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HOMOTOPIC AND TOPOLOGICAL INVARIANCE OF CERTAIN RATIONAL CLASSES OF PONTRJAGIN

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We will consider rational (or real) classes of Pontrjagin $p_i \in H^{4i}(M^n, R)$ of closed orientable manifolds. The formula of Thom-Rohlin-Hirzebruch

$$(L_k(p_1, \ldots, p_k),\ [M^{4k}]) = \tau(M^{4k})$$

is well known, where L_k is the polynomial of Hirzebruch and τ is the index of the manifold. In questions of invariance of classes only the following was known:

1. For homotopic invariance there is only the formula $(L_k, [M^{4k}]) = \tau(M^{4k})$ for simply-connected manifolds; no other relation of invariance exists (a discussion of this question can be found in [1], supplement 1).

2. For combinatorial invariance there are all classes of invariance (see [3,4]).

3. For topological invariance as shown by Rohlin in 1956 [2], the class $L_k(M^{4k+1})$ is invariant. Later it will be seen (Theorem 1) that the class $L_k(M^{4k+1})$ is even homotopic invariant, so that the result of Rohlin on $L_k(M^{4k+1})$ did not give any relations for the topological invariance of classes of different homotopy. As was determined by Thom and Rohlin [2,4], in questions of invariance it is useful to pass from the classes p_k to the classes L_k.

The results of this section consist of the following:

Theorem 1. *The class $L_k(M^{4k+1})$ is homotopic invariant.*

Theorem 2. *Let $n = 4k + 2$ and let $x \in H_{4k}(M^n, Z)$, be an indivisible element such that $Dx = y_1 \wedge y_2$ where $y_1, y_2 \in H^1(M^n, Z)$ and D is the duality operator of Poincaré. Then the scalar product (L_k, x) is homotopic invariant if the element x has the following properties: we consider a covering $\hat{M} \to M^n$ for which the closure is covered by those paths γ in M^n for which $(\gamma, y_1) = (\gamma, y_2) = 0$; it is required that the homology groups $H_{2k+1}(\hat{M}, R)$ be finite-dimensional.*

The next theorem already relates to essentially topological invariance.

Theorem 3. *Let $n = 4k + 2$, and let $x \in H_{4k}(M^n, Z)$ be an element such that $(Dx)^2 = 0$. Then the scalar product (L_k, x) is a topological invariant.*

Corollary 1. *From Theorem 2 it follows that the class $L_k(T^{4k+2})$ of the torus T^n (or $p_1(T^6)$) is homotopic invariant; moreover, the class $L_k(M^{4k} \times T^2)$ has an invariant scalar product with the cycle $M^{4k} \times O$.*

Corollary 2. *From Theorem 3 it follows that the class of Pontrjagin $p_k(S^2 \times S^{4k})$ is a topological invariant. The class $L_k(M^{4k+2})$ for $M^{4k+2} = M^{4k} \times S^2$ has an invariant scalar product with the cycle $M^{4k} \times O$.*

Corollary 3. *The problem of Hurewicz about the difference between homotopic types and homeomorphisms of closed manifolds was solved by Moise only for $n = 3$ on the basis of the "Hauptvermutung" in the 1950's.*

In addition, the examples for $n = 3$ (lenses) are not simply connected and have different "simple" homotopic type. For $n > 3$ the problem has remained open.

As mentioned, the J-functor $\widetilde{J}_r(X)$ of Atiyah is always final; we can then apply the results of the author and Browder on normal bundles (see [1], §14) for $X = S^2 \times S^{4k}$ and obtain now for $S^2 \times S^4$ an infinite number of homotopic equivalent singly-connected manifolds with distinct classes of Pontrjagin (of homotopic type $S^2 \times S^4$).

Theorem 4. *In each dimension of the form* $4k+2,\ k \geq 1$ *there are an infinite number of nonhomeomorphic manifolds of homotopic type* $S^2 \times S^{4k}$. *For* $k \geq 2$ *among these manifolds there are PL-manifolds which are not homeomorphically smooth and which have fractional class of Pontrjagin* $p_k(M^{4k+2})$.

