

## Holomorphic bundles and commuting difference operators. Two-point constructions

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This paper is closely related to our previous paper [1]. As is well known, in current mathematical physics the theory of commuting one-dimensional operators appears as an auxiliary algebraic aspect of the theory of integration of non-linear soliton systems and the spectral theory of periodic finite-zone operators [2]–[5].

As a purely algebraic problem, the problem of classifying ordinary scalar differential operators had already been posed in the 1920s by Burchinal and Chaundy [6], who advanced a long way towards a solution of the problem in the case of operators of mutually prime orders (in which the rank is always equal to 1), completed in [3]. However, they remarked that the general problem for rank  $r > 1$  seemed extraordinarily difficult.

The first steps were taken in [7] and [8]. A method of effective classification of commuting differential operators of rank  $r > 1$  in general position was created by the authors in [9] and [10]. Commuting pairs of rank  $r > 1$  depend on  $(r - 1)$  arbitrary functions of one variable, a smooth algebraic curve  $\Gamma$  with *one* distinguished point  $P$  and a set of Tyurin parameters (characterizing a framed stable holomorphic bundle). We call these *one-point* constructions.

For difference operators the whole theory, which has already become classical, of pairs of commuting operators of rank  $r = 1$  was based only on *two-point constructions* [11], [12]. Rings of such operators turned out to be isomorphic to the rings  $A(\Gamma, P^\pm)$  of meromorphic functions on an algebraic curve  $\Gamma$  with poles at a pair of distinguished points  $P^\pm$ .

In our previous paper [1] we showed that for rank  $2l \geq 2$  a broad class of commuting difference operators can be obtained from the one-point construction. As in the continuous case, these operators depend on arbitrary functions of one variable  $n \in \mathbb{Z}$ .

In the present paper we have obtained a description of a broad class of commuting difference operators constructed starting from two-point constructions. In contrast to one-point constructions, there are no arbitrary functions here; the coefficients of the operators can be calculated by means of the Riemann theta function. As in the rank 1 case, these operators lead to solutions of the equations of the 2D Toda lattice and the whole hierarchy connected with them.

We consider a smooth algebraic curve  $\Gamma$  of genus  $g$  with two distinguished points  $P^\pm$ . Let  $(\gamma)$  be a collection of  $rg$  points  $\gamma_s$ ,  $s = 1, \dots, rg$ , on  $\Gamma$ , and  $(\alpha)$  be a collection of  $(r - 1)$ -dimensional vectors:  $\alpha_s = (\alpha_{sj})$ ,  $j = 1, \dots, r - 1$ . According to [9], [10], these parameters  $(\gamma, \alpha)$  are called Tyurin parameters. In the general case they determine a stable framed bundle  $\mathcal{E}$  over  $\Gamma$  of rank  $r$  and degree  $c_1(\det \mathcal{E}) = rg$ .

**Lemma 1.** *For any choice of Tyurin parameters  $(\gamma, \alpha)$  in general position there exists a unique meromorphic vector function  $\psi_n(Q) = (\psi_n^i(Q))$ ,  $i = 1, \dots, r$ ,  $Q \in \Gamma$  such that: (i) outside  $P^\pm$  the functions  $\psi_n^i(Q)$  have no more than simple poles at the points  $\gamma_s$ , and their residues at these points satisfy the relation  $\alpha_{si} \operatorname{res}_{\gamma_s} \psi_n^i = \operatorname{res}_{\gamma_s} \psi_n^r$ ; (ii) in a neighbourhood of the distinguished points  $\psi_n(Q)^i$ ,  $n = kr + j$ ,  $0 \leq j < r$ , has the form*

$$\psi_n^i = z_\pm^{\mp k} (\xi_{n,\pm}^i + O(z_\pm)), \quad (1)$$

where  $\xi_{kr+j,+}^j = 1$ ,  $\xi_{kr+j,+}^i = 0$  for  $i > j$ , and  $\xi_{kr+j,-}^i = 0$  for  $i < j$  (here  $z_\pm$  are local coordinates in neighbourhoods of  $P^\pm$ ).

The vector function defined above is the analogue of the Baker-Akhiezer difference function for the higher-rank case.

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**Theorem 1.** Let  $\psi_n^i(Q)$  be the Baker-Akhiezer vector function corresponding to the choice of algebro-geometric data  $\{\Gamma, P^\pm, (\gamma, \alpha)\}$ . Then for any function  $f(Q) \in A(\Gamma, P^\pm)$  there exists a unique difference operator  $L_f$  of the form

$$L_f = \sum_{i=-rn_-}^{rn_+} u_i(n)T^i, \quad u_{\pm rn_\pm} = f_\pm \neq 0, \quad (2)$$

such that  $L_f \psi_n^i(Q) = f(Q) \psi_n^i(Q)$ ,  $i = 1, \dots, r$ . Here  $Ty_n = y_{n+1}$  is the shift operator and  $n_\pm$  are the orders of the function  $f(Q)$  at the points  $P^\pm$ .

Both the function  $\psi$  and the coefficients of the operators can be explicitly calculated in terms of the Riemann theta function corresponding to the curve  $\Gamma$ . The coefficients of the operators  $L_f$  are periodic functions if and only if  $A(P^+) - A(P^-)$  is a point of finite order on the Jacobian  $J(\Gamma)$ .

*Remark 1.* We call attention to the fact that there are no functional parameters in this construction even in the case of rank  $r > 1$ . It follows from (2) that among the operators  $L_f$  there are both those for which all shifts  $T^i$  are positive,  $i > 0$ , and those for which all shifts are negative,  $i < 0$ . One-point constructions never lead to such operators. Here we also arrive naturally at the construction of representations of a version of the Kac-Moody algebra  $\widehat{sl}(r, C)$  associated with the algebraic curve  $\Gamma$  with distinguished points  $P^\pm$  following the plan of [13].

*Remark 2.* For any rank  $r > 1$  the authors have also constructed two-point Baker-Akhiezer vector functions giving the solutions of the complete hierarchy of equations of the 2D Toda lattice. They depend on  $2(r-1)$  arbitrary functions. This construction contains the theory of integrable potentials of the two-dimensional Schrödinger operator connected with bundles of higher rank begun by the authors in [14].

We consider the ring  $\mathcal{D}$  of difference operators of finite order. By Theorem 1 a choice of Tyurin parameters  $(\gamma, \alpha)$  in general position defines a homomorphism  $G_{(\gamma, \alpha)}: A(\Gamma, P^\pm) \mapsto \mathcal{D}$ , whose image is a maximal commutative subring. The following theorem shows that the proposed construction describes all similar rings in general position.

**Theorem 2.** For any ring monomorphism  $G: A(\Gamma, P^\pm) \mapsto \mathcal{D}$  such that the operators  $L_f = G(f)$  have the form (2), the normalized joint eigenfunctions determine a framed holomorphic bundle of rank  $r$  and degree  $rg$ . In general position, when this bundle is described by Tyurin parameters  $(\gamma, \alpha)$ , the homomorphism  $G$  coincides with the homomorphism  $G = G_{(\gamma, \alpha)}$  defined by virtue of Theorem 1 with the aid of the corresponding Baker-Akhiezer vector function.

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