Four Lectures: Discretization and Integrability. Discrete Spectral Symmetries

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1 Introduction

In these lectures I am going to consider the integrability phenomenon as a by-product of the hidden symmetry of the spectral theory of some famous linear operators. **Our objective is to apply it to the spectral theory** of these operators. This approach (not pretending to be universal)has indeed worked well since 1974 when the so-called finite gap 1D periodic and quasiperiodic Schrodinger operators and corresponding solutions of KdV were discovered (see [1]). Recently we developed a theory based on the discrete symmetries of the continuous and discrete 1D and 2D Schrodinger operators (see [2, 3]). Some results for the 1D Schrodinger operators were obtained in the works[4, 5, 6, 7].

Going back to the famous discovery of the so-called inverse scattering transform for the KdV equation $u_t = 6uu_x - u_{xxx}$ in 1967 (see [8]), we know that it is based, in fact, on the interpretation of KdV as an isospectral deformation for the 1D Schrodinger operator $L_t = LA - AL, L = -\partial_x^2 + u(x,t)$ (see[9] where an infinite-dimensional commutative group of such deformations was found; people call it the KdV hierarchy).

We call this KdV hierarchy **a continuous spectral symmetry group** for the 1D Schrodinger operator L.

For the rapidly decreasing ("soliton-type") class of functions $u(x,t) \to 0$ when $x \to \pm\infty$, the inverse scattering problem was solved many years ago by Gelfand, Levitan, Marchenko and others. Therefore, the inverse scattering transform was considered as an application of this theory for solving the KdV equation.

However, for the x-periodic functions $u(x,t)$, no good solution of the inverse spectral problem was known before. The approach started in [1] was based on the connection of the 1D Schrodinger operator to KdV type systems ("higher KdV"), generating a KdV hierarchy. It led to the effective solution of inverse spectral problems for the so-called "finite-gap" Schrodinger operators L and to the exact solutions of the nonlinear KdV equation. The spectral theory of finite-gap operators, its connection with Riemann surfaces and completely integrable hamiltonian systems are (at least) as important as the solutions of KdV. So the continuous spectral symmetry group certainly played a fundamental role here.

During the last decade we started to study **discrete spectral symmetries**. In fact, some of these symmetries were known for many years. For example, the substitutions called today "Darboux transformations" for the 1D Schrodinger operator L were invented by Euler in 1742. There analogs for the 2D Schrodinger operators were found by Laplace. The association of the Darboux transformations with KdV was realized in the early 1970s under the name "Backlund transformations". The interesting conjecture concerning the connection of cyclic chains of such transformations with finite-gap periodic potentials was formulated in the work [4] in the 1980s.

However, the studies of the remarkable spectral properties of the low-dimensional Schrodinger operators based on the discrete spectral symmetries started only in 1990s. One can say that these investigations have roots also in the studies of the famous quantum physicists of 1930s and 1940s (Dirac and Schrodinger) who started to work with such transformations in the modern algebraic way and to use some examples of that kind for very important goals.

2 Continuous and discrete spectral symmetries of 1D systems and spectral theory of operators. 1D continuous Schrodinger operator and its discrete analogue

Let us consider a one-dimensional Schrodinger operator $L = -\partial_x^2 + u$. For the construction of the Darboux transformation B_c depending on the constant c, we factorize L in the form

$$L + c = QQ^+ = -(\partial_x + a)(\partial - a) \tag{1}$$

Such a factorization requires a solution for the Riccati equation

$$u + c = a_x + a^2 \tag{2}$$

For the real and bounded function $u(x)$ we can always find a constant c big enough, such that this factorization is possible. We call it **strong factorization**. It depends on the parameter c and also on the solution of the Riccati equation.

Any strong factorization generates a **Darboux transformation** $\tilde{L} = B_c(L)$ of the operator L by the formula:

$$\tilde{L} = Q^+Q, L + c = QQ^+ \tag{3}$$

Lemma 1 *For any solution of the spectral equation $L\psi = \lambda\psi$, the new function $Q^+\psi = \tilde{\psi}$ is a solution of the new spectral equation $\tilde{L}\tilde{\psi} = (\lambda + c)\tilde{\psi}$*

The proof of this lemma is trivial. Let us formulate some useful conclusions:

1 On the formal (local) level, the operator $\tilde{L}$ has "almost" the same eigenfunctions as L except maybe one function: the operator Q^+ has a kernel $Q^+\psi^0 = 0$ or $\psi^0_x = a\psi^0$.

2. Let us assume that we are dealing with Hilbert space $L_2(R)$. The function ψ^0 belongs to this space (i.e. it is square integrable on the real line) if and only if the spectrum of the operator L starts from the point $-c$, i.e. $\lambda \geq -c$, and ψ^0 is a ground state. Therefore there is only one choice of the constant c if operator L is semibounded.

Example 1 . *Let $L = -\partial_x^2 + x^2$ is a quantum oscillator. We have a strong factorization here*

$$L + 1 = -(\partial_x - x)(\partial_x + x); Q^+ = \partial_x + x; \psi_0 = \exp\{-x^2/2\} \in L_2(R)$$

In this case we have also the famous relations $QQ^+ - Q^+Q = -2$. All basic eigenfunctions ψ^n for this operator can be obtained by the iterations $\psi^n = Q^n\psi^0$ with eigenvalues $L\psi^n = (2n+1)\psi^n$.

As we can see for the opposite operator $L' = Q^+Q$ where $Q' = Q^+$, the equation $Q\psi = 0$ leads in this case to the function $\psi'_0 = \exp\{x^2/2\}$ which does not belong to the space $L_2(R)$. The operator L' is positive and strongly factorized but its spectrum does not start from 0 because the "instanton equation" $Q^+\psi = 0$ has no proper solutions.