The last part of the theorem follows from the combinatorial J-functor and the results of [1], supplement 2, generalizing the theorems of Browder and the author on the combinatorial case (J-functor and normal bundles). For $k = 2$ the denominator of the class $p_2(M^{10})$ can be taken equal to 7. In earlier known examples the nonsmoothability was of a coarser, that is, homotopic character.

We shall give a sketch of the proof first of Theorems 1 and 2.

I. We consider a covering $\hat{M} \xrightarrow{p} M^{4k+1}$ for which the only paths γ with closed covering are those that have a null index of intersection with the basis cycle $x \in H_{4k}(M^{4k+1})$. (An analogous cover for M^{4k+2} has already been described in the statement of Theorem 2.) The group Z acts on $\hat{M}$ generated by the mapping $T: \hat{M} \to \hat{M}$. For the case $n = 4k+2$, $Z + Z$ acts on $\hat{M}$ generated by $T_1: \hat{M} \to \hat{M}$ and $T_2: \hat{M} \to \hat{M}$.

Lemma 1. *A cycle* $\hat{x} \in H^{4k}(\hat{M})$ *can be obtained such that* $T_* \hat{x} = \hat{x}$ *for* $n = 4k+1$ *and* $T_{1*}\hat{x} = T_{2*}\hat{x} = \hat{x}$ *for* $n = 4k+2$ *and such that* $p_*\hat{x} = x$. *This element* $\hat{x}$ *is defined uniquely if it is required that* $\hat{x} \otimes [T^l],\ l = 1, 2,$ *is the cycle of all differential spectral sequences of fibres* $M^n \to T^l$ *of dual coverings. We shall call it canonical.*

For any complex X and cycle $z \in H_{4k}(X)$ it is possible to introduce an invariant $\tau(z)$: we consider the quadratic form (y^2, z) on $H^{2k}(X, R)$ and we discard its "degenerate part"; there remains the finite-dimensional nondegenerate quadratic form, and for it we must take an index, and this is $\tau(z)$.

We have already chosen the element $\hat{x} \in H_{4k}(\hat{M})$ in Lemma 1. It is defined by the element $x \in H_{4k}(M^n)$. The fundamental formula is valid for computing the class $L_k(p_1 \cdots p_k)$ by means of the homotopic invariants of the manifold M^n ($n = 4k+1,\ 4k+2$) under the hypotheses of Theorem 1 and 2:

$$(L_k(p_1, \ldots, p_k), x) = \tau(\hat{x}),$$

where $\hat{x}$ is the canonical cycle on $\hat{M}$. This formula can be interpreted as the right generalization of the formula of Hirzebruch $(L_k, [M^{4k}]) = \tau(M^{4k})$ if we count M^{4k} itself as a "trivial covering" of itself and we assume that $\hat{x} = x = [M^{4k}]$. From the formula it is obvious that the number of times a covering must be counted in the calculation of the class of Pontrjagin is equal to the co-dimension of the cycle under study.

We shall indicate one trivial property of the form and of the index $\tau(z)$.

Lemma 2. *Let* $X_1 \subset \cdots \subset X_k \subset \cdots$ *be a chain of finite complexes,* $X = \bigcup X_i$, $j_k: X_1 \subset X_k$. *Let* $z \in H_{4k}(X_1)$ *be an element such that* $j_* z \neq 0$ *in* X. *Then for the largest of the numbers* k *we* $\tau(j_{k*} z) = \tau(j_* z)$. *Moreover, the nondegenerate part of the quadratic form* $[y^2, j_{k*} z]$ *is stabilized for the largest indices* k.

