SMOOTH FOLIATIONS ON THREE-DIMENSIONAL MANIFOLDS

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We first consider a smooth (for example, closed and analytic) n-manifold M, on which these is given a non-singular 1-form ω of class C^∞. The equation

$$\omega = 0$$

defines a field of $(n-1)$-dimensional directions on M, smoothly dependent on the point, As usual, we call hypersurfaces touching our field everywhere solutions of $\omega = 0$. These hypersurfaces are called leaves and the family of them is called a foliation. Locally, in a coordinate system $(x_1, \ldots, x_n)$ near a point, the equation takes the form

$$\sum P_i (x_1, \ldots, x_n)\, dx_i = 0;$$

it is called Pfaff's equation (without singularities) and is soluble under the integrability condition of Frobenius

$$\omega \wedge d\omega = 0.$$

Foliations have been stvdied by a number of authors (Ehresmann, Reeb, Haefliger and others) in a much more general situation than that given here, they have obtained a number of results on the properties of foliations, on the behaviour of closed leaves of special kinds, on certain properties of one-dimensional curves on leaves, on the existence of analytic leaves, and have also constructed several interesting examples.

However, not a single result was known that establishes the existence of closed leaves. The simplest problem of this type is Kneser's conjecture on the existence of closed leaves of any smooth foliation on the usual 3-sphere S^3 (the leaf has dimension 2); in all known examples Kneser's existence hypothesis is satisfied. My aim is to prove a somewhat stranger statement than this conjecture.

THEOREM 1. *If the universal covering of a closed manifold M^3 is non-contractible, then any smooth orientable foliation on it has a closed leaf, and either this is a torus T^2, homologous to zero in M^3 and bounding in M^3 a full torus $D^2 \times S^1$ with a special foliation, or all the leaves are spheres S^2 or projective planes P^2 and the universal covering is the product of a 2-sphere and a line.*

In connection with the method of proof I remark that a sufficient (not necessary) characterization of a closed leaf is the property that no closed transversal passes through it; this trivial property serves as a peg to hang the proof on. To use it, a domain is constructed engulfing transversals not inclined at too small an angle to the leaf, and the closed leaf is approximated by such transversals (by decreasing this angle).

The construction of the domain engulfing the transversal is the central part of the proof from both the technical and conceptual point of view, and closed curves on the leaves figure in various ways in it. Such curves are often, for example, "limit cycles" round which the nearby leaves spiral indefinitely; they may be curves that are not limit cycles relative to nearby leaves. The latter case interests us under the condition that the curve is not homotopic to zero in its leaf, but is homotopic to zero after a displacement to an arbitrarily close leaf (reconstruction of the topology of the leaf). A leaf with a curve of this type turns out to be closed. The domain engulfing the transversals to the leaves is glued together from films stretched over this curve after all possible displacements to nearby leaves. These films "capture" each other with some period and are glued together to form a domain with the necessary properties. The boundary of this domain is the required closed leaf.

We indicate some results obtained in passing immediately following from Theorem 1.

a) A dynamical system which is a transversal to a foliation on M^3 always has a periodic orbit. Further it has a system of periodic orbits knotted in the topological sense.

A dynamical system of this sort cannot be conservative, since it enters the domain bounded by the closed leaf. The latter fact is of a global character, since a transverse foliation can always be constructed locally.

b) A pair of transverse foliations with everywhere dense leaves is connected with one interesting class of dynamical systems (U-systems and U-cascades in the sence of Anosov).

If the manifold M^n with the U-system is three-dimensional, it is easy to show that its fundamental group is infinitive (Anosov). It also turns out that

$$\pi_2(M^3) = 0.$$

If we are dealing with U-cascades, then M^3 has the topological type of the torus T^3. Some facts can also be obtained here for $n > 3$.

c) The rank of M^n is the maximum number of linearly independent vector fields on M^n whose commutators [ ] are pairwise zero (Milnor). What, for example is the rank of S^3?

V.I. Arnol'd has shown me a proof that the rank of S^3 is 1, which is a consequence of the results on foliations given above. This was conjectured by Milnor.

In addition, the rank

$$M^n \leqslant n-2,$$

if $\pi_1(M^n)$ is finite, or

$$\pi_2(M^n) \neq 0$$

(M^n is compact and closed).

d) On simply-connected manifolds M^n, $n \geqslant 3$, there are no analytic foliations, but there may be infinitely differentiable ones (Haefliger). For $n = 3$, it turns out that there cannot be orientable analytic foliations if $\pi_2(M^n) \neq 0$ (or π_1 is finite); this follows from Theorem 1. Foliations of class C^∞ always exist if M^3 is closed and orientable (Zieschang).

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