As it has been well-known for many years in the Theory of Solitons, Darboux transformations generate multisoliton solutions and a more general class of "solitons on the given background". However, only recently their connection with periodic and quasiperiodic finite-gap solutions and finite-gap Schrodinger operators was revealed. Consider now a chain of Darboux transformations

$$\ldots, L_k, L_{k+1}, L_{k+2}, \ldots; L_{k+1} = B_{c_k} L_k = \tilde{L_k} \tag{4}$$

We call chain periodic of the period N if $L_N + \sum c_k = L_0$. These chains were studied in the work [4] assuming all $c_k = 0$. In particular, an interesting conjecture was formulated that for the odd values of $N = 2M+1$ the operators L_k in the periodic chain are the finite-gap ones. This conjecture was proved in the stronger form in [5]: Let $N = 2M+1, \sum c_k = 0$. Then the operator L_k is **an algebraic operator**, i.e. there exist a differential operator A of the order $2M+1$ such that $[L, A] = 0$. According to the result of [1], such operators are finite-gap in the sense of the spectral theory if coefficients are smooth periodic or quasiperiodic. For the case $N = 2M+1, \sum c_k = c \neq 0$ it was proved in [5] that there is a differential operator A of the order $2M+1$ such that

$$LA - AL = cA \tag{5}$$

If all operators L_k in the cyclic chain are smooth, then the spectrum of all of them is equal to the union of N arithmetic progressions with the same difference. We can say that these operators are analogous to the quantum oscillator. For $N = 3$ we get new examples of operators with such remarkable properties of the spectrum. The equation for finding potential reduces in this case to the Painleve' equation. Numerical calculations made by V.Adler in his PhD show that it really has such nonsingular solutions.

Our conclusion is that **even these simple discrete symmetries of the 1D Schrodinger operator on the line lead to new interesting results in the spectral theory.**

For the even values of $N = 2M$ we don't know of any classification of periodic Darboux chains. This problem is open.

The discrete analog of the "soliton-type" theory for the 1D Schrodinger operator appeared many years ago in the theory of the so-called Toda Chain and Discrete KdV systems (see[10, 11, 12]). The operator L here acts on the functions of the discrete variable $\psi_k, k \in Z$. It has a form (for the Toda chain) in terms of the unitary shift operators $T = \exp\{\partial_x\}, T : n \to n+1, T^+ = T^{-1}$

$$L = c_n T + c_{n-1} T^{-1} + v_n; L\psi_n = c_n \psi_{n+1} + c_{n-1}\psi_{n-1} + v_n \psi_n = \lambda \psi_n \quad (6)$$

and reduction $v_n = 0$ for the discrete KdV ([11, 12]).

It is interesting to point out that the reduction to standard classical discretization $c_n = 1, n \in Z$ cannot be recognized in terms of the inverse spectral (scattering) data. It is noninvariant under the time dynamics of any nontrivial isospectral system. As we shall see, it is noninvariant also under the disrete Darboux transformations $B_c^{\pm}$. Therefore we come to the following important **Conclusion: in order to construct a right ("good") discretization of the 1D Schrodinger operator $L = -\partial_x^2 + u(x)$, we need to replace derivative ∂_x by the "covariant shift" operator $c_k T = \exp\{\partial + s(x)\}$ instead of standard shift operator $T = \exp\{\partial\}$; otherwise the class of discretized operators will not have discrete (and continuous as well) spectral symmetry transformations.**

Let us construct them using the strong factorization of the first or of the second type.

The first type discrete Darboux transformation B_c^+ has a form:

$$L = QQ^+ + c; Q = a_n + b_n T; Q^+ = a_n + b_{n-1}T^{-1}; \tilde{L}_+ = Q^+ Q \quad (7)$$

The second type B_c^- is defined in the same way but the role of T and T^{-1} are reversed:

$$L = RR^+ + c'; R = u_n + v_n T^{-1}, R^+ = u_n + v_{n+1} T; \tilde{L}_- = R^+ R \quad (8)$$

The second type transformations are inverse to the first ones.

These transformations were studied in the works [6] and [2], Appendix 2. In particular, we proved that **any cyclic sequence of the first type transformations such that $\sum c_k = 0$ leads to the finite-gap (algebro-geometric) discrete operators with Riemann surfaces of the genus**

no more than half of the period of the chain. However, if both types are involved, the classification of cyclic chains remains unclear. This problem is analogous to the classification of the periodic Darboux chains of the even length for the continuous Schrodinger operator.

Let us present here two interesting examples of the discrete 1D operators discussed from the algebraic point of view in the works [7, 6] and also in [2],Appendix 2 from the viewpoint of the spectral theory.

Example 2 . *Let $L = QQ^+ + c$ and $QQ^+ - Q^+Q = const$. The operators Q, Q^+ can be easily found in the form*

$$Q = 1 + \sqrt{a + bn}T; Q^+ = 1 + \sqrt{a + b(n-1)}T^{-1}$$

However, these operators cannot be real and adjoint to each other on the whole lattice Z because linear function $a + bn$ cannot be positive for all $n \in Z$. We require the "quantization condition" $a/b = m \in Z$ and positivity $a > 0, b > 0$. Consider these operators acting on the subspace H_+ in the Hilbert space $L_2(Z)$ such that $\psi_n = 0$ for $n \leq m$. Let $n = m+k$ and $k > 0$. The operators Q, Q^+, L are well-defined on the space H_+. The ground state $Q^+\psi^0 = 0$ is such that

$$(\psi_k^0)^2 = \frac{b^{-k+1}}{(k-1)!}$$

We can see that it is a Poisson distribution. The eigenfunctions $\psi^l = Q^l\psi^0$ are equal to the so-called Charlet polinomials in the discrete variable k multiplied by the ground state ψ_k^0. Our formula gives a good definition of these polynomials on the half-lattice Z_+ orthogonal corresponding to the Poisson weight. As far as I know, this discrete realization of the Dirac Harmonic oscillator is not mentioned in the traditional literature in quantum mechanics. The eigenvalues, of course, are the same as in the standard realization of the commutation relations: $\lambda_l = lb, l \in Z$.

