(where T is a word in F_{2n}). The set of automorphisms $\{\widetilde{\alpha}_{2n}\}$ (for a fixed $\alpha_n \in \mathrm{Sp}(n)$). is the same as the set of all the ways in which α_n can be written in $\mathfrak{G}_{2n}$ (taking into account the defining relation Q_{2n}). If two manifolds $\mathscr{M}_1^3$ and $\mathscr{M}_2^3 \in M$ are not diffeomorphic, then $\{\widetilde{\alpha}_{2n}(\mathscr{M}_1^3)\} \cap \{\widetilde{\alpha}_{2n}(\mathscr{M}_2^3)\} = \varnothing$ in $GL(2n)$. Conversely, if $\widetilde{\alpha}_{2n} \in GL(2n)$ preserves $J_{2n} \subset F_{2n}$ $(\widetilde{\alpha}_{2n}(J_{2n}) \subset J_{2n})$, then this automorphism defines a unique manifold $\mathscr{M}^3 \in M$.

We consider in $GL(2n)$ the subgroup of all automorphisms that preserve J_{2n} and denote it by $\widetilde{\mathrm{Sp}}(n)$. Clearly, $\mathrm{Sp}(n)$ lifts to $\widetilde{\mathrm{Sp}}(n)$ in $GL(2n)$. If $\widetilde{\mathrm{Sp}}$ denotes the union $\bigcup_{n=0}^{\infty} \widetilde{\mathrm{Sp}}(n)$; then $\widetilde{\mathrm{Sp}} \subset GL$.

DEFINITION. Let $\mathscr{M}^3 \in M$ and let $\widetilde{\alpha}_{2n} = \widetilde{\alpha}_{2n}(\mathscr{M}^3) \in \widetilde{\mathrm{Sp}}(n) \subset \widetilde{\mathrm{Sp}}$ (for some integer $n \geqslant 0$). Then $\widetilde{\alpha}_{2n}(\mathscr{M}^3)$ (a collection of true reduced words $(W)_{2n}$) is called a strong representation of $\mathscr{M}^3$ or simply a representation of $\mathscr{M}^3 \in M$.

Thus, there is a one-to-one correspondence between strong representations $\widetilde{\alpha}_{2n} \in \widetilde{\mathrm{Sp}}$ and collections $(W)_{2n}$ of true reduced words $\{x_1, \ldots, x_n;\ y_1, \ldots, y_n\}$ (written in $\mathscr{A}_{2n}$) such that

1) the words $\{x_i;\ y_j\}$ $(1 \leqslant i,\ j \leqslant n)$ form a basis of F_{2n};

2) there is an element $T = T(W) \in GL(2n) \subset GL$ such that
$$\prod_{i=1}^{n} (x_i y_i x_i^{-1} y_i^{-1}) = T\,[\prod_{i=1}^{n} u_i v_i u_i^{-1} v_i^{-1}]^{\varepsilon}\, T^{-1} \in J_{2n}.$$

REMARK. Clearly, a manifold $\mathscr{M}^3 \in M$ has infinitely many strong representations. The totality of all strong representations of all manifolds $\mathscr{M}^3 \in M$ splits into an infinite collection of subgroups $\widetilde{\mathrm{Sp}} = \bigcup_{n=0}^{\infty} \widetilde{\mathrm{Sp}}(n)$. We take $\widetilde{\mathrm{Sp}}$ as the list of all manifolds $\mathscr{M}^3 \in M$. We enumerate this list.

§10. A theorem of S.P. Novikov

Theorem on the impossibility of recognizing whether an n-dimensional manifold is an n-dimensional sphere $(n \geqslant 5)$ or whether a contractable domain in $(n+1)$-dimensional Euclidean space with smooth boundary is an $(n+1)$-dimensional disc.

