QUASIPERIODIC STRUCTURES IN TOPOLOGY

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Introduction

This paper describes the results of the author and some of his colleagues (mainly in Moscow) on the topology of closed 1-forms on finite-dimensional manifolds (these investigations started in 1981). This theory may be considered as a natural analog of Morse theory. There are two types of problems here:

Problem 1. To estimate the number $m_i(\omega)$ of critical points with Morse index equal to i for a closed 1-form ω on a manifold M. This problem is the direct analog of classical Morse theory. Some very nice algebraic topology appears here, including some analytical problems of unusual type about the structure of the "Morse complex".

Problem 2. To investigate the topological structure of the level surfaces $[\omega = 0]$. This looks like the typical problem in foliation theory, but our foliation $[\omega = 0]$ has very special properties. Its nondegenerate leaves are the "*quasiperiodic manifolds*" (see below).

An important special case is that of a hypersurface M embedded in the torus T^{n+1}, with ω the constant 1-form

$$\sum_{i=1}^{n+1} a_i \, d\varphi_i$$

restricted to M. For $n+1 = 3$ the surface $M^2 \subset T^3$ appears as a "Fermi-surface"; the form ω now represents the homogeneous magnetic field; the curves $[\omega = 0]$ on M^2 describe the semiclassical motion of the electron in the space of quasimomenta (see [1b]). Even in that case nontrivial unsolved problems appear about the quasiperiodic structure of the foliation $[\omega = 0]$. The asymptotics of the leaves in the covering space $\mathbb{R}^3 \to T^3$ are especially important in this last example.

I will now present some brief historical remarks, describing how this natural subject appeared. About 10 years ago (see [1a, 1b]) in 1981, I began to study the following important examples. There are Lagrangians, occurring both in problems of very classical analytical mechanics and in modern field theory, which lead to multivalued action functionals $S\{f\}$ such that the variation δS uniquely determines a closed 1-form on some functional space F of mappings $f\colon X \to Y$ (or on the space of sections of a fibre bundle with base X). All such "local" Lagrangians were classified and the following quantization principle was formulated: *for the Feynman-type quantization the so-called "Feynman amplitude" e^{iS} should be one-valued.*

This means for our theories that the contour integral of the 1-form $(2\pi\delta S)$ along a closed path γ in the functional space F should be an integral number

$$\oint_\gamma 2\pi\delta S = n(\gamma) \in \mathbb{Z}.$$

(As people say, the coupling constants should be quantized. An idea of this kind appeared in the work of Yang and Lee on the Dirac monopole (1977) but was not formulated in this sharp form.)

Important examples are: the generalized Dirac monopole, including the factorization of the classical top in a constant gravity field (or in a perfect fluid) by the area integral ($\dim X = 1$); and the multidimensional field-theoretical generalizations, very important in modern quantum field theory, which were found by the author (1981) and by Edward Witten (1983).

A very important example of a different kind (nonlocal) of a multivalued action is presented by the so-called "Chern–Simons" functional on the (nonlocal) space of gauge-equivalence classes of Yang–Mills fields; the quantization principle for this example was first discussed by Deser–Jackiw–Templeton (1982). This last example may be reduced to the previous type if the connections are parameterized not by Christoffel-type symbols (Yang–Mills fields) but by the corresponding mappings of the underlying 3-manifold X into the classifying space Y (in the style of Ehresmann, Pontrjagin and Steenrod); physicists never use this parameterization.

Much later an interesting analog of Morse theory was constructed by Floer for the Chern–Simons functional on the space M of all connections on a 3-manifold X with $H_1(X) = 0$. The Morse indices are infinite in this case. Floer theory uses the differences between the indices which are finite and well-defined mod 8. It is interesting that the Morse complexes determined by closed 1-forms on finite-dimensional manifolds are normally much more complicated analytically than the Floer complex for the Chern–Simons functional.

1. Morse complexes. Topological estimates for the critical points

Let M be an n-dimensional closed manifold and

$$(\gamma_1, \dots, \gamma_\ell), \quad \gamma_i \in H_1(M, \mathbb{Z})$$

a basis for the first homology group of M. For any closed 1-form ω we have ℓ periods

$$\kappa_j = \oint_{\gamma_j} \omega.$$

Definition. The *irrationality* $\mathrm{Irr}(\omega)$ of the 1-form co is the $\mathbb{Z}$-rank of the free abelian subgroup in $\mathbb{R}$ generated by $(\kappa_1, \dots, \kappa_\ell)$. The form ω is quantized if its class belongs to $H^1(M, \mathbb{Z})$.