II. We consider $n = 4k + 1$ and we realize the cycle $x \in H_{4k}(\hat{M}^n)$ by means of a submanifold. All paths on the submanifold $M^{4k} \subset M^n$ have closed coverings in $\hat{M}$ by the closure, so that the imbedding $j: M^{4k} \subset \hat{M}$ is defined and $TM^{4k} \cap M^{4k} = \emptyset$; moreover, between TM^{4k} and M^{4k} lies a film N, $\partial N = M^{4k} \cup TM^{4k}$. Obviously, $\hat{M} = \bigcup_i T^i N$. We put

$$M_1 = \bigcup_{i\geq 0} T^i N, \quad M_2 = \bigcup_{i<0} T^i N, \quad M_1 \cup M_2 = \hat{M}, \quad M_1 \cap M_2 = M^{4k}.$$

The imbeddings $M^{4k} \subset M_1$ and $M^{4k} \subset M_2$ are designated by j_1 and j_2. We put $J = \operatorname{Im} j^*$, $J_1 = \operatorname{Im} j_1^*$, $J_2 = \operatorname{Im} j_2^* \subset H^{2k}(M^{4k}, R)$. The form $(y^2, [M^{4k}])$ induces forms on J, J_1, and J_2 the indices of which we denote by $\tau(J)$, $\tau(J_1)$, $\tau(J_2)$. Obviously $J = J_1 \cap J_2$. We have trivially

$$\tau(J) = \tau(j_*[M^{4k}]) = \tau(\hat{x}),$$
$$\tau(J_1) = \tau(j_{1*}[M^{4k}]) = \tau(\hat{x}_1),$$
$$\tau(J_2) = \tau(j_{2*}[M^{4k}]) = \tau(\hat{x}_2),$$
$$\hat{x} = j_*[M^{4k}], \quad \hat{x}_1 = j_{1*}[M^{4k}], \quad \hat{x}_2 = j_{2*}[M^{4k}].$$

Lemma 3. $\tau(\hat{x}) = \tau(\hat{x}_1) = \tau(\hat{x}_2) = \tau(J) = \tau(J_1 \cup J_2)$.

The proof is obtained from Lemma 2 and the remark that the transformation T^l allows us to combine arbitrarily large parts M_1, M_2 and M. The conclusion about the index $\tau(J_1 \cup J_2)$ is proved on the basis of the simple algebra of quadratic forms, since the entire nondegenerate part J_i is concentrated on $J = J_1 \cap J_2$, $i = 1, 2$.

Lemma 4. $\tau(J_1 \cup J_2) = \tau(M^{4k})$.

If $\alpha \in H^{2k}(M^{4k}, R)$, where $(\alpha J_1, [M^{4k}]) = (\alpha J_2, [M^{4k}]) = 0$, then the element $\beta = \alpha \cap [M^{4k}]$ is such that $j_{1*}\beta = j_{2*}\beta = 0$. The films extended over β on the right and left define the cycle $\delta \in H_{2k+1}(M, R)$. The compact cocycle $D_k\delta$ in $\hat{M}$ is such that $j^* D_k \delta = \alpha$. Hence it follows that $\alpha \in J$. (The imbedding of compact cohomologies in the usual cohomologies is commutative with j^*.) Hence Lemma 4 follows easily.

Remark. If δ is homologous to zero, then the argument based on the duality of D_k is unnecessary, for then $\alpha \cap [M^{4k}] = 0$. Moreover, it is sufficient that δ be homologous to zero with respect to the modulus of infinity by means of a film having a compact intersection with M^{4k}. In the further applications of such arguments this fact will be very important.

Theorem 1 now follows easily from the lemmas. In fact,

$$(L_k, x) = (L_k, p_*\hat{x}) = (p^* L_k, \hat{x}) = \tau(\hat{x}).$$

III. We turn to Theorem 2. We realize the cycles Dy_1, Dy_2 of the manifolds M_1^{4k+1}, $M_2^{4k+1} \subset M^{4k+2}$ and the cycle x means of the intersection $M_1^{4k+1} \cap M_2^{4k+1} = M^{4k}$. Then we have:

a) over M^{4k} the trivial covering;

b) over M_i^{4k+1} the covering decomposes into an infinite number of coverings with group Z. We put $p^{-1}(M_1^{4k+1}) = \bigcup_q M_q$ where $T_2 M_q = M_{q+1}$, $T_1 M_q = M_q$. On M_q the cycle $t_q \in H_{4k}(M_q)$ is such that $j_{q*} t_q = \hat{x}$ where $j_q: M_q \subset M$, $T_{1*} t_q = t_q$. From the preceding we conclude that $\tau(t_q) = \tau(M^{4k})$.

since M^{4k} realizes t_q on M_q and the film between t_q and $T_1 t_q$ lies between M_q and $T_1 M_q$.

