)$ the canonical two-dimensional cobordism class. Clearly we have

$$u^k\cap\alpha_{2n-1}(1,\dots,1)=\alpha_{2(n-k)-1}(1,\dots,1),$$

where $\cap$ is the Čech "excision." We denote by $L_{2n-1}\subset BZ_p$ the $(2n-1)$-dimensional skeleton formed by the classical lens with weights $1,\dots,1$, that represents the class $\alpha_{2n-1}(1,\dots,1)$. The Poincaré–Atiyah duality operator for L_{2n-1} is denoted by $D_n\colon U_*(L_{2n-1})\to U^*(L_{2n-1})$. We observe that $U_k(L_{2n-1})=U_k(BZ_p)$ for $k<2n-1$. Therefore we can assume that $\alpha_{2(n-k)-1}(x_1,\dots,x_n)\in U_*(L_{2n-1})$ for $k>0$. We have the formulae

$$D_{n+k}\alpha_{2n-1}(1,\dots,1)=u^k,$$
$$D_{n+k}\alpha_{2n-1}(x_1,\dots,x_n)=u^kB_n(u;x_1,\dots,x_n), \tag{III}$$

where $B_n(u;x_1,\dots,x_n)$ does not depend on k. Here $u^{n+k}=0$. By definition

$$B_n(u;x_1,\dots,x_n)=\sigma_n^{-1}+\sum_{i\geq1}\gamma_i^{(n)}u^i=\frac{1}{u^k}D_{n+k}\alpha_{2n-1}(x_1,\dots,x_n).$$

Since for $B_n(u;x_1,\dots,x_n)$ only the first n terms in the power series in u make sense, we can consider $B_n(u;x_1,\dots,x_n)$ to be an infinite series and at the end take $u^n=0$.

Our problem is to find a sequence of series

$$B_1(u;x)=\frac{1}{x}+\dots,\quad B_2(u;x,y)=\frac{1}{xy}+\dots,$$
$$B_3(u;x,y,z)=\frac{1}{xyz}+\dots$$

and so on, where in calculating $\alpha_{2n-1}(x_1,\dots,x_n)$ we consider only the first n coefficients. Namely, if

$$B_n(u;x_1,\dots,x_n)=\frac{1}{x_1\circ\dots\circ x_n}+\sum_{i\geq1}\gamma_i^{(n)}u^i,$$

then

$$\alpha_{2n-1}(x_1,\dots,x_n)=\frac{1}{\sigma_n}\alpha_{2n-1}(1,\dots,1)+\sum_{i=1}^{n-1}\gamma_i^{(n)}\alpha_{2(n-i)-1}(1,\dots,1),$$

since $\alpha_{2n-2i-1}(1,\dots,1)=0$ for $i\geq n$.

The following important result holds.

Lemma 2. *The sequence of series $B_1, B_2, \dots$ is multiplicative in the sense that*

$$B_n(u; x_1, \dots, x_n) = \prod_{i=1}^{n} B(u; x_i),$$

where $B(u; x) = B_n(u; x, 1, \dots, 1)$.

Proof. We seek functions $\alpha_{2n-1}(x_1, \dots, x_n)$ as if Lemma 2 is true. Having found a solution (see Theorem 1 below) we see whether these functions satisfy the normalization and multiplicative conditions (the diagonal values will be used in determining the solution, so they will automatically be consistent). From the equations (VIII), which follow from Theorem 1 (see below), it is clear that all of the requirements are satisfied. Therefore Lemma 2 follows automatically by virtue of Lemma 1. □

Thus the problem reduces to finding the series

$$B(u, x) = \lim_{n \to \infty} B_n(u; x, 1, \dots, 1).$$

For this purpose we use the automorphism $\lambda_x \colon BZ_p \to BZ_p$, generated by multiplication by $x \in Z_p$. Recall that in cobordism theory the Adams operator $\Psi^x = \Psi^x_U$ is given by the series

$$\Psi^x(u) = x^{-1} g^{-1}(x g(u)),$$

where

$$g(u) = \sum_0^\infty \frac{[CP^n]}{n+1} u^{n+1}$$

(the "Miščenko series") and g^{-1} is the inverse series. By definition we have

$$\lambda_x^*(u) = x \Psi^x(u) = \sum_{i \geq 0} \phi_i(x) u^{i+1},$$

where $\phi_i(x) \in \Omega_U$.

We now use the general identity

$$f_*(f^*(a) \cap b) = a \cap f_*(b) \tag{IV}$$

for arbitrary transformations $f, a \in U^*$, and $b \in U_*$. In this identity we substitute $f = \lambda_x$, $a = u$, $b = \alpha_{2n+1}(1, \dots, 1)$ and observe that $f_* \alpha_{2k-1}(1, \dots, 1) = \alpha_{2k-1}(x^{-1}, \dots, x^{-1})$ for all k. Then from (IV) we obtain the equation

$$\begin{aligned}(\lambda_x)_*(x\Psi^x(u) \cap \alpha_{2n+1}(1, \dots, 1)) &= \sum_{i \geq 0} \phi_i(x) \alpha_{2n-2i-1}(x^{-1}, \dots, x^{-1}) \\ = u \cap \alpha_{2n+1}(x^{-1}, \dots, x^{-1}) &= x^{n+1} \alpha_{2n-1}(1, \dots, 1) \\ &+ \sum_i^\infty \gamma_i^{(n+1)}(x^{-1}, \dots, x^{-1}) \alpha_{2n-2i-1}(1, \dots, 1).\end{aligned} \tag{V}$$