Example 3 *Consider now a family of operators $L_c = Q_cQ_c^+ + const$ where $Q_c = 1 + ca^nT$, the constant $a \neq 0$ is fixed, $c \neq 0$. We have the following relations*

$$a^2Q_c^+Q_c = Q_{c'}Q_{c'}^+ + D, D = a^2 - 1, c' = ca^2 \tag{9}$$

where a is the same for all operators involved.

Theorem 1 *For $a > 1$ the operator $L = Q^+Q$ acting in the Hilbert space $L_2(Z)$ has a discrete spectrum $\lambda_n = 1 - a^{-2n}, n \geq 0$ for $\lambda < 1$.*

For $a < 1$ the operator $L = Q^+Q$ has a discrete spectrum $\lambda_n = 1 - a^{2n}, n > 0$ for $\lambda < 1$.

In both cases the spectrum is continuous for $\lambda \geq 1$

The investigation of the spectrum of this operator for $\lambda \geq 1$ is not done yet.

For the proof of this theorem, we solve equations $Q_c\psi^0 = 0$ and $Q_c^+\psi^0 = 0$ for all $c \neq 0$. Selecting the cases when our solution belongs to the space $L_2(Z)$, *we apply the "creation operators" $Q_{ca^{-2}}$ and get higher states for all values of c but the actual value of c is shifted every time when we apply the creation operator:*

$$\psi^0_k = (-1)^k c^{-k} a^{-(k-1)(k/2}, k \in Z \tag{10}$$

$$\psi^l = Q_c Q_{ca^2} Q_{ca^4} \dots Q_{ca^{2l-2}} \psi^0 \tag{11}$$

3 2D Schrodinger operator. Discrete spectral symmetries, spectral theory of the selected energy level, and space/lattice discretization

Already in the XVIII century Laplace invented the transformations which we are going to use later as **discrete spectral symmetries associated with one spectral level only.** Let us consider a **Hyperbolic Laplace equation** on the plane x, y:

$$L\phi = \phi_{xy} + A\phi_x + B\phi_y + C\phi = 0 \tag{12}$$

where A, B, C are some known functions. We can present it in the form (**a weak factorization of the first type**)

$$L\phi = (Q_1 Q_2 + 2W)\phi = \{(\partial_x + A)(\partial_y + B) + 2W\}\phi = 0 \tag{13}$$

where $2W = C - AB - B_x$, or in the opposite form (**a weak factorization of the second type**)

$$L\phi = (Q_2 Q_1 + 2V)\phi = \{(\partial_y + B)(\partial_x + A) + 2V\}\phi = 0 \tag{14}$$

where $2V - AB - A_y = C$. So we have $2V - 2W = A_y - B_x = 2H(x, y) = [Q_1, Q_2]$. We call the quantity H **a magnetic field or a curvature** for the operator L. There are natural **Gauge Transformations** for this operator

$$L \to e^f L e^{-f}, \phi \to e^f \phi \tag{15}$$

for any function $f(x, y)$. The quantities W (or V) and H are only invariants of the gauge transformations.

By the **Laplace transformation** we call the following map

$$L \to \tilde{L} = W Q_2 W^{-1} Q_1 + 2W, \phi \to \tilde{\phi} = Q_2 \phi \tag{16}$$

By the **opposite Laplace transformation** we call the following map

$$L \to \tilde{L}' = V Q_1 V^{-1} Q_2 + 2V, \tilde{\phi}' = Q_1 \phi \tag{17}$$

Lemma 2 *For any solution $L\psi = 0$ we have $\tilde{L}\tilde{\psi} = 0$ and $\tilde{L}'\tilde{\psi}' = 0$. These transformations are inverse to each other modulo gauge transformation. For the case of the strong factorization $W = const$ or $V = const$ these transformations transform every eigenfunction $L\psi = \lambda\psi$ of the operator L into the eigenfunction $\tilde{\psi}$ or $\tilde{\psi}'$ for the operator $\tilde{L}$ or $\tilde{L}'$, correspondingly.*

The proof of this lemma is almost obvious: the equation $L\psi = 0$ implies $Q_1\tilde{\psi} = -2W\psi$ by definition. Therefore we have $W^{-1}Q_1\tilde{\psi} = -2\psi$. Applying Q_2 to both sides and multiplying by W after that, we get the desired result. Our lemma is proved for the first type. For the second type the proof is similar. Let us prove now that they are inverse to each other: Performing the second type after the first one , we come to the operator $\tilde{\tilde{L}}' = WLW^{-1}$ Taking $W = \exp\{f\}$ we get a gauge equivalence if $W \neq 0$. Lemma is proved.

Lemma 3 *The Laplace transformations are gauge invariant. In terms of the gauge invariant quantities, they can be written in the form:*

$$\tilde{W} = W + \tilde{H}; \tilde{H} = H + 1/2\partial_x\partial_y \log W \tag{18}$$

Let us demonstrate this here by following a simple but important theorem (in fact, known already in the XIX century to Darboux).

Theorem 2 *Let an infinite Laplace chain is given*

$$\ldots, L_k, L_{k+1}, \ldots : L_{k+1} = \tilde{L_k} \tag{19}$$

Then this chain can be described by the 2D Toda Lattice System and vice-versa.

Proof. Let $W = e^f$. When we have

$$e^{f_{k+1}} = e^{f_k} + H_{k+1}$$

$$H_{k+1} = H_k + 1/2\partial_x\partial_y f_k$$

as a definition of the Laplace chain in terms of gauge invariant quantities. So we exclude magnetic field using the first equation:

$$H_{k+1} = e^{f_{k+1}} - e^{f_k}$$

After substitution of this expression into the second equation and making change of the dependent variables $f_k = g_{k+1} - g_k$, we come exactly to the famous 2D Toda lattice system

$$1/2\partial_x\partial_y g_k = e^{g_{k+1}-g_k} - e^{g_k-g_{k-1}} \tag{20}$$

People in Soliton theory found the complete integrability of this system (see[13]) but did not know about its connection with the 2D Schrodinger equation (or Laplace equation in hyperbolic case).