By an analogous reasoning we can show that $\tau(t_q) = \tau(\hat{x})$. In this second step, however, we use the finite-dimensionality of $H_{2k+1}(\hat{M}, R)$ for carrying out the argument of the type of Lemma 4 (this was indicated in the remark on Lemma 4). From the finite-dimensionality it follows that for any cycle $\delta \in H_{2k+1}(\hat{M}, R)$ there is a relation of the type $\delta = \sum_{s=1}^{N(\delta)} \lambda_s T_{2*}^s \delta$, whence we can conclude that the cycle δ is homologous to zero on $\hat{M}$ with respect to the modulus of infinity, so that the film has a compact intersection with $M_q \subset \hat{M}$. The remainder is analogous to the preceding.

Theorem 2 also follows now from the lemmas.

IV. We turn to the topological part. In one smoothness on M^{4k+2} we realize the cycle x by the submanifold with trivial normal bundle $M^{4k} \times R^2 \subset M^{4k+2}$, as may be done by virtue of the condition $(Dx)^2 = 0$. We consider the natural imbedding $M^{4k} \times S^1 \times R \subset M^{4k} \times R^2$, and in any other smoothness the same cycle will be realized in $M^{4k} \times S^1 \times R$. Let us have another smoothness. We realize the cycle $M^{4k} \times S^1$ by means of the smoothness of $W^{4k+1} \subset M^{4k} \times S^1 \times R$. We define the projection f of degree $+1$: $W^{4k+1} \to M^{4k} \times S^1$. We consider a covering over $M^{4k} \times S^1 \times R$, preserving the closure of all paths on $M^{4k} \times O$. It is easily seen that $\hat{M}$ is homeomorphic to $M^{4k} \times R \times R$ and $\hat{N} = p^{-1}(W^{4k+1})$ is a covering over W^{4k+1} with group Z, induced by the transformation $T: N \to N$. One cycle $z \in H_{4k}(N)$ is found such that $T_* z = z$, and $p_* z$ is homologous to $M^{4k} \times O$; moreover, z is homologous to $M^{4k} \times O \times O$ in $\hat{M}$. I assert that $\tau(z) = \tau(M^{4k})$ and $\tau(z) = (L_k(W^{4k+1}), p_* z)$. The second part was already proved earlier. For the first equation we consider the transformation $T': M^{4k} \times R \times R \to M^{4k} \times R \times R$, where $T'(m, s, t) = (m, s, t+1)$, $m \in M^{4k}$; moreover, we assume that W^{4k+1} lies between the levels $t = 0$ and $t = 1$. The transformation T' is possibly not smooth. We denote the imbedding $N = p^{-1}(W^{4k+1}) \subset \hat{M}$ by j. then $T'_*(j_* z) = j_* z$, and also the group $H_{2k+1}(\hat{M}) = H_{2k+1}(M^{4k})$ is invariant relative to T'_*. As above, $\tau(z) = \tau(j_* z) = \tau(M^{4k})$. Hence the statement of the theorem follows, since the normal bundle W^{4k+1} in $M^{4k} \times S^1 \times R$ (in the smoothness in question) is trivial and the class $L_k(W^{4k+1})$ is intersected by the class $L_k(M^{4k} \times S^1 \times R)$ and therefore by the class $L_k(M^{4k+2})$. The proof of the theorem is thus complete.

Note added in proof. At present the author has completely proved the topological invariance of all rational classes of Pontrjagin. A brief account appears in [5].

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