When we apply the operator D_n to this equation, we find (bearing in mind that $u^n = 0$ and $n \to \infty$):

$$\sum_{i\geq 0}^{n} \phi_i(x) B^{n-i}(u; x^{-1}) u^i = B^{n+1}(u; x^{-1}), \tag{V$'$}$$

where

$$B^{n-i}(u; x^{-1}) u^i = D_n \alpha_{2n-2i-1}(x^{-1}, \dots, x^{-1}).$$

We divide the equation (V′) by $B^{n+1}(u, x^{-1})/u$. Then we obtain

$$x\Psi^x\left(\frac{u}{B(u, x^{-1})}\right) = u, \tag{VI}$$

where

$$x\Psi^x(v) = g^{-1}(xg(v)) = \sum_{i\geq 0} \phi_i(x) v^{i+1}$$

and

$$g(v) = \sum_{n\geq 0} \frac{[CP^n]}{n+1} v^{n+1}.$$

Therefore

$$u = x^{-1}\Psi^{x^{-1}}(u) B(u, x^{-1})$$

and the series $B(u; x)$ is of the form

$$B(u; x) = \frac{u}{x\Psi^x(u)}; \quad B_n(u; x_1, \dots, x_n) = \prod_{i=1}^{n} \frac{u}{x_i \Psi^{x_i}(u)}. \tag{VII}$$

Since $\alpha_{2n-1}(x_1, \dots, x_n) = B_n(u; x_1, \dots, x_n) \cap \alpha_{2n-1}(1, \dots, 1)$, (VII) implies the following definitive result.

Theorem 1. *We have the formula*

$$\alpha_{2n-1}(x_1, \dots, x_n) = \prod_{i=1} \frac{u}{x_i \Psi^{x_i}(u)} \cap \alpha_{2n-1}(1, \dots, 1),$$

where

$$x\Psi^x(u) = g^{-1}(xg(u)), \quad g = \sum_{n\geq 0} \frac{[CP^n]}{n+1} u^{n+1}$$

(observe that by definition $au^k \cap \alpha_{2n+1}(1, \dots, 1) = a\alpha_{2n-2k+1}(1, \dots, 1)$, where $a \in \Omega_U$ is an arbitrary "scalar").

If the group Z_p acts in a complex manner on the manifold M^n with fixed points $\mathcal{P}_1, \dots, \mathcal{P}_q$, at which it has a collection of weights $x_1^{(j)}, \dots, x_n^{(j)} \in Z_p$, $j = 1, \dots, q$, then we have the equations[2]

$$\sum_{j=1}^{q} u \prod_{i=1}^{n} \frac{u}{x_i^{(j)} \Psi^{x_i^{(j)}}(u)} = 0, \tag{VIII}$$

[2]These equations will be referred to as the Conner–Floyd equations.

where u is a formal variable that generates the ring $\Omega_U[u]$ with relations $p\Psi^p(u) = 0$ and $u^{n+1} = 0$. We take p to be a prime.

III′. Implications of the Conner–Floyd equations

For transformations with isolated fixed points (q of them) the equations (VIII) are actually $S(n)$ numerical equations mod p, at least for large primes $p > n$, involving the qn variables $x_i^{(j)} \in Z_p$, where $x_i^{(j)} \neq 0 \bmod p$; here $S(n)$ is the number of symmetric polynomials in n variables of degree $\le n$ (or $S(n) = \sum_{k=0}^{n} \operatorname{rk} \Omega_U^{2k}$). We observe that the collections of weights $\{x_i^{(j)}\}$ of the action of Z_p are geometrically indistinguishable if we perform an arbitrary permutation of the indices i and the indices j and multiply the whole collection $\{x_i^{(j)}\}$ by a number $\lambda \not\equiv 0 \bmod p$. In fact the equations (VIII) are defined on projective space P^{qn-1} over Z_p with the hypersurfaces $P_{ij}[x_i^{(j)} = 0]$ eliminated. Therefore the "manifold" of types of action of Z_p lies in $P^{qn-1} \setminus \bigcup_{ij} P_{ij}$, factored by the product of the permutation groups $S_n \times S_q$ (see [2]).

From (II) we can already draw certain conclusions:

1. For $n = 2$ and $p = 3$ the number q of necessary fixed points of the action of Z_3 cannot be two. For $n = 2$ and $p \geq 5$ we can have an action of Z_3 with two fixed points.

2. For $n = 3$, unlike the case of $n = 2$, we can have an action of Z_p with two fixed points for $p \geq 3$.

3. For $n = 3$ and $q = 2$ (two points), of the four equations (II) (since $S(3) = 4$) is a consequence of the other two, since the coefficient of $\left(\frac{[CP^2]}{3} - \frac{[CP^1]^2}{4}\right)\alpha_1(1)$ is a homogeneous function of $\sigma_1, \sigma_2, \sigma_3$ of degree 1. Therefore it changes sign when the σ_i change sign. For the two points $\mathcal{P}^1$ and $\mathcal{P}^2$ the Conner–Floyd equations (II) and (VIII) require a change of sign. This fact is an accidental coincidence.

4. Since for arbitrary q the number of equations $S(n)$ grows much faster than qn (namely, $S(n) \approx \frac{1}{2\pi\sqrt{2n}} e^{\pi\sqrt{\frac{2n}{3}}}(1 + O(n^{-\frac{1}{4}}))$), for each number q there exists an $n = n(q)$ such that for $N > n(q)$ we know that every complex action of Z_p on N-dimensional complex manifolds has more than q fixed points.

However, a rigorous proof requires a demonstration of the independence of approximately $S(n)$ of the equations (VIII), at least for large p. In view of possible coincidences, as in the preceding subsection, this requires an additional algebraic analysis of the equations.

5. We consider the real case corresponding to a change from U_*-bordism to SO_*-bordism. Here we restrict our attention to groups of odd order. The corresponding functions will be denoted by

$$\alpha_{2n-1}^R(x_1, \dots, x_n) \in SO_{2n-1}(BZ_p),$$

where $\alpha_{2n-1}^R = r_*\alpha_{2n-1}$, $r_*\colon U_* \to SO_*$ is the realizing homomorphism.

Clearly the functions possess the following symmetry:

$$\alpha_{2n-1}^R(-x_1, \dots, -x_n) = -\alpha_{2n-1}^R(x_1, \dots, x_n),$$

corresponding to rotations of the sphere S^{2n}.

