RATIONAL PONTRJAGIN CLASSES. HOMEOMORPHISM AND HOMOTOPY TYPE OF CLOSED MANIFOLDS. I

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In a number of special cases it is proved that the rational Pontrjagin–Hirzebruch classes may be computed in terms of cohomology invariants of various infinitely-sheeted coverings. This proves their homotopy invariance for the cases in question (Theorems 1 and 2). The methods are applied to the problem of topological invariance of the indicated classes (Theorem 3). From the results there follow various homeomorphism and homotopy types of closed simply-connected manifolds, which yields a solution to the problem of Hurewicz for the first time in dimensions larger than three (Theorem 4). We note that in the paper [3] the author completed the proof of the topological invariance of all the rational Pontrjagin classes using a quite different method.

Introduction

As is known, already for three-dimensional manifolds homeomorphism is distinct from the homotopy type in the sense that there exist closed manifolds which are homotopically equivalent but not homeomorphic. They are distinguished by the Reidemeister invariant "torsion". It is natural to expect that also in dimensions $n > 3$ homeomorphism also will not coincide with homotopy type. For example, they are distinguished by the torsion invariant also in large dimensions, if one proves that torsion is a topological invariant. Another widely known invariant, not a horootopy invariant, but, by assumption, a topological invariant, is the Pontrjagin class, considered as rational. However in dimensions $n > 3$ no invariant has been established as topological, unless it is also obviously homotopic. It is interesting that for $n = 3$ the torsion invariant, as a means of distinguishing combinatorial lenses, has been known since the thirties, and its topological invariance was obtained only in the fifties in the form of a consequence of the "Hauptvermutung" (Moise). The situation is that in three dimensions a continuous homeomorphism may be approximated by a piece wise linear one. This can hardly be true in high dimensions, and even if it is true, at the present time there are no means in sight for the proof of this fact.

In the present paper we study the rational Pontrjagin classes as topological and homotopy invariants. It is known that for simply-connected manifolds there are no "rational relations" of homotopy invariance of classes other than the signature theorem:

$$(L_k(p_1, \dots, p_k), [M^{4k}]) = \tau(M^{4k}),$$

where $\tau(M^{4k})$ denotes the signature of the quadratic form $(x^2, [M^{4k}])$, $x \in H^{2k}(M^{4k}, R)$ and L_k are the Hirzebruch polynomials of the Pontrjagin classes. In what follows we shall speak about the classes $L_k = L_k(p_1, \dots, p_k)$ along with the

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classes p_k for manifolds, since they are convenient in the investigation of invariance. This is shown by the signature theorem presented above and the combinatorial results of Thom, Rohlin and Švarc (see [4]–[6]). The only "hole" in the theory of Pontrjagin classes, from the point of view of the problems posed, was the theorem of Rohlin, proved in 1957, establishing that the class $L_k(M^{4k+1})$ is a topological invariant, but here it was not known whether the indicated class was a homotopy invariant (see [4]). However, Rohlin's proof was in no way connected with the fundamental group. One should note that on passing to a simply-connected manifold this theorem is empty, since $H_{4k}(M^{4k+1}) = 0$.

In the present article we establish for certain cases the algebraic connection of the classes with the fundamental groups From the resulting relation it follows for example that the class $L_k(M^{4k+1})$ in reality is a homotopy invariant. The formulas (see § 3) found by the author may be considered to a certain degree as generalizations of the formula of Hirzebruch. Their connection with coverings was rather unexpected, since in the theory of characteristic classes the fundamental group had earlier played no role at all.

It was possible to apply these formulas to the question of the topological invariance of classes. Under certain conditions we were able to prove that the scalar products (L_k, x), where $x \in H_{4k}(M^n)$, $n = 4k + 2$, are topological invariants. Already for $M^n = S^2 \times S^{4k}$ this fact makes it possible to solve affirmatively the question on the distinction between the homeomorphism and homotopy type for all dimensions of the form $4k + 2$, $k \geq 1$, and also in the class of simply-connected manifolds, where the "simple" homotopy type coincides with the ordinary one.

The basic results of this paper were published in brief in [1].

We take this opportunity to express our gratitude to V. A. Rohlin for useful discussions on this work.

§ 1. Signature of a cycle and its properties

We gather in this section a number of simple algebraic facts on quadratic forms which will be used in the sequel.

We suppose that we are given a real linear space P, possibly of infinite dimension, and that on P there is given a symmetric bilinear form $\langle x, y\rangle$ with values in R. We shall be interested only in the case when P can be decomposed into a sum $P = P_1 + P_2$, where P_1 is finite-dimensional and $\langle x, y\rangle = 0$, $y \in P_2$, $x \in P$, i.e. the entire form is concentrated on a finite-dimensional subspace $P_1 \subset P$, which, it is to be understood, is chosen nonuniquely. We shall say in this case that the form is of finite type. The quadratic form $\langle x, x\rangle$ is concentrated, essentially, on P_1, and one may speak of its signature, which we shall use as the signature of the quadratic form $\langle x, x\rangle$ on P. The signature does not depend on the choice of P_1. Evidently every subspace $P' \subset P$ is such that the form $\langle x, x\rangle$ for $x \in P$ is also of finite type and has a signature in the same sense. It is easy to construct a decomposition into a sum $P' = P_1' + P_2'$, where $\langle x, y\rangle = 0$, $y \in P_2'$ and P_2' is finite-dimensional.