As is well known, for any effectively given sequence of finitely presented groups one can construct in a standard way a $(n+1)$-dimensional manifold $(n \geqslant 4)$ with boundary of the form
$$\mathscr{M}^{n+1}_{\{h,\, g\}} = [D^{n+1} \bigcup_{h_1, \ldots, h_k} D_j^n \times D_j^1] \bigcup_{g_1, \ldots, g_l} D_q^{n-1} \times D^2,$$
where $j = 1, \ldots, k$; $q = 1, \ldots, l$;
$$h_j\colon\ (D_j^n \times \partial D_j^1) \to \partial D^{n+1},$$
$$g_q\colon\ D_q^{n-1} \times \partial D_q^2 \to \partial\, [D^{n+1} \bigcup_{h_1, \ldots, h_k} D_j^n \times D_j^1]$$

and the glueing is done according to these maps with the standard smoothing. The fundamental group $\pi_1(D^{n+1} \bigcup_{h_1, \ldots, h_l} D_j^n \times D_j^1)$ is free, and $g_1, \ldots, g_l$ are relations between the free generators $h_1, \ldots, h_k$ of $\pi_1(\mathscr{M}^{n+1}\{h, g\})$, $g_q = h_{i_1}^{\alpha_1^{(q)}} \circ h_{i_2}^{\alpha_2^{(q)}} \circ \ldots \circ h_{i_s}^{\alpha_s^{(q)}}$.

According to Smale's theorem (see [1]), if $n \geqslant 5$ and $\pi_1(\mathscr{M}^{n+1}_{\{h, g\}}) = 1$, then $\mathscr{M}^{n+1}_{\{h, g\}}$ can be constructed with no handles of index 1: there are elements $\widetilde{g}_1, \ldots, \widetilde{g}_{l-k}$ such that $\mathscr{M}^{n+1}_{\{h, g\}} = \mathscr{M}^{n+1}_{\{0, \widetilde{g}\}} = D^{n+1} \bigcup_{\widetilde{g}_1, \ldots, \widetilde{g}_{l-k}} D_q^{n-1} \times D_q^2$.

We have the obvious lemma.

LEMMA 1. *If $\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(s)}$ is a sequence of finitely generated groups such that $H_1(\pi) = H_2(\pi) = 0$, then one can construct effectively a sequence of manifolds $\mathscr{M}^{n+1}_{(1)}, \ldots, \mathscr{M}^{n+1}_{(s)}$ such that $\mathscr{M}^{n+1}_{(j)}$ is a disc if and only if $\pi^{(j)}$ is trivial.*

PROOF. Suppose that the first step of the construction has already been made: if $\pi^{(j)} = \{h_1, \ldots, h_s;\ g_1^{(j)}, \ldots, g_l^{(j)}\}$, then we have a manifold $\mathscr{M}^{n+1}_{(h, g^{(j)})}$ with group $\pi_1(\mathscr{M}^{n+1}_{(h, g^{(j)})}) = \pi^{(j)} = \pi_1(\partial\mathscr{M}^{n+1}_{(h, g^{(j)})})$. Since $H_2(\pi^{(i)}) = 0$, all the cycles in $H_2(\partial\mathscr{M}^{n+1}_{(h, g^{(i)})})$ by Hopf's theorem, are spherical. Knowing this fact a priori, we realize a free basis of the cycles in $H_2(\partial\mathscr{M}^{n+1}_{(h, g^{(i)})})$ (effectively) by the spheres $S^2_\alpha \times D^{n-2}_\alpha \subset \partial\mathscr{M}^{n+1}_{(h, g^{(i)})}$ and then we glue the handle $D^3_\alpha \times D^{n-1}_\alpha$ by using this imbedding of $\partial D^2_\alpha \times D^{n-2}_\alpha$. By Smale's theorem the lemma is proved when $n + 1 \geqslant 6$.

LEMMA 2. *There is an effective sequence of finitely generated groups $\pi^{(i)}, \ldots, \pi^{(j)}, \ldots$ such that* a) $H_1(\pi^{(i)}) = H_2(\pi^{(i)}) = 0$, b) *it is impossible to recognize in this sequence whether a group is trivial.*

