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TOPOLOGICAL INVARIANCE OF RATIONAL PONTRJAGIN CLASSES

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The object of this paper is to prove the following theorem:

Theorem 1. *Let M_1^n and M_2^n be two smooth manifolds, or PL manifolds, and let $h: M_1^n \to M_2^n$ be a continuous homeomorphism. Then $h^* p_i(M_2^n) = p_i(M_1^n)$, where $p_i(M^n)$ are the Pontrjagin classes, with rational or real coefficients, of a manifold M^n.*

In this paper we give a sketch of the proof of this theorem for smooth simply-connected manifolds M_1^n and M_2^n; the non-simplyconnected case reduces to the simply-connected case. We remark that the method used in a previous paper [2] by the present author, where this proposition was proved in a special case, differs from that presented here, although the ideas in the two papers are closely related.

Theorem 1 follows from the following fundamental lemma:

Fundamental lemma. *Suppose that we are given a smooth structure on the Cartesian product $M^{4k} \times R^m$, making $M^{4k} \times R^m$ into a smooth open manifold W (here M^{4k} is a closed compact simply-connected manifold). Then the following formula holds:*

$$(L_k(W), [M^{4k}] \otimes 1) = \tau(M^{4k}).$$

Here L_k is the Hirzebruch polynomial in the Pontrjagin classes of the manifold W, and $\tau(M^{4k})$ is the signature of the manifold M^{4k}.

The proof is divided into several steps.

Step 1. We take the open subset $T^{m-1} \times R \subset R^m$, where T^{m-1} is the $(m-1)$-dimensional torus. We consider the open submanifold

$$W_1 = M^{4k} \times T^{m-1} \times R \subset W.$$

The following lemma is not difficult.

Lemma 1. *There exists a smooth closed submanifold $V_1 \subset W_1$ representing the cycle $[M^{4k} \times T^{m-1}] \otimes 1 \in H_{4k+m-1}(W_1)$, such that the natural projection $p: V_1 \to M^{4k} \times T^{m-1}$ induces isomorphisms of homotopy and homology groups in dimensions $< 2k + [(m-1)/2]$.*

Similarly, let W be a smooth manifold having the homotopy type of $M^{4k} \times T^m$, and let W_1' be a covering having the homotopy type of $M^{4k} \times T^{m-1}$. Then there exists a natural projection $f: W_1' \to M^{4k} \times T^{m-1}$ of degree +1; and $H_{4k+m-1}(W_1') = Z$.

Lemma 1$'$. *There exists a smooth submanifold $V_1' \subset W_1'$ representing the fundamental cycle of the group $H_{4k+m-1}(W_1') = Z$, such that the natural projection $f: V_1' \to M^{4k} \times T^{m-1}$ induces isomorphisms of homotopy and homology groups in dimensions $< 2k + [(m-1)/2]$.*

The proofs of the two lemmas are similar. They are accomplished by performing successive Morse surgeries on the kernel of the inclusion $V_1 \subset W_1$ or $V_1' \subset W_1'$, similarly, for example, to [1,3]. Here we make the additional remark that, since the group ring of an abelian group is noetherian, all homotopy kernels of the mapping f are finitely generated as $Z(\pi_1)$-modules, where $\pi_1 = Z + \cdots + Z$.

Thus in the dimensions indicated the homotopy kernels of the mapping f can be killed by surgery. We note that the mapping of universal coverings $\hat{f}\colon \hat{V}_1 \to M^{4k} \times R^{m-1}$ has degree $+1$ and is a proper map. Therefore the formula $f_* D f^* D = 1$ holds, where D stands for Poincaré duality. Moreover, in this case we have the following equality, which is not hard to prove:

$$\operatorname{Ker} \hat{f}_*(\pi_i) / Z_0(\pi_1) \operatorname{Ker} \hat{f}_*(\pi_i) = \operatorname{Ker} f_*(H_i)$$

(by the Hurewicz theorem) where $\operatorname{Ker} \hat{f}_*(\pi_j) = \operatorname{Ker} f_*(\pi_j) = 0$ for $j < i$, and $\operatorname{Ker} f_*(\pi_i) = \operatorname{Ker} f_*(H_i)$. This is enough to give Lemmas 1 and 1′.

Step 2. The investigation of dimension $i = [2k + (m-1)/2]$ is more difficult. For this purpose, we make some remarks on intersection numbers. Let $P\colon \hat{M} \to M$ be a covering with monodromy group $\pi\colon \hat{M} \to \hat{M}$. The group π acts on the groups $H_i(\hat{M})$. We consider two cycles $x, y \in H_*(\hat{M})$ and all elements $\alpha_i \in \pi$. Then the formula $(p_* x) \circ (p_* y) = \sum_{\alpha_i \in \pi} x \circ (\alpha_i y)$ holds, where the symbol $\circ$ stands for taking intersection numbers in M and $\hat{M}$ respectively. Suppose that, using Lemmas 1 and 1′, we have manifolds $V_1 \subset W_1$ (or $V_1' \subset W_1'$) together with a projection $W_1 \to M_{4k} \times T^{m-1}$, having degree $+1$ on V_1, which is a homotopy equivalence on W_1 and a homotopy equivalence in dimensions $< [2k + (m-1)/2]$ on V_1. If we let N be the kernel $\operatorname{Ker} f_*(\pi_l)$ for $l = [2k + (m-1)/2]$, then $\operatorname{Ker} f_*(H_l) = N/Z_0(\pi_1) N$, where $\pi_1 = Z + \cdots + Z$. An analogous formula holds for mappings of all intermediate coverings of V_1 and of $M^{4k} \times T^{m-1}$ which correspond to subgroups $\pi' \subset \pi_1$. If π' is of finite index in π_1, by the properties of mappings of degree $+1$ the kernels of the induced homomorphisms of homology groups of the corresponding coverings satisfy the usual Poincaré duality.