The following facts on the signature easily follow from the analogous facts for forms on finite-dimensional subspaces.

a) If we are given two subspaces $P' \subset P$ and $P'' \subset P$ such that every element of P is a sum $x_1 + x_2$, $x_1 \in P'$, $x_2 \in P''$, and if the form $\langle x, y\rangle$ is topologically zero on P' and on P'', then the signature of the form $\langle x, x\rangle$ is equal to zero on P. If now the forms on P' and P'' are nontrivial, then P' and P'' decompose into the sums $P = (P' \cap P'') + P_1'$ and $P'' = (P' \cap P'') + P_1''$ such that $\langle x, y\rangle = 0$, where $y \in P_1'$, $x \in P'$; $y \in P_1''$, $x \in P''$, then the signature of $\langle x, x\rangle$ on P coincides with the signature of $\langle x, x\rangle$ on $P' \cap P''$.

b) If we are given a subspace $P' \subset P$ such that $\langle x, y\rangle = 0$ for all $x \in P'$ implies $\langle y, y\rangle = 0$, then the signature of $\langle x, x\rangle$ on P' coincides with the signature of $\langle x, x\rangle$ on P.

Suppose that K is any locally finite complex and $z \in H_{4k}(K, Z)/\mathrm{Torsion}$. Consider the group $H^{2k}(K, R) = P$ and the bilinear form $\langle x, y\rangle = (xy, z)$, $x, y \in P$. It is easy to prove

Lemma 1. *The bilinear form $\langle x, y\rangle$ has finite type on the group $P = H^{2k}(K, R)$.*

Proof. One can find a finite subcomplex $K_1 \overset{i}{\subset} J$ such that in K_1 there is an element $z_1 \in H_{4k}(K_1)$ and $z = i_* z_1$. The group $H^{2k}(K_1, R)$ is finite-dimensional. The homomorphism $i^*\colon P \to H^{2k}(K_1, R)$ is defined. Since

$$((i^*x)(i^*y), z_1) = (xy, z) = \langle x, y\rangle,$$

the kernel $\operatorname{Ker} i^* \subset P$ consists only of those elements $y \in \operatorname{Ker} i^*$ for which $\langle y, y\rangle \equiv 0$. The image $\operatorname{Im} i^*$ is finite-dimensional, and therefore the form $\langle x, y\rangle$ has finite type on P. The lemma is proved. □

Thus the signature of the form on $P = H^{2k}(K, R)$ is determined.

By the *nondegenerate part* of a form of finite type on a linear space P we shall mean a subspace $P_1 \subset P$ such that the form is nondegenerate on P_1 and is trivial on the orthogonal complement to P_1. It is natural to consider P_1 as a factor of P. Evidently the signature is defined by the nondegenerate part of the quadratic form, which is uniquely defined (as a factor).

Lemma 2. *Suppose that $K_1 \subset K_2 \subset \cdots \subset K$ is an increasing sequence of locally finite complexes and $K = \bigcup_j K_j$. We shall denote the inclusion $K_1 \subset K_j$ by i_j and the inclusion $K_1 \subset K$ by i. Suppose that we are given an element $z_1 \in H_{4k}(K_1, Z)/\mathrm{Torsion}$ such that $i_{j*} z_1 \neq 0$. Consider the elements $i_{j*} z_1 = z_j$ and forms on the spaces $P_j = H^{2k}(K_j, R)$. Then the nondegenerate part of a quadratic form on P_j is one and the same for all sufficiently large indices and coincides with the nondegenerate part of the quadratic form on $P = H^{2k}(K, R)$.*

Proof. Consider the homomorphisms $i_j^*\colon P_j \to P_1$ and $i^*\colon P \to P_1$. Select in P_1 a finite-dimensional nondegenerate part $P_1' \subset P_1$. Then we may suppose that the images of all the nondegenerate parts $P_j' \subset P$ under i_j^* lie in $P_1' \subset P_1$.[1] But the image

$$\operatorname{Im} i^* = \bigcap_j \operatorname{Im} i_{j+1}^*;$$

in view of the finite dimensionality of P_1' and the inclusions

$$\operatorname{Im} i_j^* \supset \operatorname{Im} i_{j+1}^*$$

for all j, we obtain a stabilization of the images $i_j^* P_j' \subset P_1'$. And since the kernel $\operatorname{Ker} i_j^*$ consists only of the purely degenerate part, it follows that the forms are equal on P_j' and $i_j^* P_j^*$. The lemma is proved. □

[1] Here for the proof of the stabilization of the images it is convenient to select in K_j finite subcomplexes $\bar{K}_j \subset K_j$ such that $\bar{K}_j \subset \bar{K}_{j+1}$ and $\bigcup_j \bar{K}_j = K$, and carry out the argument for $\bar{K}_j$.

In what follows the signature of the natural form on $P = H^{2k}(K, R)$ for a given element $z \in H_{2k}(K, Z)/\mathrm{Tor}$ will be called the "signature of the cycle z" and will be denoted by $\tau(z)$. If $K = M^{4k}$ and $z = [M^{4k}]$, then $\tau(z) = \tau(M^{4k})$.

Evidently $\tau(-z) = -\tau(z)$ and $\tau(\lambda z) = \tau(z)$, if $\lambda > 0$.

§ 2. The fundamental lemma

We suppose that W^n is an open manifold and V^{n-1} a submanifold dividing W^n into two parts W_1 and W_2 such that $W_1 \cup W_2 = W^n$ and $W_1 \cap W_2 = V^{n-1}$. We suppose that V^{n-1} and W^n are smooth (or PL) manifolds and that the inclusion $i\colon V^{n-1} \subset W^n$ is smooth or piecewise linear. Suppose we are given a continuous (not necessarily smooth or piecewise linear) mapping $T\colon W^n \to W^n$ such that the intersection $TV^{n-1} \cap V^{n-1}$ is empty, while V^{n-1} and TV^{n-1} jointly bound a connected piece of the manifold W^n. We require further that the mapping $W^n \to W^n//T$ is a covering, so that the intersection $TN \cap N$ coincides with TV^{n-1} and so that W^n decomposes into the union

$$W^n = \bigcup_l T^l N.$$

Under these conditions we have the following lemma.