PROOF. $H_1(\pi)$ is effectively computable since $H_1 = \pi/[\pi, \pi]$. Therefore, we may assume by results of Adyan (see [7]) that $H_1(\pi^{(i)}) = 0$, but possibly $H_2 \neq 0$, in a sequence where it is impossible to recognize whether a group is trivial. We prove the fact that for any finitely presented group for which $H_1(\pi) = 0$ one can construct effectively a central extension $\widetilde{\pi}$, where the sequence $1 \to H_2(\pi) \to \widetilde{\pi} \to \pi \to 1$ is exact and $H_2(\widetilde{\pi})$ is trivial. Let $h_1, \ldots, h_k$ be the generators and $g_1, \ldots, g_l$ the relations. We consider a group $\widetilde{\widetilde{\pi}}$ with the generators $h_1, \ldots, h_k$ and relations $A_{jq} = h_j g_q h_j^{-1} g_q^{-1}$. Obviously, there is a map $\widetilde{\widetilde{\pi}} \xrightarrow{\alpha} \pi$ with kernel $H_2(\pi) \times \mathbf{Z}^k$, where $\mathbf{Z}^k$ is free Abelian with generators $\alpha^{-1}(h_1) = \widetilde{h}_1, \ldots, \alpha^{-1}(h_k) = \widetilde{h}_k$. Since $H_1(\pi) = 0$, we can choose elements $b_j = \prod_{j=1}^{l} g_j^{\varkappa_{\gamma j}};\ \gamma = 1, \ldots, k$ such that $b_\gamma = h_\gamma$ in $\pi/[\pi, \pi]$ $(\gamma = 1, \ldots, k)$. We consider the group π with the generators $h_1, \ldots, h_k$ and relations

$$A_{jq} = h_j g_q h_j^{-1} g_q^{-1} = [h_j;\, g_q] = 1, \quad b_\gamma = \prod_{j=1}^{l} g_j^{\varkappa_{\gamma j}} = 1.$$

Using homological algebra it is easy to show that $\widetilde{\pi}$ has the requisite properties. To prove Lemma 2, we take Adyan's sequence of groups $\pi^{(1)}, \ldots, \pi^{(s)}, \ldots$, assuming that $H_1(\pi^{(1)}) = 0$, and construct effectively the required sequence $\widetilde{\pi}^{(1)}, \widetilde{\pi}^{(2)}, \ldots, \widetilde{\pi}^{(s)}, \ldots$ This proves Lemma 2. Our theorem now follows obviously from Lemmas 1 and 2.

Added in proof

Our algorithm for discriminating a standard three-dimensional sphere can be used to construct an algorithm for discriminating a trivial knot in a three-dimensional sphere (or in Euclidean space); This algorithm, in contrast to Haken's well known algorithm, is effective in practice and has a very simple realization on an electronic computer. In particular, there is an estimate of the order of n^4 for the number of operations in this algorithm, where n is the number of vertices of the knot. It is convenient to give the list of knots as the elements of a stable braid group; the braids in their turn correspond to automorphisms of a Riemann surface which give three-dimensional manifolds (see [6]). From Waldhausen's theorem it follows that the relevant automorphism of a Riemann surface is spherical (that is, determines a three-dimensional sphere) if and only if the original knot is trivial. Consequently, the final algorithm for discriminating a three-dimensional knot is as follows:

a) for each simplicially realized knot an element of the braid group is constructed (non-uniquely), represented as a word in its generators σ_i;

b) according to [6] each generator σ_i corresponds to a Dehn operation, which simulated in §8 on nets that is, from the word obtained in terms of the σ_i, a net is constructed (determining a three-dimensional manifold),

c) by means of the algorithm (A) (which is defined on all nets), it is checked whether the net so obtained is spherical.

We note that we describe the set of all knots from the set of all their representations in terms of the braid group; therefore, choosing an element of this group at random determines, generally speaking, a non-trivial knot; the percentage of trivial knots constructed by a random walk is approximately equal to that of representations of a three-dimensional sphere in the class of all representations of all three-dimensional manifolds.

References

[1] J. Milnor, Lectures on the h-cobordism theorem, Univ. Press, Princeton, N.J., 1965. MR **32** # 8352. Translation: *Teorema ob h-kobordizme,* Mir, Moscow 1969. MR **42** # 5272.

[2] Marshall Hall, Jr, The theory of groups, Macmillan, New York 1959. MR **21** # 1996 Translation: *Teoriya grupp*, Izdat. Inost. Lit., Moscow 1962.