Already in the XIX century, geometers like Darboux with his pupils and others started to use hyperbolic Laplace transformations for the needs of the theory of surfaces imbedded in the euclidean space R^3. They considered also Laplace chains and periodic chains in particular. Simple calculations show that for the period $N=2$ where $L_2=L_0$, we come to the equation

$$\partial_x\partial_y G(x,y) = -8sinh\{G(x,y)\} \tag{21}$$

For the period $N=3$ assuming that the magnetic field is equal to zero $L_0 = L_3 = \partial_x\partial_y + 2W(x,y)$, we come to the equation

$$\partial_x\partial_y G = e^G - e^{-2G} \tag{22}$$

Both of these systems are well-known in the theory of completely integrable systems and were obtained in completely different way, with no relationship to the 2D (linear) Schrodinger operator.

We are going to apply these ideas to the spectral theory of the 2D Schrodinger operator. Let us consider now the **elliptic Schrodinger operator** written in the weakly factorized form through the complex derivatives $\partial = \partial_x - i\partial_y, \bar\partial = \partial_x + i\partial_y$:

$$L = (-\partial + A)(\bar\partial + B) + 2W \tag{23}$$

We call operator L **physical** if magnetic field $H = 1/2(A_{\bar z} - B_z)$ and potential $W = \exp\{f\}$, both are real. We call the operator **periodic** if both of them are smooth and double periodic on the plane R^2. We call the periodic operator **topologically trivial** if the magnetic flux $[H]$ through the elementary cell K is equal to zero:

$$[H] = \bar H|K| = \int\int_K H(x,y)dxdy = 0 \tag{24}$$

We call the operator **quantized** if $[H] \in 2\pi Z$ From the formulas for the Laplace transformations written through the gauge invariant quantities above, we deduce the following

Lemma 4 *For the smooth physical double-periodic operators we have for the fluxes through the elementary cell:*

$$[\tilde H] = [H]; [\tilde W] = [W] + [H] \tag{25}$$

These changes lead only to the replacement of the operators ∂_x, ∂_y by the complex ones $\partial, \bar\partial$ in all formulas above for the Laplace transformations and Laplace chains. All formal calculations remain unchanged. However, the equations responsible for the periodicity property of chains became elliptic. In the global double periodic problems on the plane R^2, this fact led to the important conclusions (see[2]):

Theorem 3 *Let a periodic elliptic Laplace chain is given such that all 2D Schrodinger operators L_k in this chain are smooth periodic in R^2 and physical. Then all these operators are topologically trivial. All of them have a family of Bloch-Floquet solutions $L\psi = 0$ parametrized by the points of some Riemann surface of finite genus with two marked points ("infinities").These solutions can be found explicitly. This family contains a subfamily of the bounded functions on R^2 providing a basis for the spectral (i.e. energy) level $\lambda = 0$ in the Hilbert space $L_2(R^2)$. This class of 2D Schrodinger operators was invented in the work [15] in 1976.*

Nothing like that exists in the hyperbolic case. The reason for this is that in the smooth elliptic case any nonlinear system on compact manifold (2-torus here) may have only a finite-dimensional family of global solutions. For the 1+1 dimensional completely integrable systems describing the periodicity property of Laplace chains, this fact leads to the linear dependence of the higher flows in the corresponding hierarchy and finally to the Riemann surfaces of finite genus, exactly as it was found for KdV in 1974 (see in [16]).

Example 4 *Let $N = 2$ is the period. We come to the equation*

$$\Delta f_0 = -8sinh\{f_0\}; W_0 = e^{f_0}; f_0 = -f_1; H_0 = 2sinh\{f_0\} \tag{26}$$

Exactly this equation appeared in the theory of the toroidal surfaces in R^2 with constant mean curvature $k_1 + k_2 = const$ (see in[17] the details and the authors of this discovery). It was observed in this theory that all of them can be obtained from the Riemann surfaces of finite genus like in the periodic theory of solitons. Our theorem can be considered as a natural extension of that technical result with a completely different interpretation.

We introduced also a notion of **semi-cyclic chain** $L_0, \dots, L_N$ satisfying to the identity:

$$L_0 = L_N + C \tag{27}$$

The most interesting new class of Laplace chains $L_0, \dots, L_N$ leading to the operators with very specific anomalous spectral properties is the class of **the quasi-cyclic chains** $L_0, \dots, L_N$, such that the boundary operators are strongly factorizable (all factorizations on the boundary are assumed to be the first type, and the Laplace transformations are assumed to be of the second type):

$$L_0 = -(\partial + A)(\bar{\partial} + B), L_N = (\partial + A')(\bar{\partial} + B') + C_N \tag{28}$$

where $C = const$.

Lemma 5 *Both semicyclic and quasicyclic Laplace chains of the length equal to one $N = 1$ lead to the Landau Operator QQ^+ with magnetic field equal to constant $H_0 = const$ and $W_0 = 0, V_0 = H_0$. Let $H_0 > 0$. Its spectrum consists of the infinite number of highly degenerate Landau levels $\Lambda_k, k \geq 0, \lambda_k = kH_0$, isomorphic to each other by the operator Q:*

$$Q = \partial + A(z,\bar{z}) : \Lambda_k \to \Lambda_{k+1}; \tag{29}$$

where $Q^+\Lambda_0 = 0$ *and* $Q^+ = \bar{\partial} + B(z,\bar{z})$

Let a quasicyclic chain of the length N is given, $H_0 > 0$ and all operators $L_k = Q_k^+ Q_k + 2V_k$ in the chain are smooth physical and double-periodic on the plane R^2 where $Q_k = (\partial + A_k)$ and $Q^+ = (\bar{\partial} + B_k)$. It is convenient sometimes in the physical case to choose gauge conditions such that Q and $-Q^+$ are adjoint to each other, i.e. $\bar{A} = -B$. We have always

$$V_0 - 1/2\Delta \log V_0 + H_0 = V_1; H_1 = H_0 - 1/2\Delta \log V_0$$

for the second type Laplace transformation.