As we already remarked, intersection numbers in the homotopy kernels of π'-coverings can be obtained from intersection numbers in N by the following formula:

$$(qx) \circ (qy) = \sum_{\alpha_i \in \pi'} x \circ (\alpha_i y), \quad q\colon N \to N / Z_0(\pi') N.$$

It is also important to note that in this case the kernels of the homotopy homomorphisms for all the coverings split into the sum of the "right kernel" and the "left kernel" from V_1 to W_1 (or from V_1' to W_1').

We now proceed as follows:

1. For odd values of the integer $2l = 4k + m - 1$, by performing surgery on V_1 inside W_1 (only to one side of V_1) operating on a $Z(\pi_1)$ basis of the "right kernel" in homotopy, we can get the "right kernel" in homotopy to be trivial in V_1 in dimension $l = [2k + (m-1)/2]$. Then, (similarly to [1]) the homotopy kernels in this dimension will be free on all coverings with finitely many sheets.

2. For even values of $2l = 4k + m - 1$ one can show that $N/N_\infty = F \oplus \ldots \oplus F$, where N_∞ is the kernel of projection onto all coverings with finitely many sheets and $F = \sum_{\alpha_i \in \pi'} \alpha_i Q$, $\dim Q = 2$; the intersection matrix for Q is of the form $\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$ and $\alpha_i Q \cdot \alpha_j Q = 0$ for $\alpha_i \neq \alpha_j$, where π' is a subgroup of finite index in π_1.

3. We continue in both cases ($4k + m - 1$ even and odd) by performing surgery on the basis elements of the $Z(\pi_1)$-module N to get the kernels of the homotopy mappings on all coverings with finitely many sheets to be trivial, without changing the homotopy in dimensions $< l$.

4. However, at this stage the surgeries are already being carried out not inside the manifold W_1 (or W_1'), but abstractly on a π'-covering.

It is easily seen that after these operations the scalar product of the class $L_k(V_2)$ with the cycle $f_*^{-1}([M^{4k}] \otimes 1)$ is still the same as the scalar product $(L_k(W_1), f_*^{-1}[M^{4k}] \otimes 1)$, since this cycle lies in V_1 and V_1 has a trivial normal bundle in W after the last modification.

Step 3. The following fact is of the greatest importance: after the last modification, we obtain $N = \text{Ker } f_*(\pi_l) = 0$.

This is not obvious, since all we knew was that $N/N_\infty = 0$, where N_∞ is the kernel of all projections on coverings with finitely many sheets. Since the ring $Z(\pi_1)$ is noetherian, the module N is finite-dimensional over $Z(\pi_1)$. By the Hurewicz theorem, the module N_∞ can be defined here in a purely algebraic manner. By an easy argument the problem can be reduced to one-dimensional modules in the case $\pi_1 = Z$, when it takes the following form: there exists a polynomial $P(x)$, with integer coefficients, such that $P(0) \neq 0$ and $P(1) = 1$ and for any root of unity $\zeta^i = 1$ the value $P(\zeta)$ is invertible among the integral elements of the corresponding cyclotomic field. One must show that $P(x) \equiv 1$. This proposition is proved using elementary algebraic number theory.* A similar assertion holds for polynomials mod p.

It follows that, after carrying out modifications on V_1 (or V_1'), we obtain $\text{Ker } f_*(\pi_l) = 0$.

Therefore the modified mapping $f\colon V_1 \to M^{4k} \times T^{m-1}$ is a homotopy equivalence in dimensions $\leq [2k + (m-1)/2]$. Using Poincaré duality, the Hurewicz theorem and the proposition on polynomials mod p stated above, one can show that the mapping f is a homotopy equivalence in all dimensions.

Step 4. We are now ready for the fundamental lemma.

We consider $W = M^{4k} \times T^{m-1} \times R$ with some smoothing, and using the preceding work we construct a manifold V_1 having the homotopy type of $M^{4k} \times T^{m-1}$ such that its class $L_k(V_1)$ is the same as $L_k(W)$.

Furthermore, we consider the covering of V_1 with group Z: $V_1 = W_2 \to V_1$, and a mapping f_2: $V_1 \to M^{4k} \times T^{m-2}$. Using our preceding work, from $V_1 = W_2$ we can obtain a similar manifold V_2 having the same class $L_k(V_2)$ together with a mapping $V_2 \to M^{4k} \times T^{m-2}$ of degree $+1$, and so forth, until we reach dimension $4k$. We obtain a manifold V_m of dimension $4k$ having the homotopy type of M^{4k} and whose class $L_k(V_m)$ is the same as $L_k(W)$; more precisely, the scalar product of the class L_k with the corresponding cycle is the same. Using Hirzebruch's formula, we conclude that $L_k(V_m) = (L_k(W), f_*^{-1}([M^{4k}] \otimes 1)) = \tau(M^{4k})$. This gives the fundamental lemma.

Remark 1. The author's method does not, so far, give a means of defining Pontrjagin classes of topological microbundles and of topological manifolds, since the whole argument was based on a smooth structure; how such a definition can be given constitutes the most immediate problem suggested by the results of this paper.

Remark 2. The proof of invariance seems somewhat artificial to the author, since the surgeries were obviously not essential. It should become clearer when the connection between the classes L_k and coverings is explained; this is shown in a special case by the proof in [2], and by hitherto unpublished results concerning these problems which have been obtained by the present author and by V. A. Rohlin.

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