

ON EMBEDDING SIMPLY-CONNECTED MANIFOLDS IN EUCLIDEAN SPACE

S. P. NOVIKOV

We shall consider compact smooth manifolds and smooth mappings. As usual, a mapping $f: M^n \rightarrow W^m$ will be called *regular* if its Jacobian has rank n at every point, and *completely regular* if the inverse image $f^{-1}(w)$ of every $w \in W^m$ contains no more than two points. We shall study the possibility of smoothly embedding $M^n \subset E^{2n-1}$. Our method generalizes Whitney's in [2], where he considers embeddings $M^n \subset E^{2n}$, and is based on an idea of Pontryagin's concerning homotopy groups of spheres [1].

Theorem 1. *Every simply-connected odd-dimensional M^n ($n > 6$) can be smoothly embedded in E^{2n-1} .*

The proof depends on a series of lemmas on regular mappings in the large. By means of a familiar technique elaborated by Whitney, one easily proves

Lemma 1.* *For every regular $f: M^n \rightarrow E^{2n-k}$ with $k < [\frac{n}{2}]$ there exists a regular $g: M^n \rightarrow E^{2n-k}$, close to f in the C^1 topology, such that*

- 1) *The equation $g(x) = g(y)$ defines a compact submanifold $\tilde{M}_g^k \subset M^n \times M^n - \Delta(M^n)$, where Δ is the diagonal map;*
- 2) *The projection $p: M^n \times M^n \rightarrow M^n$, restricted to \tilde{M}_g^k , is a smooth mapping;*
- 3) *g is completely regular; the restriction of g to the singular submanifold $M_g^k = p(\tilde{M}_g^k) \subset M^n$ is a two-sheeted covering.*

It follows from the lemma that the singular manifold M_g^k decomposes into some number s of special pairs of mutually homeomorphic connected components

$$\bigcup_{i=1}^s (M_{g,1}^{k,i} \cup M_{g,2}^{k,i})$$

such that $g(M_{g,1}^{k,i}) = g(M_{g,2}^{k,i})$, and some number t of connected manifolds $\bigcup_{j=1}^t M_g^{k,j}$ on which g is a nontrivial 2-covering. Thus

$$M_g^k = \left(\bigcup_{j=1}^t M_g^{k,j} \right) \cup \left(\bigcup_{i=1}^s (M_{g,1}^{k,i} \cup M_{g,2}^{k,i}) \right).$$

Definition. M^n will be called *k-parallelizable* if an ϵ -neighborhood $U_\epsilon^{(k)}$ of the k -skeleton of a differentiable triangulation of M^n is parallelizable, for some sufficiently small ϵ .

It is obvious that for $n > 2k + 2$ our definition does not depend on the triangulation, and that a k -connected manifold is k -parallelizable. For $k = 1$, our definition gives simply orientability. Note also that a k -parallelizable manifold is $(k - 1)$ -parallelizable. Let us now take $n \geq 2k + 3$.

Lemma 2. *If M^n is k -parallelizable, then the singular submanifold M_g^k has trivial normal bundle in M^n and is a π -manifold (i.e., in an embedding $M_g^k \subset E^m$ the normal bundle is trivial if $m \geq 2k + 3$).*

The proof is based on the fact that $M^n \times M^n - \Delta(M^n)$ is also k -parallelizable, while the normal

*For $k = 1$ this lemma is contained in Whitney's work, for example, in [3].

bundle of a submanifold of dimension $\leq k$, contained in a k -parallelizable manifold of greater dimension, is constructed exactly as in a euclidean space of the same dimension.

Now suppose n even, $k = 1$, M^n orientable.

Lemma 3. *The singular manifold $M_g^1 \subset M^n$ consists only of singular pairs of circles.*

Suppose on the contrary that M_g^1 contains a circle $S_g^1 \subset M_g^1$ on which g is a connected 2-covering. Obviously $g(S_g^1) = S^1 \subset E^{2n-1}$. Choose a system (W_1, \dots, W_{n-1}) of independent vector fields tangent to M^n and transversal to $S_g^1 \subset M^n$.