Fundamental Lemma. *If we have an element $z \in H_{4k}(V^{n-1}, Z)/\mathrm{Tor}$ such that $i_* z \neq 0 \mod \mathrm{Tor}$, $T_* i_* z = i_* z$ and the film between z and Tz lies in N, then*

$$\tau(z) = \tau(i_* z)$$

provided that one of the following conditions is satisfied:

a) *$n = 4k+1$, V^{n-1} is compact, $z = [V^{n-1}]$*;

b) *n is arbitrary, but the group $H_{2k+1}(W^n, R)$ has no T-free elements* (*this means that for any $\alpha \in H_{2k+1}(W^n, R)$ there is an index $q = q(\alpha)$ such that*

$$\alpha = \sum_{l=1}^{q} \lambda_l T_*^l \alpha.$$

For example this is satisfied if the group $H_{2k+1}(W^n, R)$ is finite-dimensional).

Proof. Denote by i_1 and i_2 respectively the inclusions $V^{n-1} \subset W_1$ and $V^{n-1} \subset W^2$ and by $J_l \subset H^{2k}(V^{n-1}, R)$ the image $i_l^* H^{2k}(W_l, R)$. On J_1 the form $(x^2, z) = \langle x, x\rangle$ is defined, and its signature, as indicated in § 1, coincides with the signature of the cycle $i_{l*} z \in H_{4k}(W_l)$. We have

Lemma 3. $\tau(i_{l*} z) = \tau(i_* z)$, $l = 1, 2$.

The proof of Lemma 3 follows from Lemma 2. Indeed, for the proof of the equation $\tau(i_{l*} z) = r(i_* z)$ we need to put

$$K_1 = N \cup T^{-1} N,\ \ldots,\ K_i = K_{i-1} \cup T^{i-1} N \cup T^{-i} N,\ \ldots,\ K = W^n,$$

and analogously decompose

$$W_2 = \bigcup_j K'_j, \quad K'_i = T^{-i} K_i, \quad W_2 = K',$$

take into account that the transformation T^q homeomorphically maps K'_q onto K_q and to apply Lemma 2 on the stabilization of the nondegenerate parts of forms, defining the signature.[2]

From the proof of Lemma 3 it follows that

Lemma 3′. *Suppose that J is the image of $i^* H^{2k}(W^n, R)$. Then the nondegenerate part of a form on J_l, $l = 1, 2$, may be chosen entirely on $J = J_1 \cap J_2$.*

In order to finish the proof of the basic lemma, we need to establish that the signature of a quadratic form on J coincides with the signature of a quadratic form on the entire group $P = H^{2k}(V^{n-1}, R)$.

1) Suppose first that $n = 4k + 1$ and $z = [V^{n-1}]$. Suppose that $\alpha \in P$ and $\langle a, x\rangle = 0$, $x \in J_1$. Then the element $\alpha \cap [V^{n-1}] = \beta \in H_{2k}(V^{n-1}, R)$ obviously satisfies $(\beta, x) = 0$, $x \in J_1$. That means that $i_{1*}\beta = 0$. Since $i_{1*}\beta = 0$, the selfintersection index $\beta \circ \beta = 0$. Therefore

$$(\alpha^2, [V^{n-1}]) = \beta \circ \beta = 0.$$

From the algebraic properties of the signature (see § l, b)) we conclude that the signature of the form on J_1 coincides with the signature of the form on P, which is equal to $\tau(z) = \tau(V^{n-1})$.

For $n = 4k + 1$ the lemma is proved.

2) Now suppose that $n > 4k + 1$. From Lemmas 3 and 3′ and the properties of the signature (see § 1, a)) we conclude that the signature $\tau(i_* z)$, which coincides with the signature of the form on $J \subset P$, is equal simultaneously to the signature of the form on the space P', representing the linear envelope of J_1 and J_2.

Now suppose that $\alpha \in P$ and $\langle \alpha, x\rangle = 0$, $x \in P'$. Consider the element $\beta = \alpha \cap z \in H_{2k}(V^{n-1}, R)$. Since $(\beta, x) = 0$, $x \in P'$, we have $i_{1*}\beta = i_{2*}\beta = 0$. The films ∂_1 and ∂_2, stretched on the cycle (representative of an element of W_1 and W_2 respectively), define together a cycle $\delta = \partial_1 - \partial_2$, which we shall consider as an element of $\delta \in H_{2k+1}(W^n, R)$. Since by hypothesis

$$\delta = \sum_{l=1}^{q(\delta)} \lambda_l T_*^L \delta,$$

there exists a $2k + 2$-dimenslonal chain c_0 in W^n whose boundary defines that relation. We put

$$c = c^0 + \sum_{l=1}^{q(\delta)} \lambda_l T^l c_0 + \cdots + \sum_{l_1,\ldots,l_m} \lambda_{l_1} \cdot \lambda_{l_2} \cdot \ldots \cdot \lambda_{l_m} T^{l_1+\cdots+l_n} c_0 + \cdots .$$

Although c is a noncompact chain, its compact boundary is δ and the intersection $c \cap V^{n-1}$ is compact. However the boundary of the intersection, $\partial(c \cap V^{n-1})$, is exactly β. Therefore

$$\beta = \alpha \cap z = 0 \quad \text{and} \quad (\alpha^2, z) = 0.$$

The lemma is proved. □

[2]Here of course, the basic role is played by the T-invariance of the cycle $i_* z$, and also the condition on the film between z and Tz, where $z \in H_{4k}(V)$, $Tz \in H_{4k}(TV)$.

