Algebraic properties of two-dimensional difference operators

S. P. Novikov

We consider the two-dimensional lattice $\mathbb{Z}^2$ with basic vector-periods T_1, T_2. The points of the lattice are numbered by pairs of integers $n = (n_1, n_2)$, $n+T_1 = (n_1+1, n_2)$, $n+T_2 = (n_1, n_2+1)$. We shall also use T_1, T_2 to denote the corresponding shift operators on functions. In the present paper we shall only consider operators with real coefficients.

The discretization of the two-dimensional Schrödinger operator on a square lattice, where a vertex interacts with the four nearest neighbours, does not admit transformations of Laplace type. As the author has shown in [1], Appendix 1, we must consider a regular triangular lattice, where a vertex has six nearest neighbours, since the periods T_1, T_2, $T_1^{-1}T_2$, T_1^{-1}, T_2^{-1}, $T_1T_2^{-1}$ all have the same length. The purely real self-adjoint operator

$$L = a_n + b_nT_1 + c_nT_2 + b_{n-T_1}T_1^{-1} + c_{n-T_2}T_2^{-1} + d_{n-T_2}T_1T_2^{-1} + d_{n-T_1}T_2T_1^{-1}$$

admits a factorization of the form $L = QQ^\dagger + w_n$, where $Q = x_n + y_nT_1 + z_nT_2$, $T_j^\dagger = T_j^{-1}$, $x_n^\dagger = x_n$, $y_n^\dagger = y_n$, $z_n^\dagger = z_n$, $(ab)^\dagger = b^\dagger a^\dagger$, and w_n is the operator of multiplication by a real function.

Laplace transforms. If $L\psi = 0$, then we put $\widetilde{\psi} = Q^\dagger\psi$, $\widetilde{L} = Q^\dagger w_n^{-1}Q + 1$. It is easy to see that $\widetilde{L}\widetilde{\psi} = 0$. If $w_n = w = \text{const}$, then the Laplace transform can be applied to any eigenfunction. Let $w = 0$, $L = QQ^\dagger$. We put $\widetilde{L} = Q^\dagger Q$, and also $\widetilde{\psi} = Q^\dagger\psi$ for any eigenfunction $L\psi = \lambda\psi$.

Generally speaking the Laplace transform cannot be applied a second time to the whole spectrum, except in special cases, which we shall also consider here.

Remark. In all there are six Laplace transforms, chosen as the basic periods of an arbitrary pair (T_1^*, T_2^*), running consecutively in a cyclic sequence

$$\dots, T_1, T_2, T_1^{-1}T_2, T_1^{-1}, T_2^{-1}, T_1T_2^{-1}, T_1, \dots$$

(above we chose $T_1^* = T_1$, $T_2^* = T_2$). The group of Laplace transforms has not been studied. Arbitrary words in it may be used to define 'cyclic chains', quasicyclic chains, and so on. This gives rise to a difference analogue of the two-dimensionalized Toda chain, by analogy with [1] and [2].

In the present paper we shall construct distinctive q-analogues of the well-known Schrödinger–Landau operator on a homogeneous magnetic field. This is a direct development of the idea of joint work with Taimanov [1], Appendix 2, where q-analogues of a one-dimensional harmonic oscillator were constructed (the earlier history is given in [1] together with the work of other authors in the one-dimensional case).

Theorem 1. *For the operators Q, $Q^\dagger$, where*

$$Q = Q_{c,d,u,v} = 1 + cu^{n_1}v^{n_2}T_1 + d(u^2v^{-1})^{n_1}u^{n_2}T_2,$$

we have the relations

$$Q_{c,d,u,v}Q^\dagger_{c,d,u,v} - 1 = q\bigl(Q^\dagger_{c',d',u,v}Q_{c',d',u,v} - 1\bigr),$$

where $q = u^{-2}$, $c = qc'$, $d = qd'$.