The Conner–Floyd equations (VIII_R) are obtained from (VIII) by means of the realizing operator r_*. These are actually $S(k)$ numerical equations for dimensions $n = 4k + 2$ and $n = 4k + 4$, since

$$S(k) = \sum_{i=0}^{n-1} \operatorname{rk} \Omega^i_{SO} \otimes Z_p.$$

If there is no general interdependence between these equations apart from the symmetry indicated above, then, as in subsection 5, this implies the following result:

a) If the number of points q is odd, then there exists a dimension $n(q)$ such that on manifolds of dimension $N > n(q)$ there is no action of Z_p with q isolated fixed points.

b) If $q = 2t$, then for dimensions $N > n(q)$ every action of Z_p is such that the points $\mathcal{P}_1, \dots, \mathcal{P}_q$ split into pairs $\mathcal{P}_1, \bar{\mathcal{P}}_1, \dots, \mathcal{P}_t, \bar{\mathcal{P}}_t$ for which the collections of weights of the points $\mathcal{P}_j$ and $\bar{\mathcal{P}}_j$ are opposite in sign.

Here p is odd and the manifolds are even-dimensional and orientable. The dimension $n(q)$ can be calculated if we know the number of independent equations in (VIII_R). If this number increases faster than linearly, then the result is true. If it increases more slowly than pn, then $n(q) = \infty$.

Apparently this number increases as $S(n/4)$, but I have not been able to prove it;

$$S\left(\frac{n}{4}\right) \sim \frac{1}{4\pi\sqrt{2n}} e^{\pi/2\sqrt{\frac{2n}{3}}}.$$

IV. Numerical realizations of the Conner–Floyd equations

How can we find these equations most effectively? Here

$$\Psi^x = x^{-1} g^{-1}(x g(u)),$$

where

$$g = \sum \frac{[CP^n]}{n+1} u^{n+1}.$$

Since CP^n is algebraically independent in Ω_U, then g is practically a series with arbitrary coefficients. But the series $\prod_{i=1} \frac{u}{x_i \Psi^{x_i}(u)}$ is of the form $\sum_{i \ge 0} \gamma_i^{(n)} u^i$, and sometimes it is possible to calculate effectively some of the Chern numbers of the coefficients $\gamma_i^{(n)}$, which gives various numerical realizations of the equations (VIII). We introduce the following important examples of such realizations:

a) ***T*-genus** (corresponding to the homomorphism $U^* \to K^*$). Since $T[CP^n] = 1$ and $T(M \times N) = T(M)T(N)$, $g(u)$ goes into

$$g_T(u) = \sum \frac{u^{n+1}}{n+1} = -\ln(1-u).$$

We have

$$g_T^{-1}(u) = 1 - e^{-u}$$

and

$$T(x\Psi^x(u)) = 1 - (1-u)^x.$$

Finally we obtain the equation

$$\sum_j u \prod_i \frac{u}{1-(1-u)^{x_i^{(j)}}} = 0, \tag{VIII$_T$}$$

where

$$u^{n+1} = 0, \quad u^k(1-(1-u)^p) = 0, \quad k \geq 0.$$

b) L**-genus** (signature). Since $L[CP^{2n}] = 1$, $L[CP^{2n+1}] = 0$, g goes into

$$g_L = \sum \frac{u^{2n+1}}{2n+1} = \frac{1}{2} \ln \left(\frac{1+u}{1-u} \right)$$

and

$$g_L^{-1} = \operatorname{th} u.$$

Therefore

$$g_L^{-1}(xg_L(u)) = \operatorname{th} \left(\frac{x}{2} \ln \left(\frac{1+u}{1-u} \right) \right) = \frac{(1+u)^x - (1-u)^x}{(1+u)^x + (1-u)^x}.$$

We have the equation

$$\sum_j u \prod_i u \frac{(1+u)^{x_i^{(j)}} + (1-u)^{x_i^{(j)}}}{(1+u)^{x_i^{(j)}} - (1-u)^{x_i^{(j)}}}, \tag{VIII$_L$}$$

where

$$u^{n+1} = 0, \quad u^k \frac{(1+u)^p - (1-u)^p}{(1+u)^p + (1-u)^p} = 0, \quad k \geq 0.$$

c) **The Euler characteristic** c (of the tangent bundle). Since $c_n[CP^n] = n+1$, it follows that

$$g_c(u) = \frac{u}{1-u} \quad \text{and} \quad g_c^{-1} = \frac{u}{u+1}.$$

Therefore

$$g_c^{-1}(xg_c(u)) = \frac{ux}{1-u+ux}.$$

We have the equation

$$\sum_j u \prod_i \frac{1-u+ux_i^{(j)}}{x_i^{(j)}} = 0, \tag{VIII$_c$}$$

where

$$u^{n+1} = 0, \quad u^k \frac{pu}{1-u+pu} = 0, \quad k \geq 0.$$

d) **The** t**-characteristic, which selects polynomial generators in** Ω_U**,** $t_k = c_\omega$**,** $\omega = (k)$**.** We observe that $t(M^i \times N^j) = 0$ for $i > 0$ and $j > 0$. Since $t[CP^n] = n+1$, it follows that

$$g_t(u) = \frac{u}{1-u}, \quad (g^{-1})_t = v - \frac{v^2}{1-v}.$$

$$[g^{-1}(xg(u))]_t = xu \left(\frac{1}{1-u} - \frac{xu}{1-xu} \right)$$

and

$$\left[\frac{u}{g^{-1}(xg(u))}\right]_t = \frac{1}{x}\left(1 - \frac{u}{1-u} + \frac{ux}{1-ux}\right),$$

$$\left[\prod_{i=1}^{n} \frac{u}{x_i \Psi^{x_i}(u)}\right]_t = \frac{\sum_{i=1}^{n}\left(1 - \frac{u}{1-u} + \frac{ux_i}{1-ux_i}\right) - (n-1)}{x_1 \circ \cdots \circ x_n}.$$