Remark. As V. A. Rohlin has indicated to me, in the part of the basic lemma relating to $n = 4k + 1$ it is essentially proved that if M^{4k} is one of the components of the edge of any (for example, open) manifold W_1^{4k+1}, then

$$\tau(M^{4k}) = \tau(i_{1*}[M^{4k}]);$$

the conclusion concerning the signature $\tau(i_* z)$ in the union $W = W_1 \cup W_2$ is therefore proved by use of the transformation $T\colon W \to W$. This may be bypassed and an analogue of the lemma established for the case when W is an open manifold and M^{4k} is a compact cycle separating it, so that in reality the transformation T does not play a large role here. However, for $n = 4k + 2$ this method of reasoning without using T has not been successfully applied in the homotopy theorem.

§ 3. Theorems on homotopy invariance. Generalized signature theorem

Consider a closed manifold M^n, where $n = 4k + m$. Suppose that we are given an element $z \in H_{4k}(M^N, Z)/\mathrm{Tor}$ such that the dual element $Dz \in H^m(M^n, Z)$ is the product of indivisible elements $Dz = y_1 \dots y_m \mod \mathrm{Tor}$, $y_i \in H^1(M^n, Z)$. We define the covering $p\colon \hat{M} \to M^n$, under which those and only those paths $\gamma \subset M^n$ for which $(\gamma, y_j) = 0$, $j = 1, \dots, m$ are covered by closed sets. Evidently on the manifold $\hat{M}$ there operates a monodromy group generated by the commutative transformations $T_1, \dots, T_m \colon \hat{M} \to M$.

Lemma 4. *There exists an element $z \in H_{4k}(\hat{M}, Z)$, such that $T_* \hat{z} = \hat{z}$, $j = 1, \dots, m$, and $p_* \hat{z} = z$.*

Proof. We realize the cycle $Dy_j \in H_{n-1}(M^n, Z)$ by the subsets $M_i^{n-1} \subset M^n$ and the cycle z by its intersections

$$M^{4k} = M_1^{n-1} \cap \dots \cap M_m^{n-1}.$$

It is easy to see that all the paths lying in M^{4k} are covered by closed sets. Thus there is defined a covering inclusion $M^{4k} \subset \hat{M}$, which gives us the required element $\hat{z}$. The lemma is proved. □

Now consider a Serre fibering

$$q\colon M^n \xrightarrow{\hat{M}} T^m,$$

where the basis is of the topological type of the torus T^n, the space is of type M^n and the fiber of type $\hat{M}$. This fibering is dual to the covering. It is defined homotopically invariantly. Evidently the term $E_2^{m,4k}$ of the spectral sequence in homologies is isomorphic to the subgroup $H_{4K}^{\mathrm{inv}} \subset H_{4k}(\hat{M}, Z)$, consisting of elements invariant relative to the monodromy group. The subgroup $E_\infty^{m,4k} \subset E_2^{m,4k}$, consisting of cycles of all differentials of the spectral sequence of the covering, is defined.

Lemma 5. *The subgroup $E_\infty^{m,4k}$ is infinite cyclic. It is precisely the group $H_n(M^n) = Z$, and*

$$E_\infty^{m-1,4k+1} = \dots = E_\infty^{1,4k+m-1} = 0.$$

Proof. The fact that $E_\infty^{m,4k}$ is a factor of $H_n(M^n)$ is a consequence of the definition of filtration in the spectral sequence of homologies. Therefore it is a cyclic group. We note now that $E_\infty^{m,4k}$ is infinite and the corresponding element was constructed in Lemma 4. Therefore $E_\infty^{m-s,4k+s}$ for $s > 0$ is trivial. The lemma is proved. □

As a consequence of Lemmas 4 and 5 we obtain

Lemma 6. *There exists a unique element* $\hat{z} \in H_{4k}(\hat{M}, Z)$ *such that* $T_*\hat{z} = \hat{z}$, $p_*\hat{z} = z$, *and in the term* $E_2^{m,4k}$ *of the spectral sequence of the covering the element* $\hat{z} \otimes [T^m]$ *lies in the group* $E_\infty^{m,4k} = Z$, *m i. e.* $\hat{Z} \otimes [T^m]$ *is the cycle of all differentials. Here* $[T^m]$ *is the fundamental cycle of the torus.*

Lemma 6 is a unification of Lemmas 4 and 5 with the additional observation that in Lemma 4 an element of $E_\infty^{m,4k}$ was concretely constructed. The element $\hat{z}$ indicated in Lemma 6 will be called canonical.

Theorem 1. *For* $m = 1$ *and* $m = 2$ *with the additional condition that the group* $H_{2k+1}(\hat{M}, R)$ *is finite-dimensional, we have the formula for an indivisible* $z \in H_{4k}(M^n, \hat{Z})$, $Dz = y_1 \dots y_m$:

$$(L_k(M^n), z) = \tau(\hat{z}),$$

where $\hat{z}$ *is a canonical element. In particular, this scalar product is a homotopy invariant.*

Corollary 1. *The rational class* $L_k(M^{4k+1})$ *is a homotopy invariant.*

We note for example that if $\pi_1(M^5) = Z$ and $p_1(M^5) \neq 0$, then the group $\pi_2(M^5)$ is infinite, although in homologies this may in no way be reflected. The resulting formula makes it possible to define $L_k(M^{4k+1})$ for any homology manifolds.

Corollary 2. *The class* $L_k(M^{4k+2})$ *of a manifold of the homotopy type of the torus* T^{4k+2} *is trivial. The scalar product of* $L_k(M^{4k} \times T^2)$ *with the cycle* $z = [M^{4k}] \times 0$ *is homotopically invariant and equal to* $\tau(M^{4k})$.

It would be mteistma to deal up the question as to whether there exist invariant relations on the stable tangent bundle other than those which are given by the J-functor and Theorem 1 for $n = 4k + 1$ under the assumption that the group π_1 is commutative and $H^{4i}(M^n) = 0$, $i < k$.