Theorem 4 *The operator* $L_N - C_N = Q_N Q_N^+$ *has a highly degenerate space of ground states*

$$Q_N^+ \psi^0 = 0; \lambda_0 = 0$$

isomorphic to the Landau level Λ_0*. It has also a second highly degenerate level* $\Lambda_N; \lambda_N = C_N = N[H_0]$*, isomorphic to the Landau level. The second level can be obtained from the solutions* $Q_0^+ \phi^0 = 0$ *belonging to the space* $L_2(R^2)$*, by the formula:*

$$\psi = (\partial + A_{N-1}) \dots (\partial + A_0)\phi^0$$

The exact elliptic formulas for the functions ϕ^0 and ψ^0 can be extracted from the work [2] for the case where the magnetic flux is quantized. This formula is based on the result of [18] for the strongly factorized Schrodinger operators where these eigenfunctions of the ground level were calculated. The result itself remains true in the case of the irrational fluxes as well, because we may use a completely localized basis in the space of groundstates instead of the magnetic Bloch functions used in these works.

Example 5 *Consider the case* $N = 2$*. The condition of the strong factorization of the boundary operators leads to the equation*

$$\Delta g(z,\bar{z}) = 4e^g - 2C_2; V_0 = H_0 = \exp\{g\} \tag{30}$$

We have also $C_2 = W_2 = V_1$ *and* $H_2 = H_1 = H_0 - 1/2\Delta g = C_2 - H_0$*. We can see that this equation has a lot of periodic smooth solutions depending on one variable. It is not hard to prove that it has a lot of smooth double-periodic solutions essentially dependent on both variables* x, y*.*

4 Discretization of the 2D Schrodinger operators and Laplace transformations on the square and equilateral lattices

In the continuous case all formal calculations for the hyperbolic and elliptic cases were identical. The difference between them originated in the global properties only. For the difference operators these cases look completely different even on the formal level.

I. Let us start with the **hyperbolic case**. The discrete Schrodinger (or Laplace) equation is defined for the function ψ_n where $n = (n_1, n_2)$ on the square lattice $n \in Z^2$ on the plane by the formula:

$$0 = L\psi_n = a_n\psi_n + b_n\psi_{n+T_1} + c_n\psi_{n+T_2} + d_n\psi_{n+T_1+T_2} \tag{31}$$

where $n+T_1 = (n_1+1, n_2); n+T_2 = (n_1, n_2+1)$. The operator L is well-defined modulo **gauge transformations**

$$L \to f_n L g_n; \psi_n \to g_n^{-1}\psi_n \tag{32}$$

where f_n, g_n are nonzero functions.

There exists a unique **weak factorization** of this operator written in the form

$$L = f_n[(1 + u_n T_1)(1 + v_n T_2) + w_n] = f_n[Q_1 Q_2 + w_n] \tag{33}$$

(this is a first type factorization). It generates a (first type) Laplace transformation

$$L \to \tilde{L} = w_n Q_2 w_n^{-1} Q_1 + w_n; \tilde{\psi} = Q_2\psi \tag{34}$$

up to gauge transformation. As in the continuous case, the coefficients u_n, v_n, w_n can be easily found by elementary algebraic formulas. It was observed, in fact, in 1985 (see [19]) that the equation $L\psi = 0$ on the square lattice (above) has a nice family of algebra-geometric exactly solvable cases. Such solvable cases and discrete spectral symmetries normally appear exactly for the same classes of operators.

There are many orthonormal bases T'_1, T'_2 equivalent to each other in the square lattice. We can take any one of them: $(T'_1, T'_2) = (T_i^{\pm 1}, T_j^{\pm 1}$ where $i =\neq j$ and $i, j = 1, 2$. Any choice of basis defines a Laplace transformation

$$L \to \tilde{L}'; \tilde{\psi}' = Q'_2\psi$$

through the weak first type factorization of the form

$$L = f'_n[Q'_1 Q'_2 + w'_n]; Q'_1 = 1 + u'_n T'_1; Q'_2 = 1 + v'_n T'_2 \tag{35}$$

We have a total number of eight for the Laplace transformations defined in this way.

Lemma 6 *The Laplace transformations defined above generate a group with four generators $B_{\pm,\pm}$ corresponding to the bases $T'_1 = T_1^{\pm 1}, T'_2 = T_2^{\pm 1}$. The Laplace transformations correspondent to the basis T'_1, T'_2 are inverse to the Laplace transformation correspondent to the basis T'_2, T'_1 modulo gauge transformations.*

This statement can be checked by elementary calculation.

As in the continuous case, we have gauge invariant quantities.

Lemma 7 *A pair of gauge invariant quantities (the 'discrete curvatures") is defined*

$$K_{1n} = \frac{b_n c_{n+T_1}}{d_n a_{n+T_1}}$$

$$K_{2n} = \frac{c_n b_{n+T_2}}{d_n a_{n+T_2}}$$

All other gauge invariants, including the potential w_n, can be expressed through them. In particular, the potential w_n has a form $K_{1n} = (1+w_n)^{-1}$. A "magnetic field"

$$H_n = \frac{v_n u_{n+T_2}}{u_n v_{n+T_1}}$$

can be expressed through the quantities K_1, K_2. They also can be expressed in terms of w_n, H_n.

As in the continuous case, it is convenient to write Laplace transformation in terms of w_n, H_n:

Lemma 8 *The Laplace transformation can be written in the form*

$$1 + \tilde{w}_{n+T_1} = (1 + w_{n+T_2}) \frac{w_n w_{n+T_1+T_2}}{w_{n+T_1} w_{n+T_2}} H_n^{-1}$$

$$\tilde{H}_n = \frac{1 + w_{n+T_2}}{1 + \tilde{w}_{n+T_2}}$$

For the infinite Laplace Chain

$$\tilde{H}^k = H^{k+1}, \tilde{w}^k = w^{k+1}$$

we can express H^k through w^k, w^{k+1} as in the continuous case. It leads to the completely discrete 2D Toda lattice (it is a discrete 3D system found by Hirota many years ago from completely different ideas)

$$\frac{(1 + w^{k+2}_{n+T_1})(1 + w^{k+1}_{n+T_2})}{(1 + w^{k+1}_{n+T_1})(1 + w^{k}_{n+T_2})} = \frac{w^k_{n+T_1} w^k_{n+T_2}}{w^k_n w^k_{n+T_1+T_2}}$$

Its reduction for the periodic Laplace chains of the length $N = 2$ leads to the nice analog of the sinh-Gordon equation (see [3]). In the discrete case we have a big group of Laplace transformations generated by the four generators (above). This group has not been studied yet.