Finally we have the equation

$$\sum_j u \frac{\sum_i \left(1 - \frac{u}{1-u} + \frac{ux_i^{(j)}}{1-ux_i^{(j)}}\right) - (n-1)}{x_1^{(j)} \circ \cdots \circ x_n^{(j)}}, \qquad (\mathrm{VIII}_t)$$

where

$$u^{n+1} = 0, \quad u^k[p\Psi^p(u)]_t = 0, \quad k \geq 0.$$

IV′. Q-genera of general form

We obtained four types of numerical equations. It is possible to deduce a general numerical equation corresponding to a multiplicative Hirzebruch sequence of Chern classes $Q(z) = 1 + a_1 z + a_2 z^2 + \dots$. Of special interest are the A-genera

$$A = Q(z) = \frac{z/2}{\operatorname{sh} z/2} \quad \text{and} \quad \bar{A} = \frac{2z}{\operatorname{sh} 2z}.$$

where $\bar{A} = e^{c_1/2}T$. We can only use an A-genus for odd p in view of the lack of spinors. For an arbitrary multiplicative sequence $W(z)$ we have a Q-genus $Q[M^n]$, and we must calculate

$$g_Q = \sum_{n \geq 0} \frac{Q[CP^n]}{n+1} u^{n+1},\ g_Q^{-1},\ g_Q^{-1}(xg_Q) \text{ and } u/g_Q^{-1}(xg_Q).$$

Since $Q[CP^n]$ is the component of z^n in the series $Q^{n+1}(z)$, it follows that

$$Q[CP^n] = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{Q^{n+1}(z)}{z^{n+1}} dz.$$

Hence we obtain

$$\frac{dg_Q}{du} = \sum_{n \geq 0} Q[CP^n] u^n = \frac{1}{2\pi i u}\sum_{n \geq 0}\int_{|z|=\varepsilon}\left(\frac{Q(z)u}{z}\right)^{n+1} dz$$

$$= \frac{1}{2\pi i}\int \frac{Q(z)/z}{1 - \frac{uQ(z)}{z}}, dz = \frac{1}{2\pi i}\int \frac{dz}{z/Q(z) - u}, \quad \left|\frac{uQ}{z}\right| < 1.$$

Therefore

$$g_Q(u) = \frac{1}{2\pi i}\int_0^u \int_{|z|=\varepsilon} \frac{dz\, du}{z/Q(z) - u}$$

for small u. Integrating with respect to u, we find that

$$g_Q(u) = \frac{1}{2\pi i}\int_{\substack{|z|=\varepsilon,\\ |u|<\left|\frac{z}{Q(z)}\right|}} -\ln\left(1 - \frac{uQ(z)}{z}\right) dz = \left(\frac{z}{Q(z)}\right)^{-1}(u).$$

Let $\phi(z) = \frac{z}{Q(z)}$ and let $\phi^{-1}(v)$ be the inverse function. From the form of $g_Q(u)$ we have that

$$g_Q(u) = \phi^{-1}(u), \quad g_Q^{-1}(v) = \frac{v}{Q(v)}.$$

Therefore

$$g_Q^{-1}(xg_Q(u)) = \frac{xg_Q}{Q(xg_Q)}.$$

Consequently, the general Conner–Floyd equation for a Q-genus takes the form

$$\begin{gathered}\sum_j u \prod_i u \frac{Q(x_i^{(j)} g_Q(u))}{x_i^{(j)} g_Q(u)} = 0, \\ u^{n+1} = 0, \quad u^k \frac{pg_Q(u)}{Q(pg_Q(u))} = 0, \quad k \geq 0,\end{gathered} \tag{VIII$_Q$}$$

where

$$g_Q(u) = \sum_{n \geq 0} \frac{Q[CP^n]}{n+1} u^{n+1}, \quad \frac{g_Q(u)}{Q(g_Q(u))} = u.$$

We consider the case $Q = \bar{c} = \frac{1}{1+z}$, $Q = A$ and $Q = \bar{A}$. The functions are everywhere to be understood as formal series in u, where

$$(1+y)^\alpha = 1 + \alpha y + \frac{\alpha(\alpha-1)}{2} y^2 + \cdots .$$

e) **The normal Euler characteristic $\bar{c}$.** Since $Q = \frac{1}{1+z}$, it follows that

$$g_Q(u) = \frac{1}{2}(\sqrt{1+4u} - 1).$$

Therefore

$$g_Q^{-1}(xg_Q) = \frac{x}{4}(\sqrt{1+4u} - 1)(2 + x(\sqrt{1+4u} - 1)).$$

We have the equation

$$\sum_j u \prod_i 4u \frac{(2 + x_i^{(j)}(\sqrt{1+4u} - 1))^{-1}}{x_i^{(j)}(\sqrt{1+4u} - 1)} = 0. \tag{VIII$_{\bar{c}}$}$$

where

$$u^{n+1} = 0, \quad u^k g_Q^{-1}(pg_Q(u)) = 0, \quad k \geq 0.$$

f) ***A*-genera:**

$$Q = A = \frac{z/2}{\operatorname{sh} z/2}, \quad \bar{Q} = \bar{A} = \frac{2z}{\operatorname{sh} 2z}.$$

For an A-genus we obtain

$$g_A(u) = 2 \ln \left(\frac{u}{2} + \sqrt{1 + \frac{u^2}{4}} \right),$$

$$g_A^{-1}(u) = \frac{u}{Q(u)} = 2 \operatorname{sh} \frac{u}{2}.$$

Furthermore

$$g_A^{-1}(xg_A) = 2\,\mathrm{sh}\left(x\ln\left(\frac{u}{2}+\sqrt{1+\frac{u^2}{4}}\right)\right)$$
$$= \left(\frac{u}{2}+\sqrt{1+\frac{u^2}{4}}\right)^x - \left(\frac{u}{2}+\sqrt{1+\frac{u^2}{4}}\right)^{-x}.$$

Finally we have the following equation for odd p:

$$\sum_j u\prod_i \frac{u}{\left(\frac{u}{2}+\sqrt{1+\frac{u^2}{4}}\right)^x - \left(\frac{u}{2}+\sqrt{1+\frac{u^2}{4}}\right)^{-x}} = 0.$$
$$u^{n+1}=0, \quad u^k g_A^{-1}(pg_A(u))=0, \quad k\geq 0. \tag{VIII$_A$}$$

By analogy we write down the equations (VIII$_{\bar c}$), (VIII$_T$), (VIII$_L$), (VIII$_c$), (VIII$_t$), (VIII$_A$), (VIII$_{\bar A}$) for the case in which we have an action of the group Z_p for which all fixed manifolds have a trivial normal bundle.