Proof of Theorem 1. First we consider the case $m = 1$, $n = 1 + 4k$. In this case the elements z and $\hat{z}$ are indivisible. From the fundamental lemma,

$$\tau(\hat{z}) = \tau(M^{4k}),$$

where $M^{4k} \subset^i \hat{M}$ and $\hat{z} = i_*[M^{4k}]$. On the other hand, $z = p_*\hat{z}$ and

$$L_k(\hat{M}) = p^* L_k(M^{4k+1}).$$

Therefore

$$(L_k(\hat{M}), \hat{z}) = \tau(M^{4k}) = \tau(\hat{z}) - (L_k(M^{4k+1}), z).$$

For $m = 1$ the theorem is proved.

Now we turn to the case $m = 2$. We recall first chat the element z is divisible, where $Dz = y_1 y_2$. The indivisible elements Dy_1, Dy_2 are realized by submanifolds M_1^{n-1} and M_2^{n-1}, and the element z by their intersection

$$M^{4k} = M_1^{n-1} \cap M_2^{n-1}.$$

Consider the covering $p\colon \hat{M} \to M^n$ defined earlier. The manifold $M^{4k} \subset M_1^{n-1}$ defines an indivisible element $z_1 \in H_{4k}(M_1^{n-1})$. By the preceding formula for $m = 1$ we conclude that on $i\colon \hat{M}_1^{n-1} \subset \hat{M}$, covering M_1^{n-1}, there is one cycle $\hat{z}_1$ such that

$$\tau(\hat{z}_1) = (L_k(M_1^{n-1}), z_1).$$

The transformation $T_2\colon \hat{M} \to \hat{M}$ is such that the fundamental lemma may be applied to the ball $\hat{M} \supset \hat{M}_1^{n-1}$ and to the elements $\hat{z}_1, i_*\hat{z}_1$. Thus

$$\tau(\hat{z}_1) = \tau(i^*\hat{z}_1).$$

Accordingly

$$\tau(i_*\hat{z}_1) = (L_k(M_1^{n-1}), z_1) = \tau(M^{4k}).$$

But $i_*\hat{z}_1 = \hat{z}$ and $\tau(M^{4k}) = (L_k(M^n), z)$, so that Theorem 1 results for the indivisible cycle z. The theorem is proved. □

Now suppose that $z = \lambda z'$ and $Dz = y_1 y_2$, where y_1, y_2 are indivisible elements of the group $H^1(M^n, z)$. As earlier, we suppose that $M^{4k} = M_1^{n-1} \cap M_2^{n-1}$ and that on M_1^{n-1} and M_2^{n-1} the manifold M^{4k} realizes respectively the elements z_1 and z_2. If at least one of z_1 or z_2 is indivisible, then the preceding considerations retain their force. Moreover, if $z_1 = \lambda_1 z_1'$ and $z_2 = \lambda_2 z_2'$, then for $\hat{M}_1^{n-1}$ and $\hat{M}_2^{n-1}$ we will have

$$(L_k(M_l^{n-1}), z_l) = \lambda_l (L_k(M_l^{n-1}), z_l') = \lambda_l \tau(\hat{z}_l') = \lambda_l \tau(i_{l*}\hat{z}_l') = \lambda_l \tau(\hat{z}).$$

since $\tau(\mu\hat{z}) = \tau(\hat{z})$ for $\mu > 0$, $l = 1, 2$, $\lambda_l > 0$. Therefore $\lambda_1 = \lambda_2$, if $\tau(\hat{z}) \neq 0$. Thus the cycles z_1 and z_2 are divisible by one and the same number $\mu = \lambda_1 = \lambda_2$.

Remark. M^{4k} divides each of the manifolds M_1^{n-1} and M_2^{n-1} into exactly μ pieces, respectively $a_1, \ldots, a_\mu$ and $b_1, \ldots, b_\mu$, where

$$M_1^{n-1} = \bigcup_j a_j$$

and

$$M_2^{n-1} = \bigcup_j b_j.$$

The pieces a_j and b_j are cyclically ordered. Therefore the boundary of each of those pieces divides into two pieces ∂_j' and ∂_j'' for a_j and δ_j' and δ_j'' for b_j, passing one after another in cyclic order.

From the preceding we obtain the following theorem.

Theorem 2. *If the element $z \in H_{4k}(M^n, Z)$ is divisible by λ, where $Dz = y_1 y_2$, and y_1, y_2 are indivisible elements of the group $H^1(M^n, Z)$, then the scalar product $(L_k(M^n), z)$ is equal to $\mu\tau(\hat{z})$, where $\hat{z}$ is a canonical element and μ is divisor of λ.*

Corollary 3. *If $\tau(\hat{z}) = 0$, then the scalar product $(L_k(M^n), z)$ is homotopically invariant and is equal to zero. Since z/λ is an integral indivisible class, then*

$$(L_k(M^n), z/\lambda) = \mu/\lambda\tau(\hat{z}).$$

If $\tau(\hat{z})$ is mutually prime with λ, $\mu = \lambda$. The scalar product $(L_k(M^n), z)$ may take on only a finite number of values, equal to $\mu_i\tau(\hat{z})$, where μ_i are divisors of the number λ.

Remark. Here it was shown that if we have two indivisible cycles $M_1^{n-1}, M_2^{n-1} \subset M^n$, $n = 4k + 2$, and their intersection is divisible by λ, and is not equal to zero, then in each of them this cycle, the intersection of z, is divisible by one and the same number μ, under the condition that $\tau(\hat{z}) \neq 0$. Moreover,

$$\mu = (L_k(M^n), z)/\tau(\hat{z})$$

and therefore μ is topologically invariant (see the preceding section). Is it possible to prove that always $\mu = \lambda$?