Corollary 1. *The spectrum of the operators $L = QQ^\dagger$, $\widetilde{L} = Q^\dagger Q$ for all $c \neq 0$, $d \neq 0$, $u > 0$, $v > 0$, acting on the Hilbert space $\mathcal{L}_2(\mathbb{Z}^2)$, is discrete for all eigenvalues $\lambda < 1$ and can lie only at the points $\lambda_j = 1 - u^{2j}$, where $j \geqslant 0$ for $u < 1$ and $j \leqslant 0$ for $u > 1$ (the spectrum may not cover all these points: see below).*

Theorem 2. *The equations $Q\psi_0 = 0$ and $Q^\dagger\psi_0 = 0$ have non-zero solutions $\psi_0 \in \mathcal{L}_2(\mathbb{Z}^2)$ in the following cases:*

(1a) $u^3 > v > u > 1$; (1b) $u > \max(v, v^{-1}) > 1$, $Q\psi_0 = 0$;
(2a) $u^{-3} > v^{-1} > u^{-1} > 1$; (2b) $u^{-1} > \max(v, v^{-1}) > 1$, $Q^\dagger\psi_0 = 0$.

The solution spaces are infinite-dimensional in all these cases, with bases $\psi_0^{(k)} \in \mathcal{L}_2(\mathbb{Z}^2)$:

(1a) $\psi_0^{(k)} = c^{-n_1} d^{k-n_2} (-\sqrt{u})^{n_1+n_2-k} t^{-kn_1} \chi_{k-n_2}(t) e^{-Q_2(n)}$;
(1b) $\psi_0^{(k)} = c^{k-n_1} d^{-n_2} (-\sqrt{u})^{n_1+n_2-k} t^{kn_2} \chi_{k-n_1}(t^{-1}) e^{-Q_2(n)}$;
(2a) $\psi_0^{(k)} = c^{n_1} d^{n_2-k} (-\sqrt{u})^{k-n_1-n_2} t^{kn_1} \chi_{n_2-k}(t) e^{-Q_2(n)}$;
(2b) $\psi_0^{(k)} = c^{n_1-k} d^{n_2} (-\sqrt{u})^{k-n_1-n_2} t^{-kn_2} \chi_{n_1-k}(t^{-1}) e^{-Q_2(n)}$;

here $k \in \mathbb{Z}$, $t = v^2u^{-2}$, $Q_2(n) = \alpha n_1^2 + 2\beta n_1 n_2 + \alpha n_2^2$, and the functions $\chi_m(t)$ have the form:

$$\chi_0(t) = 1, \qquad \chi_m(t) = 0, \quad m < 0, \qquad \chi_m(t) = \prod_{j=1}^{m} (1 - t^j), \quad m > 0.$$

The parameters α, β are such that:

(1a) $u = e^{2\alpha}$, $u^2 v^{-1} = e^{2\beta}$; (1b) $u = e^{2\alpha}$, $v = e^{2\beta}$, $|\beta| < \alpha$;
(2a) $u = e^{-2\alpha}$, $u^2 v^{-1} = e^{-2\beta}$; (2b) $u = e^{-2\alpha}$, $v = e^{-2\beta}$, $|\beta| < \alpha$.

Remark. These bases are not orthonormal, and their completeness has not been proved. Other cases are harder, and have not yet been investigated.

Corollary 2. *In cases* (1a) *and* (1b) *the spectrum of the operators $Q^\dagger Q$ takes up all the points $\lambda_j = 1 - u^{-2j}$, $j \geqslant 0$. The spectrum of the operators $QQ^\dagger$ takes up all points except one: $\lambda_j = 1 - u^{2j}$, $j \geqslant 1$. In cases* (2a) *and* (2b), *on the contrary, the spectrum of the operators $QQ^\dagger$ takes up all the points $\lambda_j = 1 - u^{2j}$, $j \geqslant 0$, while the spectrum of the operators $Q^\dagger Q$ omits zero: $\lambda_j = 1 - u^{2j}$, $j \geqslant 1$.*

The corresponding eigenfunctions are obtained from $\psi_0^{(k)}$ by applying the iteration of the operators $Q^\dagger$ for the case (1) *and of the operators Q in the case* (2), *with the corresponding parameters c_j, d_j.*

Bibliography

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Steklov Mathematical Institute of
the Russian Academy of Sciences;
University of Maryland,
College Park, USA

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