Then, roughly speaking, there results a decomposition of the normal bundle of $g(S_g^1) \subset E^{2n-1}$ into a sum of 2-plane bundles $\mu_i^{(2)}$, $i = 1, \dots, n-1$, generated by the vectors W_i . Each of the bundles $\mu_i^{(2)}$ is transversal to the circle $g(S_g^1)$ and has transition matrix $A_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, that is, is nonorientable. The Whitney sum of an odd number of such bundles is also nonorientable, therefore nontrivial. But since $n-1$ is odd, while the normal bundle of a circle in euclidean space must be trivial, we obtain a contradiction, and the lemma is proved.

More generally, when $k = 1$, we have

Lemma 4. *The number $t(g)$ of connected singular coverings is always even. The map g is regularly homotopic to a map g_1 having no singular nontrivial 2-coverings.*

The proof of this lemma is of rather different character, and is based on studying the projection $M^n \xrightarrow{\tilde{g}} E^{2n} \xrightarrow{\pi} E^{2n-1}$ where $g = \pi \circ \tilde{g}$ and \tilde{g} is completely regular. (We can always reach this situation by means of a small deformation of the mapping in E^{2n} , and a projection of this small deformation into E^{2n-1} , which obviously preserves the properties mentioned in Lemma 1.)

The behavior of this projection is further described in the following (trivial) lemmas:

Lemma 4a. *If $\tilde{g}: M^n \rightarrow E^{2n}$ and $\pi\tilde{g}: M^n \rightarrow E^{2n-1}$ are regular, then \tilde{g} is regularly homotopic to an embedding and has an even number of pairs of singular points.*

Lemma 4b. *A connected singular covering for g can arise under the projection only from an odd number of pairs of singular points of \tilde{g} .*

A singular pair can arise in the projection only from an even number of pairs of singular points of \tilde{g} .

Lemma 4c. *There exists a regular homotopy \tilde{g}_t of $\tilde{g} = \tilde{g}_0$ such that*

- 1) \tilde{g}_t and $\pi\tilde{g}_t$ are regular for $t \leq 1$, and completely regular for $t = 1$; $\pi\tilde{g}_1$ satisfies Lemma 1;
- 2) coverings of the map $\pi\tilde{g}_1$ can arise under the projection from a pair of singular points, while singular pairs of circles can come from nothing.

From now on we consider only mappings $g: M^n \rightarrow E^{2n-1}$ which have no connected singular coverings. Suppose also that $\pi_1(M^n) = 0$. Following Pontryagin [1], we define an invariant of a singular pair and an invariant of the mapping g .

Definition of the invariant of a singular pair. Let $S_1^1, S_2^1 \subset M^n$, $g(S_1^1) = g(S_2^1)$. Consider a pair of discs $\sigma_1^2, \sigma_2^2 \subset M^n$ such that $\sigma_1^2 \cap \sigma_2^2 = \emptyset$ and $\partial\sigma_1^2 = S_1^1$, $\partial\sigma_2^2 = S_2^1$. We define a system of vector fields $W_j^{(i)}$, $i = 1, 2$; $j = 1, \dots, n-2$ on M^n , orthogonal to the σ_i^2 . We put $W_{n-1}^{(i)} = \partial\sigma_i^2 / \partial t$ where t denotes the radii of the films (i.e., these are transversal to the set S_i^1 and to the $W_j^{(i)}$, $j \leq n-2$). We obtain vectors $\dot{g}(W_j^{(i)}) = V_{j+(i-1)}$, transversal to $g(S_i^1)$ and independent. These correspond to an element $\alpha \in \pi_1(GL(2n-2)) = Z_2$.

Lemma 5. *If the generator of $H^n(M^n, Z_2)$ has the form $Sq^2(x)$, $x \in H^{n-2}(M^n, Z_2)$, then the discs*

σ_i^2 and the fields $\mathbb{W}_j^{(i)}$ can be chosen that $\alpha = 0$.