Example 1. In connection with Theorem 1 there may arise the legitimate question: why is the formula $(L_k(M^n), z) = \tau(z)$ not true, rather than the formula $(L_k(M^n), z) = \tau(\hat{z})$? *A priori* it would be natural to expect just such a formula.

In connection with this I wish to show on the simplest examples that such a formula is false "as a rule". We shall say that the manifold M_1^n has the "homology type" of M_0^n if there exists a mapping $f\colon M_1^n \to M_0^n$ inducing an isomorphism of all the homology groups.

We consider $M_0^n = S^1 \times S^{4k}$ and show that there exist infinitely many manifolds M_0^n of the homology type of $S^1 \times S^{4k}$ and with distinct Pontrjagin classes $p_k(M_i^n)$ such that $\pi_1(M_i^n) = Z$ and all $\pi_l(M_i^n) = 0$, $1 < l < 2k$. Moreover, for $k \geq 2$, among the manifolds M_i^n there are those for which the class $p_k(M_i^n)$ is fractional and therefore they are homotopically nonequivalent to smooth manifolds.

Consider the functor $J_{PL}(M_0^n)$ and select a stable microbundle η_{PL}, J-equivalent to the trivial bundle. We form its Thom complex T_N. Since the fundamental cycle for it is spherical, we may by the now customary method reconstruct the preimages under the mapping $S^{N+n} \to T_N$, pasting together the kernel of the mapping onto π_1, all the groups of this preimage up to $l = 2k-1$ and the kernel of the mapping in dimension $l = 2k$, but only in homologies. We obtain a preimage M_i^{4k+1} the given "normal" microbundle. Since the functor J_{PL}^0 is finite, we have also obtained the required result—the class p_k may be varied very freely. By the Poincaré duality principle, the homology type of the preimage M_i^{4k+1} is the one that we need.

Example 2. In an analogous way we now show that in the portion of Theorem 1 touching on co-dimension 2, it is not possible to remove the restriction of finite-dimensionality of the group $H_{2k+1}(\hat{M}, R)$.

Consider the direct product of $T^2 \times S^{4k}$ and its J-functor. We again select a J-trivial bundle over $T^2 \times S^{4k}$ and denote its Thom complex by T_N. We select an element $\alpha \in H^{-1}[T_N]$ and representative $f_\alpha\colon S^{N+n} \to T_N$ of the element α. We may find a Morse reconstruction over

$$M_\alpha^n = f_\alpha^{-1}(T^2 \times S^{4k})$$

so that

$$\pi_1(M_\alpha^n) = Z + Z$$

and

$$\pi_i(M_\alpha^n) = 0, \quad i \leq 2k.$$

However if we chose a J-trivial bundle such that $p_k \neq 0$, we would have

$$p_k(M_\alpha^n) \neq 0,$$

and at the same time $\tau(\hat{z}) = 0$, since

$$H_{2k}(\hat{M}) = \pi_{2k}(\hat{M}) = 0,$$

where $\hat{M}$ is a universal cover of the manifold M_α^n. Therefore we can deduce that $\pi_{2k+1}(\hat{M}) = H_{2k+1}(\hat{M})$ is of infinite type given that $p_k(M_\alpha^n) \neq 0$.

§ 4. Theorem on topological invariance

We consider a cycle $x \in H_{4k}(M^n, Z)$ for $n = 4k + 2$ such that $(Dx)^2 = 0$ mod Tor. Under these conditions we have

Theorem 3. *The scalar product $(L_k(M^n), x)$ is a topological invariant. Here we may suppose that M^n is a complex which is a homology manifold over Q.*

Proof. We find an integer λ such that $(D(\lambda x))^2 = 0$. We realize the cycle λx by the submanifold $M^{4k} \subset M^n$. As is known, under these conditions the normal bundle to M^{4k} in M^n is trivial. The inclusion $M^{4k} \times R^2 \subset M^n$ is defined, and is an open neighborhood $U = M^{4k} \times R^2$ of the manifold M^{4k}. Evidently

$$(L_k(M^n), x) = \frac{1}{\lambda}\tau(M^{4k}).$$

Now we select on the manifold another smooth (or PL) structure. We denote the class in the new smooth (or PL) structure by $L'_k(M^n)$. We shall show that

$$(L'_k(M^n), \lambda x) = \tau(M^{4k}).$$

The new structure induces a structure on the neighborhood $U = M^{4k} \times R^2$ and the neighborhood $W = U \setminus (M^{4k} \times 0)$, since U and W are open. W is homeomorphic to $M^{4k} \times S^1 \times R$. The coordinate along S^1 will be denoted by ϕ, and the coordinate along R by t. The system of coordinates (m, ϕ, t) is not smooth in the new smoothness, $m \in M^{4k}$. Evidently $H_{4k+1}(W) = Z$ is also generated by the cycle $M^{4k} \times S^1 \times 0$. We realize this cycle by a smooth submanifold $V^{4k+1} \subset W$ in the new smoothness. There is defined a projection of degree $+1$:

$$f\colon V^{4k+1} \to M^{4k} \times S^1, \quad \hat{f}\colon \hat{V} \to M^{4k} \times R.$$

Therefore on V^{4k+1} there is at least one $4k$-dimensional cycle $z \in H_{4k}(V^{4k+1})$ such that $f_* z = [M^{4k}_* \times 0]$. However, the scalar product $(L_k(V^{4k+1}), z)$ does not depend on the choice of such a cycle z.