II. **The elliptic case** is especially interesting. It turns out that in this case the right discretization of the second order elliptic real selfadjoint operators (i.e. operators of the form $L = -\Delta + U(x, y)$) admitting Laplace transformations should be constructed on the **equilateral triangle lattice**. So in this case even the form of discretized elliptic operators has nothing to do with the hyperbolic case described above.

For the equilateral triangle lattice we have a basis T_1, T_2 such that the shift operator $T_1T_2^{-1}$ has the same length. Therefore any vertex $n = (n_1, n_2)$ in the lattice has exactly six closest neighbors $n + T'$ where $T' = T_1^{\pm 1}$ or $T' = T_2^{\pm 1}$ or $T' = (T_1T_2^{-1})^{\pm 1}$. We write a real selfadjoint operator in the form:

$$L = a_n + b_nT_1 + c_nT_2 + d_{n-T_2}T_1T_2^{-1} + adjoint$$

We consider the **zero level gauge transformations** preserving a form of the operator and a zero spectral level $L\psi = 0$:

$$L \to f_nLf_n, \psi_n \to f_n^{-1}\psi_n$$

Lemma 9 *Any real selfadjoint operator of this form with nonzero coefficients* b_n, c_n, d_n *can be presented in the weakly factorized form of the first type*

$$L = QQ^+ + w_n; Q = x_n + y_nT_1 + z_nT_2$$

where $T_i^+ = T_i^{-1}, i = 1, 2; (AB)^+ = B^+A^+$. *This form is unique if the coefficients* c, b, d, x, y, z *are positive.*

Any equivalent basis T_1', T_2' with angle equal to $2\pi/3$ defines the analogous Laplace transformation. There is no difference between the pairs T_1', T_2' and T_2', T_1' in this factorization. So we have six different pairs:

$$(T_1, T_2), (T_2, T_2T_1^{-1}), (T_2T_1^{-1}, T_1^{-1}), (T_1^{-1}, T_2^{-1}), (T_2^{-1}, T_1T_2^{-1}), (T_1T_2^{-1}, T_2^{-1})$$

Lemma 10 *For the nonzero potential* w_n *a Laplace transformation is defined*

$$\tilde{L} = w_n^{1/2}Q_1w_n^{-1}Q_2w^{1/2} + w_n; \tilde{\psi} = w^{-1/2}Q_1\psi$$

and the operator $\tilde{L}$ *is real selfadjoint. The Laplace transformations correspondent to the inverse bases* T_1, T_2 *and* T_1^{-1}, T_2^{-1} *are inverse to each other. Therefore the group of Laplace transformations is generated by three generators.*

In the work [3] we calculated how these three generators can be expressed through the first one and rotations of the lattice. Therefore there is essentially one Laplace transformation only in this case.

Let us consider now a special class of the purely factorizable operators in the strong sense:

$$L = QQ^+ + const$$

("white factorization") or

$$L = Q^+Q + const$$

("black factorization") where $Q = x_n + y_nT_1 + z_nT_2$. Especially interesting here is the case when **the white triangle equation**

$$Q^{+}\psi = 0 \tag{36}$$

for the first case, or **the black triangle equation**

$$Q\psi = 0 \tag{37}$$

for the second case, has nontrivial solutions belonging to the space $L_2(Z^2)$.

Example 6 *Let $Q_{c,d} = 1 + ce^{l_1(n)}T_1 + de^{l_2(n)}T_2$ where l_1, l_2 are the linear forms in the variables n_1, n_2 with real coefficients*

$$l_i = \sum_j l_{ij} n_j; i, j = 1, 2; n_j \in Z \tag{38}$$

Theorem 5 *The black triangle equation $Q\psi = 0$ has an infinite dimensional subspace of solutions belonging to the space $L_2(Z^2)$, if one of the following conditions is satisfied*

$$a) l_{ii} > 0, i = 1, 2; l_{11}l_{22} - l_{12}^2 > 0$$

$$b) l_{ii} > 0, i = 1, 2; l_{11}l_{22} - l_{21}^2 > 0$$

$$c') l_{11} > 0; l_{11}l_{22} - l_{12}^2 > l_{11}(l_{21} - l_{12})$$

$$c'') l_{22} > 0; l_{11}l_{22} - l_{21}^2 > l_{22}(l_{21} - l_{12})$$

The operator $L = Q^{+}Q$ has a zero point $\lambda = 0$ as a point of discrete spectrum in these cases, such that its multiplicity is infinite.

There is also a similar statement for the white triangle equation.

For the proof of the theorem, we make a substitution:

$$\psi_n = e^{-K_2(n)}\eta_n$$

where $K_2(n)$ is a quadratic form in the variables n_1, n_2. After that we assume that coefficients of the equation for the quantity η_n either depend on the variable n_1 only (this is the case a) above) or depend on the variable n_2 only (this is the case b) above) or depend on the variable $n_1 + n_2$ only (this assumption leads to the cases c') or c") above).

In case a) we are looking for the solutions of the form

$$\eta_n = w^{n_2}\phi_{n_1}$$

Let $l_{21} > l_{12}$. We choose the value of $w = w_q$ such that $\phi_{n_1} = 0$ for $n_1 > q; q \in Z$. This assumption leads to the solutions belonging to the space $L_2(Z^2)$. Other cases can be considered in a similar way–see details in [3].