V. Global invariants of the manifold carrying the action of Z_p

Here we consider the problem of calculating the characteristic numbers and integral Q-genera modulo p of the manifold M^n that carries the action of Z_p. For simplicity we limit our attention, as before, to the case of isolated fixed points $\mathcal{P}_1,\dots,\mathcal{P}_q$ of the whole group as the only singularities. Let the weights (for the point $\mathcal{P}_j$) be $x_1^{(j)},\dots,x_n^{(j)}$. This problem was solved for a number of cases by Conner and Floyd. However, due to its simplicity, we shall indicate its solution here. In fact, the answer follows easily from an old work of Tamura [1] in which he constructed classes (Pontrjagin, Chern) of p-manifolds. In addition, I may remark that if, in the case considered by Tamura, the action without fixed points can be extended from the boundary ∂M onto M, then all the Tamura characteristic numbers are equal to zero mod p.

We consider the Tamura p-manifold $M^n(x_1,\dots,x_n) = D^{2n}$, $Z_p(\partial D^{2n})$, where Z_p acts linearly on S^{2n-1} with weights $x_1,\dots,x_n$. Tamura calculated his classes for $M^2(x_1,x_2)$, writing down an awkward answer. The general answer for the Chern–Tamura number $c_\omega[M^n(x_1,\dots,x_n)]$ is of the form

$$c_\omega = \frac{v_\omega(x_1,\dots,x_n)}{x_1\circ\cdots\circ x_n},$$

where $\dim\omega = n$ and $v_\omega = \sum x_{i_1}^{a_1}\circ\cdots\circ x_{i_n}^{a_n}$ is a symmetrized ω-monomial. Hence we obtain in a simple way a formula for the action of Z_p on M^n:

$$\sum_{j=1}^{q}\frac{v_\omega(x_1^{(j)},\dots,x_n^{(j)})}{x_1^{(j)}\circ\cdots\circ x_n^{(j)}} = c_\omega[M^n] \pmod p, \tag{IX}$$

For large primes $p > n+1$, this formula also correctly describes T-genera, L-genera and A-genera. For primes $p \leq n+1$, the formula is incorrect for these

Q-genera. For example the formula

$$T = \sum_j \left[\prod_{i=1}^n \frac{x_i^{(j)}}{1 - e^{-x_i^{(j)}}} \right]_n \cdot \frac{1}{\sigma_n^{(j)}}$$

contains p in the denominator for $p \leq n+1$. Thus it must be interpreted as follows: if the $\bar{x}_i^{(j)}$ are integers such that $\bar{x}_i^{(j)} \pmod p = x_i^{(j)}$, then we have the formula

$$\left(\sum_j \frac{1}{\sigma_n^{(j)}} \left[\prod_{i=1}^n \frac{\bar{x}_i^{(j)}}{1 - e^{-\bar{x}_i^{(j)}}} \right]_n \right)_{(\mathrm{mod}\ p)} = T[M^n] \pmod p, \qquad (\mathrm{IX}_T)$$

where the divisibility by p of the expression in parentheses is a consequence of equations (VIII) for the weights $x_i^{(j)}$ of the fixed points. In the holomorphic and real cases, the formulae (IX), (IX_T), etc., can apparently be derived from the results of Atiyah and Bott [3] and some methods from number theory.

For the L-genus, the A-genus and any other Q-genus the proof is similar.

VI. The action of a circle with fixed points

We shall consider complex actions of the circle S^1 on a manifold M^n, having only isolated fixed points for the whole group (for subgroups there may also be other singularities). At each of these fixed points $\mathcal{P}$ the action of S^1 on the tangent space $T_{\mathcal{P}}$ is given by a diagonal matrix $A_{\mathcal{P}}(\phi)$, where the $a_{ij}(\phi) = e^{ix_j\phi}$, $\phi \in S^1$, x_j are integers.

We let π denote the collection of all primes that divide the collection $\{x_j\}$, and we let $\hat{Z}_\pi$ denote the completion of the integers in the topology in which the open sets are the ideals generated by all the numbers that are relatively prime to the collection π. Let $G_\pi = \operatorname{Char} \hat{Z}_\pi$ be the group of characters.

It is easy to see that the representation $A_{\mathcal{P}}(\phi)$ determines a quasi-complex action without fixed points of the group G_π on the sphere S^{2n-1}, enclosing the singular point $\mathcal{P}$, with weights x_j, where the action of a character $h \in G_\pi = \operatorname{Char} Z_\pi$ is given by the action of the element $h(1) \in S^1$. This action is determined by the collection of integers $\{x_j\}$, which may depend on the collection of primes π.

By definition, the group of complex bordisms BG for the group G is the collection of pairs (M, G) of quasi-complex manifolds with the corresponding action of G without fixed points, identifying pairs with respect to cobordism. We call $\dim U - \dim G$ the dimension of a pair.

Suppose that the action of S^1 on the manifold M^n has fixed points $\mathcal{P}_1, \ldots, \mathcal{P}_q$ with weights $x_i^{(j)}$, $i = 1, \ldots, n$, $j = 1, \ldots, q$. Then we take π to be a collection from all the primes that divide the set $x_i^{(j)}$ and define $\hat{Z}_\pi$ analogously. We set $G_\pi = \operatorname{Char} Z_\pi$.

The linear representation $A_{\mathcal{P}_j}(\phi)$ at the point $\mathcal{P}_j$ on the sphere S^{2n-1} determines an element $\alpha(x_1^{(j)}, \ldots, x_n^{(j)}) \in U_*(BG_\pi)$.

Clearly we have the ("Conner–Floyd") relation

$$\sum_{\mathcal{P}_j} \alpha(x_1^{(j)}, \dots, x_n^{(j)}) = 0,$$

where $G_\pi = \operatorname{Char} \hat{Z}_\pi$.