We consider the covering $p\colon \hat{W} \to W$, under which all the paths on $M^{4k} \times 0$ are preserved, as closed. Evidently $\hat{W}$ is homeomorphic to $M^{4k} \times R \times R$. The complete preimage $\hat{V} = p^{-1}(V^{4k+1})$ also covers V^{4k+1} with the same monodromy group. There exists one invariant cycle $\hat{z} \in H_{4k}(\hat{V})$ such that

$$f_* p_* \hat{z} = [M^{4k} \times 0]\ ^3, \quad \hat{z} = D\hat{f}^* D[M^{4k}].$$

From Theorem 1 we conclude that

$$\tau(\hat{z}) = (L_{4k}(V^{4k+1}), p_*\hat{z}) = (L'_k(M^n), \lambda x).$$

Since $V = V^{4k+1}$ is compact, we may suppose that $\hat{V}$ lies between the levels $t = 0$ and $t = 1$ on $\hat{W}$.

Consider the (nonsmooth) transformation $T'\colon \hat{W} \to \hat{W}$ such that

$$T'(m, \phi, t) = (m, \phi, t + 1).$$

The inclusion $\hat{V} \subset \hat{W}$ will be denoted by i. Evidently $T'_* i_* \hat{z} = i_* \hat{z}$ and the group $H_{2k+1}(\hat{W}) = H_{2k+1}(M^{4k})$ is finite-dimensional. In view of the fundamental lemma, we conclude that

$$\tau(\hat{z}) = \tau(i_*\hat{z}).$$

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[3] The cycle $z = p_*\hat{z} \in H_{4k}(V)$ is obtained by intersecting $(M^{4k} \times 0 \times R) \cap V$ and V from the homology point of view. The same is true for $\hat{z}$ on $\hat{V}$.

However $i_*\hat{z}$ realizes the cycle $M^{4k} \times 0 \times 0$ on $\hat{W} = M^{4k} \times R \times R$. Therefore

$$\tau(i_*\hat{z}) = \tau(M^{4k}).$$

Since $\tau(\hat{z}) = (L'_k(M^n), \lambda x)$, we have also found that

$$(L'_k(M^n), \lambda x) = \tau(M^{4k}).$$

The theorem is proved. □

Remark. Rohlin drew my attention to the fact that for the manifold $V = V^{4k+1} \subset^i W$, constructed in the proof of Theorem 3, one has a cycle $z \in H_{4k}(V, Z)$ such that

$$\tau(i_*z) = \tau(z) = \tau(M^{4k}).$$

This shows that $\tau(z) = \tau(\hat{z})$ for the case at hand, which, generally speaking, is not true for arbitrary $(4k+1)$-dimensional manifolds, as was already shown in § 3 on simple examples. However it is interesting that we are all the same forced to turn to coverings, since the formula proved in § 3 for $L_k(V)$ refers to the cycle $\hat{z}$, and we use it in the proof.

§ 5. Consequences of the theorem on topological invariance

We collect in this section some consequences of Theorem 3. Obviously one has the following corollary.

Corollary 4. *The class $L_k(M^{4k+2})$ is topologically invariant on the subgroup $H \subset H_{4k}(M^{4k+2})/\mathrm{Tor}$, which admits a basis $x_1, \dots, x_s \in H$ such that $Dx_j^2 = 0 \mod \mathrm{Tor}$. Here M^{4k+2} is a smooth (or PL) manifold. For example, for an M^{4k+2} which is a direct product of any collection of spheres, this is always so.*

Now suppose that M^{4k+2} is any simply-connected manifold for which the subgroup $H \subset H_{4k}(M_i^{4k+2})/\mathrm{Tor}$ is nontrivial. Since the functor $J^0_{\mathrm{PL}}(M^{4k+2})$ is always finite, we may apply the "theorem of realization" of tangent bundles and obtain an infinite collection of PL-manifolds M_i with distinct values of the class $L_k(M_i^{4k+2})$ on the subgroup H, so that there does not exist any mapping $M_i^{4k+2} \to M_j^{4k+2}$ carrying a class into a class. If we wish to obtain smooth manifolds, then we must use the functor $J^0 = J^0_{S_0}$. Here, however, for $k \neq 1, 3$ we are obstructed by the Arf-invariant of Kervaire (for these results see [2], § 14 and Appendices I and II). This may be avoided if instead of M^{4k+2} one chooses the homotopy type $M^{4k+2} \# M^{4k+2}$ (in the class of PL-manifolds the Arf-invariant does not obstruct the construction of such manifolds). Thus one obtains the following theorem.

Theorem 4. *If the subgroup $H \subset H_{4k}(M^{4k+2}, Z)/\mathrm{Tor}$ for the simply-connected manifold M^{4k+2} is nontrivial, then there exists an infinite family of PL-mamfolds of homotopy type M^{4k+2}, nonhomeomorphic to one another. If $n = 6$ or 14, this is also true in the class of smooth manifolds. In the class of smooth manifolds there exists an infinite collection of pairwise nonhomemorphic manifolds of homotopy type $M^{4k+2} \# M^{4k+2}$.*

If for example $M^{4k+2} = S^2 \times S^4$, then for $k \geq 2$ one may indicate among these manifolds those which will have a fractional Pontrjagin class, and accordingly will be nonhomeomorphic to smooth manifolds, although their homotopy type is $S^2 \times S^{4k}$.

Remark. For $S^2 \times S^4$ such manifolds may be obtained by the Morse reconstruction over different Haeflinger nodes $S^3 \subset S^6$. If we choose these manifolds for the type $S^2 \times S^{4k}$ and carry out the Morse reconstruction over S^2, then for distinct values of the class p_k we will obtain distinct nodes $S^{4k-1} \subset S^{4k+2}$.

We define the concept of a "topological node with a trivial microbundle". Namely, it consists of the inclusion

$$S^n \times R^l \subset S^{n+k},$$

and equivalence consists of homeomorphisms with fiberwise preservation of the structure around $S^n \times 0$.