Consider now a special subcase of this example where

$$2l_{11} = 2l_{22} = l_{12} + 1_{21} \tag{39}$$

Lemma 11 *The operators* Q, Q^+ *satisfy to the following relations*

$$Q_{c,d}Q^+_{c,d} - 1 = u^{-2}(Q^+_{c',d'}Q_{c',d'} - 1) \tag{40}$$

where $u = e^{l_{11}}, v = e^{l_{12}}, c' = u^2 c, d' = u^2 d$.

Using these relations and the groundstates found before, we come to the following

Theorem 6 *The spectrum of operators* $L = QQ^+$ *and* $\tilde{L} = Q^+Q$ *under the conditions above, is discrete for* $\lambda < 1$ *and can lie in the following points only:*

$$a)\lambda_j = 1 - u^{2j}, j \geq 0, u < 1$$

$$b)\lambda_j = 1 - u^{-2j}, u > 1$$

In the following cases the spectrum of operator L *occupies all these points, and the spectrum of operator* $\tilde{L}$ *occupies all these points except* $\lambda_0 = 0$

$$u^{-3} > v^{-1} > u^{-1} > 1$$

$$u^{-1} > max(v, v^{-1}) \geq 1$$

The replacement $u \to u^{-1}$ *in these conditions leads to the interchange between* L *and* $\tilde{L}$ *in the theorem. All these levels are infinitely degenerate ("The discrete analogs of Landau levels").*

Nothing is known about the spectrum for $\lambda \geq 1$. *It is certainly continuous. The interesting multidimensional analogs of the operators satisfying to the relation above were found in the work [3] for the multidimensional analogs of the equilateral lattice, but there spectrum is not found yet.*

In the special case $u = 1$ of the example above, we have

$$Q_{c,d}Q^+_{c,d} = Q^+_{c,d}Q_{c,d}$$

Here we should consider both (white and black triangle equations) simultaneously:

$$Q\psi = 0; Q^+\psi = 0$$

This situation can be naturally extended to the more general pair of equations (black and white):

$$Q_1\psi = 0; Q_2\psi = 0$$

This pair leads to the "discrete curvature" making an obstacle for the local existence of solutions around every vertex. These ideas were developed in [3] in a much more general situation.

5 2D manifolds with the colored black-white triangulation. Integrable systems on a trivalent tree.

In the work [3] a theory of Laplace transformations was developed on the **2D manifolds with the colored "black-white triangulation"**. We assume that a color (black or white) is assigned to every triangle in the triangulation such that any triangles with common edge should have opposite colors. The black triangle operator Q can be defined by the field associating number $b_{P:T}$ to the pair P, T where T is a black triangle, and P is its vertex $P \in T$. We define operator Q by the formula

$$\tilde{\psi}_T = Q\psi_T = \sum_P b_{P:T}\psi_{P:T}$$

It maps the space of functions on the set of vertices into the space of functions on the set of black triangles. The factorized operators have a form $L = Q^+Q$; their zero modes satisfy to the black triangle equation

$$Q\psi_T = 0$$

This structure permits to define **combinatorial geodesics** consisting of edges and passing every vertex "as a straight line" (i.e. the numbers of triangles from both its sides should be equal to each other). The right (left) horocycles are such lines that there is exactly three triangles from the right (left) side of it in every vertex. The right (left) curvature of the combinatorial line is measured by the number of triangles from the left (right) side of it in the vertices. This structure imitates somehow conformal geometry. In particular, the black (or white) triangle equation can be considered as reasonable discrete analogs of the complex (covariant) $\partial + A$ and $\bar{\partial} + B$ operators: they factorize the second order elliptic operators (it does not matter that complex numbers are not involved in their definition); they are "more elliptic" than any other first order discrete operators known until now.

Example 7 *Let $b_{P:T} = 1$ for the operator Q. The operator $L = Q^+Q$ can be compared with Laplace-Beltrami operator $L_0 = dd^*$ where d is a standard boundary operator, and d^* is a coboundary operator. If R_P is a number of triangles entering the vertex P, we have*

$$L_0\psi_P = -\sum_{P'}\psi_{PP'} + R_P\psi_P$$

$$L\psi_P = \sum_{P'}\psi_{PP'} + R_P\psi_P$$

Therefore we conclude that there is an equality

$$L = -L_0 + 2R_P$$

The case $R_P = 6$ corresponds to euclidean geometry. In principle, a quantity like R_P corresponds to something like the **scalar curvature**.

Boundary problems of the Dirichlet type for the triangle equations can be posed for the bounded simply-connected domains on the plane with the black-white triangulation. However, careful analysis of the admissible boundary functions is required.

Example 8 *Let me remind here that in the euclidean plane with equilateral lattice Z^2 the black triangles have a form $n, n+T_1, n+T_2$ for all $n \in Z^2$. Consider any lattice straight line Z' dividing Z^2 into the parts*

$$Z^2 = R_+ \bigcup R_-; R_+ \bigcap R_- = Z'$$

where R_+ touches its boundary Z' by the black triangles. Starting with arbitrary data $\phi_n, n \in Z'$, we always can find unique solution to the black triangle equation $Q\psi_n = 0$ in the domain $n \in R_+$ such that $\psi = \phi$ on the boundary. This initial value problem is hyperbolic. However, the initial value problem in the other direction R_- is parabolic: for finding a solution in the domain R_- to the black triangle equation $Q\psi = 0$ such that $\psi_n = \phi_n$ for $n \in Z'$ we should require some decay for the Cauchy data ϕ_n on the line Z'. The operator expressing the solution ψ on any line parallel to R through the initial value ϕ became nonlocal in this domain: you have to integrate along the whole line Z'.

Now let us consider **a plane R^2 with a colored black-white triangulation**. Studying the Dirichlet-type boundary problems, we start with some simply connected bounded triangulated sub-domain D in it with the **thin** boundary polygon $\Gamma = \partial D$. It means that there are no triangles in D whose vertices all belong to the polygon Γ. We call a boundary edge **white** if its white side lies inside of the domain D, otherwise we call a boundary edge **black**. We have

$$|\Gamma| = \Gamma_b + \Gamma_w$$

where Γ_b and Γ_w are exactly the numbers of black (white) boundary edges in Γ.