In case $H^n(M^n, Z_2)/\text{Im } Sq^2 = Z_2$ the invariant α of the singular pair does not depend on the choice of the discs σ_i^2 . In this case the sum $\sum_k \alpha_k$ of the invariants of all the singular pairs $S_k = (S_{g,1}^{1,k} \cup S_{g,2}^{1,k})$ can be considered as an invariant of $g: M^n \rightarrow E^{2n-1}$ at least when the latter has no connected singular coverings.

Lemma 6. *If M^n is simply-connected and $n = 4l + 3$, then the invariant $\sum_k \alpha_k$ vanishes for any regular $g: M^n \rightarrow E^{2n-1}$ with the properties of Lemma 1.**

This lemma is an important step, and its proof, which is direct and geometrical, is rather complicated. However, it also follows easily from recent work by Hirsch on regular mappings [4].

Let S_1, S_2 be two singular pairs of the mapping $g: M^n \rightarrow E^{2n-1}$ such that $\alpha(S_1) = \alpha(S_2)$.

Lemma 7. *There exists a regular homotopy g_i of $g = g_0$ such that g_1 satisfies Lemma 1 and has two fewer singular pairs than $g = g_0$.*

The proof generalizes a familiar proof of Whitney's [2] for pairs of singular points. We attach rings $B_1 = S_1^1 \times I$ and $B_2 = S_2^1 \times I$ to M^n in such a way that $S_i^1 \times \epsilon$ form the pair S_1 and $S_i^1 \times (1 - \epsilon)$ form S_2 . We can arrange $B_1 \cap B_2 = \emptyset$. On these rings we take vector fields $\mathbb{W}_j^{(i)}$, $i = 1, 2$; $j = 1, \dots, n - 2$ extending those on the discs which induce the invariants $\alpha(S_i)$. We can easily ensure that the frames $(\tau_i, \dot{g}(\mathbb{W}_j^{(1)}), \dot{g}(\mathbb{W}_j^{(2)}))$ (where the τ_i are the fields tangent to the $g(S_i)$) define opposite orientations, for $i = 1, 2$. Now we attach a "Whitney cell" $\psi: \sigma^2 \times S^1 \rightarrow E^{2n-1}$ such that

$$\psi(\sigma^2 \times S^1) \cap g(M^n) = g(B_1) \cup g(B_2);$$

we must also choose ψ to satisfy certain compatibility conditions on the boundaries. Now, since the invariants of S_1 and S_2 coincide, we can pick in a small neighborhood $U(\psi(\sigma^2 \times S^1))$ a suitable system of coordinates, one of which is the coordinate on the circle, two others are the standard 2-frames on σ^2 , and the remainder satisfy our boundary conditions. With these coordinates in $U(\psi(\sigma^2 \times S^1))$, we can perform Whitney's deformation, for a constant circle-coordinate.

Repeating this construction and applying Lemma 6, we obtain finally a map $g_S: M^n \rightarrow E^{2n-1}$ which has singular pairs with zero invariant only.

With g_S we may now proceed in either of two ways: following [2], we may attach complementary pairs with zero invariants, and apply Lemma 7, or we may simply carry out a direct separation of a pair with zero invariants.** In either case we arrive at an embedding. The theorem thus follows from the preceding lemmas.

Note that the lemmas imply the following conditional

Theorem 2. *Suppose $n = 2l$, $n \geq 6$, $\pi_1(M^n) = 0$. There is an embedding $M^n \subset E^{2n-1}$ if and only if there is an immersion $M^n \rightarrow E^{2n-2}$. (See [4].)****

V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received 20/JAN/61

*If $n = 4l + 1$ one can assert the existence of a $g: M^n \rightarrow E^{2n-1}$ with zero invariant, since there exists an immersion $M^n \rightarrow E^{2n-2}$ [4].

**A method for separating a singular pair with zero invariant has been indicated to me by D. B. Fuks, who kindly took an interest in the present article.

***Closer study shows that for $n = 4l + 2$, the invariant α is a homotopy invariant of M^n : it does not depend on the immersion $g: M^n \rightarrow E^{2n-1}$ when $n \not\equiv 1 \pmod{4}$.