We have the following simple

Lemma 3. *For the natural homomorphism $G_\pi \to S^1$ the induced mapping*

$$\rho\colon U_*(BG_\pi) \otimes \hat{Z}_\pi \to U_*(CP^\infty) \otimes \hat{Z}_\pi$$

is an isomorphism.

Later we shall consider the functions

$$\rho\alpha(x_1^{(j)}, \dots, x_n^{(j)}) \in U_*(CP^\infty) \otimes \hat{Z}_\pi$$

and denote them by $\beta(x_1^{(j)}, \dots, x_n^{(j)})$, where $x_1^{(j)}, \dots, x_n^{(j)}$ are integers that are invertible in $\hat{Z}_\pi$.

How can we calculate the functions $\beta(x_1, \dots, x_n)$?

If $u \in U^2(CP^\infty)$ is a canonical element and the $\alpha = CP^i \in U_{2i}(CP^\infty)$ are standard bordisms, then we have the following obvious formulae:

a) $u^j \cap \alpha_i = \alpha_{i-j}$;

b) $\beta(1, \dots, 1) = \alpha_{i-1}$;

c) $\beta\left(\frac{1}{x}, \dots, \frac{1}{x}\right) = [\lambda_x]_* \beta(1, \dots, 1)$,

where $\lambda_x\colon CP^\infty \to CP^\infty$ is generated by multiplication by x in the group S^1.

d) $\lambda_x^*(\eta) = \eta^x$, where $\eta \in K(CP^\infty)$ is a canonical one-dimensional bundle;

e) $\lambda_x^*(u) = x\Psi^x(x)$, where $u \in U^2(CP^\infty)$ is a canonical element.

As for the case Z_p, so also here we seek a series $B(u; x)$ such that

$$\beta(x_1, \dots, x_n) = \prod_{i=1}^n B(u, x_i) \cap \alpha_{n-1},$$

where

$$B(u; x) = \frac{1}{x} + \dots.$$

By repeating the arguments employed in the proof of Theorem 1, we finally obtain the following assertion.

Theorem 1a. *We have the formula*

$$\beta(x_1, \dots, x_n) = \prod_{i=1}^n \frac{u}{x_i \Psi^{x_i}(u)} \cap \alpha_{n-1},$$

where the x_i are invertible elements in $\hat{Z}_\pi$, and

$$\beta(x_1, \dots, x_n) \in U_{2n-2}(CP^\infty) \otimes \hat{Z}_\pi, \quad \alpha_i = [CP^i].$$

Let us make a few remarks about other groups. For a torus the investigation is completely similar. Also, it is easy to carry it out for compact (connected) commutative groups.

VII. Arbitrary finite groups

It is known (Zassenhaus) that groups G that can act discretely and orthogonally on spheres have cyclic Sylow p-subgroups for $p > 2$ and highly special 2-subgroups ("generalized quaternions"). Therefore their homologies can be analyzed very simply. Let G be such a group and let $\{\Delta_1, \dots, \Delta_m\}$ be its complete set of unitary irreducible representations, acting discretely on spheres. Then all representations Δ such that $\Delta = \sum k_j \Delta_j$, $k_j \geq 0$ also have this property. We denote this semigroup by R_G^+. The following functions are defined on R_G^+:

$$a(\Delta) \in U_{2n-1}(BG), \quad \Delta \in R_G^+, \quad n = \dim_c \Delta.$$

The function of the Euler class in cobordism theory (or the older Chern class), $\sigma_n(\Delta) \in U^{2n}(BG)$, $\Delta \in K(BG)$, is also defined. We observe that for sums of one-dimensional fibers $\eta = \sum \eta_i$ we have

$$\sigma_n(\eta) = \prod_{i=1}^{n} \sigma_1(\eta_i),$$

and if $\eta_i = \eta^{x_i}$, then, by definition,

$$\sigma_n(\eta) = \prod_{i=1}^{n} \sigma_1(\eta^{x_i}) = \prod_{i=1}^{n} x_i \Psi^{x_i}(\sigma_1(\eta)).$$

Theorem 1b. *We have the following general formula*:

$$\alpha(\Delta_2) = \frac{\sigma_n(\Delta_1)}{\sigma_n(\Delta_2)} \cap \alpha(\Delta_1) \in U_{2n-1}(BG), \tag{X}$$

where $\dim_c \Delta_1 = \dim_c \Delta_2 = n$*,* $\Delta_i \in R_G^+$.

For commutative groups we proved this formula earlier (cf. Theorem 1), since for the canonical representation η of the group Z_p we have $\sigma_1(\eta) = u \in U^2(BG)$ and $\sigma_1(\eta^x) = x\Psi^x(u)$.

In the general case the proof can be reduced to Theorem 1 by the usual arguments involving restrictions to the Sylow p-subgroups. Since for all $p > 2$ they are cyclic, the formula (X) will have been deduced from Theorem 1 if we tensor multiply it by all the rings of p-adic integers for $p > 2$. For $p = 2$, we must analyze the 2-groups, i.e., the groups of generalized quaternions G_2^t, $t \geq 2$:

$$a, b \in G_2^t, \quad a^{2^t} = b^4 = 1, \quad b^2 = a^{2^{t-1}}, \quad aba^{-1} = b^{-1}.$$

In view of the simple structure of these 2-groups, the proof can be carried out by direct calculations, and we complete it below in examples 1 and 3.

Remark. The first coefficient $\sigma_n(\Delta_1)/\sigma_n(\Delta_2) = \gamma(\Delta_1, \Delta_2) + \dots$, where γ is a scalar, determines the degree of the equivariant sphere mapping $S^{2n-1} \to S^{2n-1}$ with respect to the actions of Δ_1, Δ_2.

