Real finite-zone solutions of the sine–Gordon equation: a formula for the topological charge

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A solution of the SG equation $4(u_{tt} - u_{xx}) = u_{\eta\xi} = 4\sin u$ is said to be *periodic* with respect to x if the quantity e^{iu} is periodic. The *topological charge* is defined to be the integer $n = [u(x+T) - u(x)]/2\pi$, where T is the period. The *density of the topological charge* is defined to be the ratio $\overline{n} = n/T$. This quantity is well defined also for quasi-periodic functions. Finite-zone solutions of the SG equation are defined in terms of an algebraic curve Γ, $\mu^2 = \prod_{k=0}^{2g}(\lambda - E_k)$, where $E_0 = 0$. These solutions are real and non-singular if the branch points E_k either are real and negative, $E_j < 0$ for $j = 1, \dots, 2m$, or form complex-conjugate pairs $\overline{E}_{2m+2j-1} = E_{2m+2j}$, $j = 1, \dots, g-m$; furthermore, the divisor $D + \tau D$ must be the divisor of zeros of a meromorphic differential Ω with first-order poles only at the points $(0, \infty)$, where $\tau(\lambda, \mu) = (\overline{\lambda}, \overline{\mu})$.

Lemma 1. *The differential* Ω *has the form*

$$\Omega = \bigl(1 - \lambda P_{g-1}(\lambda) R(\lambda)^{-1/2}\bigr)\, d\lambda,$$

where the polynomial $P_{g-1}(\lambda)$ *is real and such that the equation* $R(\lambda)^{-1/2} = \lambda P_{g-1}(\lambda)$ *has the points* $[(0,0), \gamma_1, \dots, \gamma_g, \tau\gamma_1, \dots, \tau\gamma_g]$ *as its solutions.*

All such divisors are 'admissible', that is, they yield real non-singular solutions of the SG equation; for a given Γ we have 2^m real tori [1]. The Θ-function description of these tori in [2], [3] does not give anything for the topological charge. A different approach was therefore proposed in [4]. However, gaps were revealed in this approach [5]. Here we develop a method giving a complete proof of the formula in [4] for the topological charge.

Lemma 2. *No point of the graph of the polynomial* $P_{g-1}(\lambda)$, $\lambda \in \mathbb{R}$, *falls inside the ovals of the curve* $y = \pm R(\lambda)^{-1/2}\lambda^{-1}$. *Here the values* λ *are such that* $R(\lambda) \geqslant 0$. *The inequality* $|P_{g-1}(\lambda)| \leqslant R(\lambda)^{-1/2}\lambda^{-1}$ *holds for* $\lambda > 0$. *All polynomials with this property correspond to admissible divisors, and the single polynomial* $P_{g-1}(\lambda)$ *determines* 2^g *admissible divisors, which are obtained by choosing one point arbitrarily in each pair* $(\gamma_j, \tau\gamma_j)$.

The 'topological type' of the polynomial $P_{g-1}(\lambda)$ is defined to be the tuple $[s_1, \dots, s_m]$ of signs $s_j = \pm 1$: $s_k = 1$ if the graph of $y = P_{g-1}(\lambda)$ goes over the oval with index k, that is, $P_{g-1}(\lambda) \geqslant R(\lambda)^{-1/2}\lambda^{-1}$, and $s_k = -1$ if the graph of $y = P_{g-1}(\lambda)$ goes under the oval with index k, that is, $P_{g-1}(\lambda) \leqslant R(\lambda)^{-1/2}\lambda^{-1}$ for $\lambda \in [E_{2k}, E_{2k-1}]$, $k = 1, \dots, m$.

Lemma 3. *All topological types are non-empty and vary continuously with the branch points of the curve* Γ. *In each type the set of admissible polynomials is convex; it is possible to choose a representative of the form*

$$P_{g-1}(\lambda) = \sqrt{-E_1 - E_{2m}} \times (-1)^{s_1} \prod_{k=2}^{m} \bigl(\sqrt{E_{2k-2}E_{2k-1}} - (-1)^{s_k s_{k-1}}\lambda\bigr) \prod_{k=m+1}^{g} \bigl(\sqrt{E_{2k-1}E_{2k}} - \lambda\bigr).$$

Lemma 4. *The* λ*-projections of the points* $\gamma_j = (\lambda_j, \mu_j)$ *of an admissible divisor are such that* λ_j *cannot lie in the segments* $[E_1, E_0]$, $[E_3, E_2]$, $\dots$, $(-\infty, E_{2m}]$, *where* $0 > E_1 > E_2 > \cdots$.

We now consider the 'stable case', where $m = g$. We choose a basis of a-cycles, where a_j encompasses the segment between the branch points E_0 and E_{2j+1}, $j = 0, \dots, g-1$. The canonical

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basis of differentials of the first kind with $\oint_{a_j} \omega_k = \delta_{jk}$ has the form

$$\omega_k = iR(\lambda)^{-1/2}\left(\sum_{j=0}^{g-1} D_j^k \lambda^j\right) d\lambda, \qquad D_j^k \in \mathbb{R}, \quad k = 0, \dots, g-1. \tag{1}$$

Theorem. *The density of the topological charge has the form*

$$2\pi\overline{n} = \pi \sum_{k=0}^{g-1} (-1)^{k-1} s_k \widetilde{D}^k$$

for all finite-zone real solutions of the SG equation in the stable case, where $\widetilde{D}^k = D_{g-1}^k + \prod_{q=1}^{2g} E_q^{-1/2} D_0^k$.

From the properties of the Abel transformation it follows that the density of the topological charge has the form $\overline{n} = \sum_{k=0}^{g-1} \widetilde{D}^k n_k$, where the numbers n_k are the analogues of the topological charges corresponding to the 'time' shifts t_k for the basis cycles of an Abelian torus. These are integers that must be computed.

Let us remove the segments $\{[E_1, E_0], [E_3, E_2], \dots, (-\infty, E_{2m}]\}$ from Γ. After this, $\Gamma \setminus \{\cdot\}$ decomposes into two mutually disconnected sheets G_+ and G_-. For any $g > 0$ we choose Γ and admissible divisors D in each component in such a way that to a motion in the time t_k for the cycle with index k of the Abelian torus there corresponds a motion of the point γ_k only on the sheet G_+ or G_-; the points γ_j, $j \neq k$, move along contractible cycles. These flows are non-local. All real solutions are non-singular. In these examples we get that $n_k = (-1)^{k-1} s_k$, $n_j = 0$, $j \neq k$. Here we use the 'Dubrovin equations' for the motion of the divisor D with respect to the indicated 'times'. Varying the parameters (Γ, D) along a curve in general position, we encounter singular divisors. The curves $\gamma_s(t_k)$ inevitably encounter places where $\gamma_j(t_k) = \gamma_s(t_k)$. However, the symmetric expression $e^{iu} = \text{const} \cdot \gamma_1 \cdots \gamma_g$ varies continuously. This follows from the Θ-function formulae and enables us to extend the formula for the topological charge to the whole function space. The use of the times t_k and the passage through a singularity play a decisive role.

Following our scheme, we shall present all the details in another paper along with the action-angle variables.

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