The elliptic-type Dirichlet boundary problem is to find a solution to the black triangle equation $Q\psi = 0$ in the domain D such that $\psi_P = \phi_P$ on the boundary $P \in \partial D = \Gamma$. It turns out that for the correct solution of this problem we should start with the boundary function ϕ given in some part of the boundary only:

1. The total number of known values $\phi_P, P \in \Gamma$ should be equal to the number $V - T_b$ where V is the number of vertices in D, $T_b(T_w)$ is the number of black (white) triangles, and $T_b + T_w = T, T_w = T_b + \Delta$ by definition.

Lemma 12

$$\Delta = -(\Gamma_b - \Gamma_w)/3$$

$$V - T_b = 1 + (|\Gamma| + \Delta)/2$$

The proof of this statement follows easily from the topology of the plane. Let us denote by the letters $V, E, T = 2T_b + \Delta$ the numbers of vertices, edges, triangles and black triangle T_b correspondingly in the domain D. From the Euler identity and elementary combinatorics we have

$$V - E + T = 1; E = 3/2T + |\Gamma|/2; \Delta = -(\Gamma_b - \Gamma_w)/3$$

The total number of unknown quantities is equal to V. The number of equations is equal to T_b where $T = T_b + T_w, T_w = T_b + \Delta$. So the number Q of independent data should be equal to

$$Q = V - T_b = 1 + (|\Gamma| + \Delta)/2$$

Lemma is proved.

2. For the "elliptic–type" boundary problems the set of known values should never contain both boundary vertices $P_1 \cup P_2 = \partial l$ of any black edge l on the curve $\Gamma = \partial D$. We are going to develop this subject in the next work.

III. Let us consider now **a trivalent tree** following the work [20]. Many people studied the second order (Laplace-Beltrami) difference operators on the trees, but nothing like hidden integrability of the soliton type was found for them. We are going to consider graphs (one-dimensional simplicial complexes) with the natural geodesic metric such that the length of every edge $d(PP')$ is equal to one, and every edge has exactly two vertices PP' . There are no cycles in the trees by definition.

The operator L acting on the functions of vertices

$$L\psi_P = \sum_Q b_{PQ}\psi_Q$$

is **real** if all coefficients are real. It has **an order** k equal to the maximal diameter of the interaction domain in the vertices P, i.e. $k = \max_P d(Q_1Q_2)$ such that $b_{PQ_1} \neq 0, b_{PQ_2} \neq 0$. The real operator is symmetric or **selfadjoint** if $b_{PQ} = b_{QP}$. A selfadjoint operator should have an even order $k = 2l, l = 0, 1, 2, \ldots$. For the second and fourth order cases we frequently numerate the highest order coefficients by the pair of adjusting edges $b_{PP''} = b_{RR'}$, and the second order terms by one edge $b_{PP'} = b_R$. Consider now the set of all fourth order real selfadjoint operators L on the trivalent tree such that the highest order coefficients are always positive

$$b_{PP''} > 0; d(P, P'') = 2$$

$$L\psi_P = \sum b_{PP''}\psi_{P''} + b_{PP'}\psi_{P'} + w_P\psi_P$$

where $d(PP'') = 2, d(PP') = 1$. Let me remind that in 1976 the so-called **L-A-B-triples** were invented and studied in the works [23, 15] as completely integrable soliton systems associated with the zero level of the 2D Schrodinger operator on the Euclidean plane R^2. Their discretization on the regular lattices Z^2 was discussed above.

Trivalent tree Γ has a geodesic structure analogous to the 2D hyperbolic (noneuclidean) plane. As we shall see, nontrivial $L - A - B$ triples appear here for the fourth order selfadjoint operators. Nothing like that exists here for the second order difference operators.

Theorem 7 *There exist a nontrivial time dynamics of the form*

$$L_t = LA - BL$$

where the difference operators A, B have second order and $B = A^t$

$$A\psi_P = \sum c_{PP'}\psi_{P'}$$

The coefficients $c_{PP'}$ for the edges $R = PP'$ can be calculated by the following formula. Fix some "initial" point $P_0 \in \Gamma$; For every point $P \in \Gamma$ there is a unique simple path $\gamma = [P_0, \dots, P]$ consisting of the edges $R_0, \dots, R_k$ and joining the initial point with point P. We introduce a multiplicative one-cocycle $\Psi(R)$ whose value for the oriented edge $R = Q_1Q_2$ can be described in the following way. Let the edges R_1', R_2' enter the first vertex Q_1, and the edges R_1'', R_2'' come out of the second vertex Q_2, not one of these edges coincides with R. We define this cocycle and the coefficients c

$$\Psi(R) = -\frac{b_{RR_1''}b_{RR_2''}}{b_{R_1'R}b_{R_2'R}}$$

$$c_R = -\frac{1}{b_{R_1'R_2'}}\left(\prod_{R_i \in \gamma}\Psi(R_i)\right)$$

where $R = PP'$

There is nothing surprising here that this expression is nonlocal: let me remind that for the best known hierarchy (the so-called "Novikov-Veselov" hierarchy [21, 22]) associated with the 2D Schrodinger operator L, such nonlocality also is presented. It is presented also in the famous KP hierarchy, so it always appears for the $2+1$-systems.

Theorem 8 *The generic real fourth order operator L on the trivalent tree Γ admits a one-parametric family of factorizations through the second order operators*

$$L = Q^tQ + u_P$$

where $Q\psi_P = \sum_Q d_{PQ}\psi_Q + v_P\psi_P$ and $d_{PQ} > 0$,

$$b_{PP''} = d_{P'P}d_{P'P''}; b_{PP'} = d_{P'P}v_{P'} + d_{PP'}v_P$$

$$w_P = v_P^2 + \sum_{P'} d_{P'P}^2 + u_P$$

Therefore the Laplace transformation are defined for this class of operators.

Recently in the work [24] these results were extended to all trees: the last theorem is not true anymore for the generic operators, but for the subclass of factorizable real selfadjoint fourth order operators L the analog of the first theorem remains true.

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