Example 1. Let $G_2 = G_2^2$ be the group of quaternions (of order eight) with generators a and b and relations $a^2 = b^2 = (ab)^2$ (center), $a^4 = b^4 = 1$ and $aba^{-1} = b^{-1}$. There exists a unique irreducible representation $\Delta \in R_G^+$, given by the Pauli matrices and acting discretely on S^3. The ring of representation has a basis $1, \Delta, a, b, c \in R_{G_2}$, where $\dim \Delta = 2$, $\dim a = \dim b = \dim c = 1$, and $a^2 = b^2 = c^2 = 1$, $ab = c$, $\Delta^2 = (1+a)(1+b)$, $a\Delta = b\Delta = \Delta$. There are defined elements $a_{4k-1}(k\Delta) \in U_{4k-1}(BG_2)$ and the dual elements w^j, where $\omega = \sigma_2(\Delta)$. The element $a \in G_2$ generates the group Z_4, and when restricted to Z_4, the representation Δ becomes $\rho + \rho^{-1}$, where ρ is the basis representation $(e^{2\pi i/4})$. The order of the element $\sigma_2(\rho+\rho^{-1})$ in the group $U^4(L_{4k+3})$ equals 2^{2k+1}, because of previous results about Z_4, where L_{4k+3} is the lens of dimension $4k+3$ over Z_4. Clearly

$$\sigma_2(\rho + \rho^{-1}) = D\alpha_{4k-1}(1, -1, \dots, 1, -1).$$

From the properties of the ring R_G it is easy to conclude that the orders of the elements

$$\sigma_1(\Delta) = \sum a_j w^j \quad (a_j \in \Omega_U), \quad \sigma_2(\Delta) = w$$

do not exceed 2^{k+1}. Hence $2^{2k} w \neq 0$ and $2^{2k+1} w = 0$. Therefore the restriction of this cyclic group to $U^*(L_{4k+3})$ is a monomorphism. We must consider the elements $\alpha(k\Delta) \in U_{4k-1}(BG_2)$. On the basis of the above, their orders equal 2^{2k+1}, and $w^j \cap \alpha(k\Delta) = \alpha((k-j)\Delta)$. Since the restriction of the elements w^i to Z_4 equals

$$\sigma_2^i(\rho + \rho^{-1}) = [u^2 \Psi^1(u) \Psi^{-1}(u)]^i,$$

from the equation $4\Psi^4(u) = 0$ it is easy to derive relations in the Ω-module between the w^j for various j, and apply this to the action of G_2 on M^n, having the manifolds M^i of fixed points of various dimensions with trivial normal bundles (i.e., of the form $[M^j] \times \alpha((n-j)\Delta)$.

Example 2. Let G be a group of order 120, acting discretely and unitarily on S^3, $G/[G,G] = 1$ and $1 \to Z_2 \to G \to S_5^+ \to 1$, where S_5^+ is the group of even permutations of five elements. The Sylow subgroups of G are G_2, Z_3, and Z_5, and the corresponding "Weyl groups" $W_p(G)$ of inner automorphisms of G that preserve the Sylow p-subgroups G_2, Z_3, and Z_5 are of the form: $W_2 = \operatorname{Aut} G = S_3$, $W_3 = W_5 = Z_2$, where $W_3(G)$ and $W_5(G)$ act by the automorphism $x \to x^{-1}$ on Z_3 and Z_5. We observe that the restrictions of cohomologies, K-theory, and U-theory to Sylow subgroups are invariant with respect to $W_p(G)$. There are two irreducible unitary representations $\Delta_1, \Delta_2 \in R_G$ that act discretely on spheres (namely, on S^3). Their restrictions to G_2, Z_3, and Z_5, respectively, are of the form

$$\begin{aligned} \Delta_1 &\to [\Delta(G_2), x + x^{-1}(Z_3), t + t^{-1}(Z_5)], \\ \Delta_2 &\to [\Delta(G_2), x + x^{-1}(Z_3), t^2 + t^{-2}(Z_5)] \end{aligned} \tag{XI}$$

where $x = (e^{2\pi i/3})$, $t = (e^{2\pi i/5})$ are basis representations. There exists an outer automorphism $*\colon G \to G$ such that $*\Delta_1 = \Delta_2$. We observe that $H^*(BG) = Z_{120}[y]$, $\dim y = 4$. The following "Conner–Floyd elements" are feasible: $\alpha(k\Delta_1 + l\Delta_2) \in U_{4k+4k-1}(BG)$. By virtue of the structure of the ring $H^*(BG)$, the collection of elements $\alpha(n\Delta_1) \in U_{4n-1}$ is a complete basis of $U_*(BG)$ as an Ω-module. Let $w = \sigma_2(\Delta_1)$. Clearly

$$w^j \cap \alpha(n\Delta_1) = \alpha((n-j)\Delta).$$

From the formula (XI) and Theorem 1b we conclude that

$$\alpha_{4k-1}((k-l)\Delta_1 + l\Delta_2) = \frac{w^l}{\sigma_2(\Delta_2)^l} \cap \alpha(k\Delta_1).$$

where this formula may be restricted to Z_5, since Δ_1 and Δ_2 coincide when restricted to G_2 and Z_3. After restriction to Z_5 we obtain

a) $(w/\sigma_2(\Delta_2))^l \to \left(\frac{u\Psi^{-1}(u)}{4\Psi^2(u)\Psi^{-2}(u)}\right)^l \in U^*(BZ_5)$,
b) $w/\sigma_2(\Delta_2) = 1 + \gamma$,

where γ has order $5^{h(n)}$ in any n-dimensional skeleton of BG; on such a subgroup the restriction to BZ_5 is an isomorphism.

When we write $U_*(BG)$ as the sum of its p-adic parts for $p = 2, 3, 5$, then, in principle, we can use the preceding formulas to carry out all the necessary ring calculations and draw conclusions about the fixed points.

Example 3. Let $G = G_2^t$, $t > 2$ be the group of generalized quaternions with generators a, b ($a^4 = b^{2^t} = 1$, $bab^{-1} = a^{-1}$ and $b^{2^{t-1}} = a^2$). Since $G/[G,G] = Z_2 \times Z_{2t-1}$ with generators a and b, the group H has 2^t irreducible one-dimensional representations

$$1, \eta, \rho, \rho^2, \dots, \rho^{2^{t-1}}, \eta\rho, \dots, \eta\rho^j,$$

where $\eta(a) = -1$, $\eta(b) = 1$ and $\rho(a) = 1$, $\rho(b) = e^{\frac{2\pi i}{2^{t-1}}}$.