From our results it follows that the nodes

$$S^{4k-1} \times R^3 \subset S^{4k+2}, \quad k \geq 1,$$

distinguished by the class p_k of the reconstructed manifold of homotopy type $S^2 \times S^{4k}$, are nonequivalent as topological nodes taking account of the microbundle.

We note finally that for certain manifolds, for example for the homotopy type $S^2 \times S^{4k}$ and their sums connected with one another, the "Hauptvermutung" follows from Theorem 3. Here the point is that from the results of Appendix II of [2] one may extract the fact that the rational Pontrjagin class in this case is a complete combinatorial invariant. Since it is topologically invariant, we also find by using a simple comparison of invariants that from the existence of a continuous homeomorphism there follows the existence of a piecewise linear homeomorphism. However, no such approximation theorems are proved here. From the homeomorphism, we have used for the proof of the theorem only the fact that open sets, common to both smoothnesses, are smooth open manifolds and carry the same collection of cycles. Moreover, our method makes it possible to define the classes L_k of the topological manifold M^{4k+2}. In essence the proof is only that for an arbitrary introduced smoothness the scalar product of the class L_k with a cycle is one and the same. But this introduction of smoothness is necessary, since it makes it possible to discover a large collection of submanifolds realizing cycles. This is hardly the case for purely topological manifolds.

SUPPLEMENT (V. A. ROHLIN)[4]
Diffeomorphism of the manifold $S^2 \times S^3$

I want to indicate one further application of the theorem on the topological invariance of the class L_k in codimension two:[5] *there exist diffeomorphisms of smooth manifolds, for example diffeomorphisms of the manifold*

$$V = S^2 \times S^3,$$

which are homotopic, but topologic ally are not isotopic.

The following elementary considerations are necessary for the proof. To each mapping $f: V \to V$ there corresponds the composite mapping

$$S^3 \to V \xrightarrow{f} V \to S^2,$$

[4] From a letter of January 20, 1965, from V. A. Rohlin to the author. The letter was an answer to my having sent the note [1], and is published with Rohlin's permission. (This and the footnotes which follow are due to S. P. Novikov.)

[5] I.e. Theorem 3 of the present paper.

where the first arrow denotes the natural mapping of the sphere S^3 onto any fiber $a \times S^3$ of the product $S^2 \times S^3$, and the third the projection of this product onto the first component. The absolute value of the Hopf invariant of this composite mapping is defined by the homotopy class of the mapping f and will be denoted by $\gamma(f)$. The number of homotopy classes of mappings $f\colon V \to V$ with the given value of $\gamma(f)$ is infinite, but becomes finite if we restrict ourselves to classes consisting of homotopic equivalences. In particular, there exists only a finite number of pairwise nonhomotopic diffeomorphisms $f\colon V \to V$ with a given value of $\gamma(f)$.

Now consider the manifold $W = S^2 \times D^4$ with edge V and denote by M_f the smooth manifold obtained from two exemplars of W by pasting by means of the diffeomorphism $f\colon V \to V$. The homology groups of M_f do not depend on f, i.e. they are the same as those of the product $S^2 \times S^4$ (which corresponds to the identity diffeomorphism $V \to V$), and the multiplicative structure of the integer-valued cohomology ring is defined by the formula

$$u_2^2 = \pm\gamma(f)u_4,$$

where u_3 and u_4 are the generators of the groups $H^2(M_f;Z)$ and $H^4(M_f;Z)$. In particular, $\gamma(f)$ is a homotopy invariant of the manifold M_f.

Denote by K the class of all manifolds diffeomorphic to the manifolds M_f, and by K_0 the class of smooth six-dimensional manifolds topologically equivalent to the product $S^2 \times S^4$.

Lemma. $K_0 \subset K$.

Proof. Suppose that $M \in K_0$. Then the generator of the group $H^2(M)$ is realized by a smooth imbedding of a sphere, and the normal bundle of this sphere, having the invariant homotopy type of the manifold M, is trivial. Accordingly, a tubular neighborhood of this sphere is diffeomorphic to W. If one diffeotopically carries this sphere beyond the limits of this tubular neighborhood, it keeps a trivial normal bundle, and, as is shown by standard calculations, its imbedding into the closed complement of the tubular neighborhood will be homotopic to an equivalence. From Smale's theorem it therefore follows that also this closed complement is diffeomorphic to W, so that $M \in K$. □

Proof of the theorem. Suppose that $M_1, M_2, \dots$ are manifolds lying in K_0 and pairwise nonhomeomorphic. [6]

From the lemma, there exist also diffeomorphisms $f_n\colon V \to V$ such that the manifolds M_n and M_{f_n} are diffeomorphic. Since $\gamma(f)$ is a homotopy invariant of the manifold M_f, we have $\gamma(f_n) = 0$, and since there are only a finite number of pairwise nonhomotopic diffeomorphisms

$$f\colon V \to V \quad \text{with} \quad \gamma(f) \to 0,$$

it follows that there exist indices k, l such that the diffeomorphisms f_k and f_l are homotopic. They are not isotopic, and moreover the diffeomorphism $f_k f_l^{-1}\colon V \to V$ does not extend to a homeomorphism of the manifold W, since in the contrary case the manifolds M_{j_k} and M_{j_l} would be homeomorphic. □

This proof can be made more effective, replacing the rough considerations of finiteness by a precise homotopy classification of diffeomorphisms of the manifold V.

[6]See § 5 of the present paper.

One can also give a complete homotopy and differential classification of manifolds of class K. As to the topological classification, it coincides with the differential (as holds for manifolds of the class K_0) if the class $p_1(M_f)$ is topologically invariant. The obvious generalization of the preceding lemma shows that the class K contains all the smooth six-dimensional manifolds homotopically equivalent to the total manifolds of orthogonal bundles with basis S^4 and fiber S^2.

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