Also, the group G has 2^{t-2} irreducible representations $\Delta_1, \dots, \Delta_{2t-2}$ of dimension two, acting discretely on S^3. Namely, $\Delta_j(a) = i\sigma_x$; $\Delta_j(b) = i\alpha_j\sigma_y$, where $\sigma_x, \alpha_y, \sigma_z$ are the Pauli matrices, and the α_j are numbers such that $\alpha_j^{2^{t-1}} = -1$ for $t > 2$, $\sigma_x^2 = \sigma_y^2 = 1$. There exist in all 2^{t-2} distinct roots α_j of -1 to within sign $(\alpha_j, -\alpha_j)$, determining the representations Δ_j, $j \leq 2^{t-2}$. The restriction of the representation Δ_j to the cyclic group $Z_2^t = (b)$ produces an element $p^{2j-1} + p^{1-2j}$, where α_j is the $(2j-1)$-th power of a primitive root of 1 of degree 2^t. The general form of the elements is $\alpha(\sum k_j\Delta_j) \in U_*(BG)$, where the "initial" elements may be taken as the collection $\alpha(k\Delta_1) \in U_{4k-1}(BG)$; all the remaining elements are expressed in terms of these with coefficients in Ω_U. Similarly we introduce the elements $w^k = \sigma_2^k(\Delta_1)$ that are dual to the elements $\alpha(l\Delta_1)$. It can be shown that on the Ω_U-module $\Omega[w] \in U^*(BG)$ the restriction to the cyclic subgroup $Z_2^t = (b)$ is a monomorphism. This follows from the Atiyah identity $K(BG) = R_{\hat{G}}$ and the Conner–Floyd homomorphism $\sigma_1 \colon K^0 \to U^2$ by analogy with Example 1. Thus, the order of the element

$$\alpha_{4k-1}(2x_1 - 1, 1 - 2x_1, \dots, 2x_k - 1, 1 - 2x_k) \in U_{4k-1}(BZ_{2^t})$$

equals precisely $2^{t+2(k-1)}$, which is also the order of $\alpha(\sum \Delta_{x_j})$. Therefore all the calculations connected with $\Omega_U[w]$ can be performed after restriction to $U^*(BZ_{2^t})$, and w_j becomes $-(2j-1)^2\Psi^{2j-1}(u)\Psi^{1-2j}(u)$, where $u = \sigma_1(\rho) \in U^2(BZ_{2^t})$.

Examples 1 and 3 prove the formula (X) of Theorem 1b, since it was proved earlier for cyclic groups (in Theorem 1).

A final remark on the actions and possible singular orbits of these actions for finite groups G on complex manifolds (we assume that all singular orbits are isolated): we must consider stationary subgroups of orbits $G_{\mathcal{P}} \subset G$ at their fixed points $\mathcal{P}$ such that $G_{\mathcal{P}}$ acts on the tangent sphere S^{2n-1} at the point $\mathcal{P}$ without fixed points (and $G_{\mathcal{P}}$ is maximal at the point $\mathcal{P}$), defining the invariants $\alpha_{\mathcal{P}} \in U_{2n-1}(BG_{\mathcal{P}})$.

Then it is easy to obtain the "general Conner–Floyd equations" for the action of the group G on M^n with isolated singularities. By what we have shown above, it suffices to consider only maximal cyclic subgroups $G'_{\mathcal{P}} \subset G_{\mathcal{P}}$ in the final calculations (cf. Examples 1 and 3). From the class of conjugate points $\mathcal{P}_i$ with respect to $G/G_{\mathcal{P}}$ we must select one. The imbedding $G_{\mathcal{P}} \to G$ means that the homomorphisms $\rho_*\colon U_*(BG'_{\mathcal{P}}) \to U_*(GB)$ are defined. The "general Conner–Floyd equations" are of the form:

$$\sum_{G_{\mathcal{P}}} \rho_*(\alpha_{\mathcal{P}}) = 0 \in U_*(BG), \tag{XII}$$

where for each $G_{\mathcal{P}}$ (to within a transformation) we must in (XII) select one point from the collection $g(\mathcal{P})$, $g \in G/G_{\mathcal{P}}$.

VIII. Another application of Adams operators in cobordism theory

Theorem 2. *The Thom complex $M(\xi)$ of an arbitrary element $\xi \in K(BU^{(n)})$ is homotopically equivalent to MU with preservation of the homological Thom isomorphism if and only if the difference $\xi - \eta$ is of the form*

$$\xi - \eta = \sum a_i k_i^{N_i} (\Psi^{k_i} - 1)\xi_i,$$

where η is a universal bundle, $BU^{(n)}$ is a finite skeleton of BU, and Ψ^k is the usual Adams operator in K-theory.

This theorem unfortunately does not imply the well-known Adams hypothesis. Therefore we shall not give a complete proof.

Roughly speaking, the proof is based on the fact that the operators $\Psi^{1/k}$ in U^*-theory are automorphisms of MU in the category $S \otimes Z(1/k)$, such that $\Psi_*^{1/k}/H_{2(N+i)}(MU_n)$ is multiplication by k^{-i}. On the other hand, for a universal element $\eta \in K(BU)$ there exists a transformation $\rho_k\colon M(\Psi^k\eta) \to MU$, which, in the homologies $H_{2(N+i)}$ is multiplication by k^i. Therefore the morphism $\Psi^{1/k} \cdot \rho_k\colon M(\Psi^k\eta) \to MU$ commutes with the Thom isomorphism.

Bibliography

[1] I. Tamura, *Characteristic classes of M-spaces.* I, J. Math. Soc. Japan **11** (1959), 312–342. MR 22 #5046.

[2] S. P. Novikov, *Methods of algebraic topology from the point of view of cobordism theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967) 855–951 = Math. USSR Izv. **1** (1967), MR 36 #4561.

[3] M. F. Atiyah and R. Bott, *Notes on the Lifschitz fixed point theorem for elliptic complexes*, Harvard Univ., 1964, preprint; Russian transl., Matematika **10** (1966), no. 4, 101–139.