The *direction* of the form in the group H^1 is specified completely by the point

$$\kappa = (\kappa_1 : \kappa_2 : \cdots : \kappa_\ell) \in \mathbb{R}P^{l-1}.$$

There is a minimal free abelian covering $p\colon \bar{M} \to M$ such that the form $p^*\omega$ is exact:

$$p^*\omega = dS.$$

The monodromy group is $\mathbb{Z}^k$, where $k = \mathrm{Irr}(\omega)$, generated by the covering transformations $\bar{T}_i\colon \bar{M} \to \bar{M}$, $i = 1, \dots, k$, satisfying $\bar{T}_i^* S = S + \kappa_i$.

In this case we may choose the basis

$$(\gamma_1, \dots, \gamma_l) \subseteq H_1(M, \mathbb{Z})$$

in such a way that

$$\kappa_i = 0, \quad \text{for } i \geqslant k+1.$$

A smooth metric on the manifold M generates a gradient flow for the 1-form ω which lifts to a flow with the ∞-continuation property on the covering space $\bar{M}$: each trajectory ends in a critical point or intersects all level-surfaces of the function S on $\bar{M}$. We choose a "generic" metric so that the Morse complex (see below) will be correctly denned. The cells of this complex are as usual the surfaces of steepest descent starting from a critical point (we choose exactly one critical point in $\bar{M}$ in the inverse image $p^{-1}(x)$ of each critical point x of ω on M). This gives a collection of cells

$$\sigma_q^i, \quad q = 1, \dots, m_i(\omega).$$

Next we define a boundary operator in the natural standard way (only the algebraic organization here is nonstandard):

$$\partial \sigma_q^i = \sum a_{pq}^{(i)} \sigma_p^{i-1}.$$

Here the coefficients $a_{pq}^{(i)}$ are formal power series, with integral coefficients, in the variables $(t_1, \dots, t_k, t_1^{-1}, \dots, t_k^{-1})$, series which belong to the ring K_κ described below. They do not depend on the metric.

Definition. A formal power series a in the variables $(t_1, \dots, t_k, t_1^{-1}, \dots, t_k^{-1})$ belongs to the ring K_κ iff the following conditions are satisfied:

(1) There exists a number $N(a)$ such that, writing

$$a = \sum_{n_j \in \mathbb{Z}} u_{(n_1, \dots, n_k)} t_1^{n_1} \dots t_k^{n_k},$$

the coefficient $u_{(n_1, \dots, n_k)} = 0$ if $\sum_j \kappa_j n_j < N(a)$.

(2) There are only a finite number of nonzero coefficients in any domain

$$N_1 \leqslant \sum_j \kappa_j n_j \leqslant N_2.$$

Remark 1. The stable variant K_κ^{st} of this type of ring is defined by requiring that Condition 1 hold for all κ^* such that $|\kappa - \kappa^*|$ is small enough in $\mathbb{R}P^{l-1}$. Condition 2 will be automatically satisfied in that case.

Remark 2. For $k = 1$ we have $K_\kappa = K_\kappa^{\text{st}} = K$ which does not depend on $\kappa \neq 0$. This ring contains all formal series with finite negative part:

$$K = \left\{ a = \sum_{n \geqslant N(a)} u_n t^n, \ u_n \in \mathbb{Z} \right\}.$$

The variables t_j correspond geometrically to the shifts

$$t_i \leftrightarrow \bar{T}_j \colon \bar{M} \to \bar{M}.$$

There is a natural embedding of the group ring of the monodromy group $\mathbb{Z}^k$ in K_κ^{st}:

$$\mathbb{Z}[t_1, \dots, t_k, t_1^{-1}, \dots, t_k^{-1}] \to K_\kappa^{\text{st}} \subset K_\kappa.$$

The last embedding generates the locally constant sheaves on the manifold M with coefficients in the rings K_κ^{st}, K_κ and the corresponding homology (cohomology) groups

$$H^*,\ H_*(M,[\omega]),\ H_*^{\mathrm{st}}(M,[\omega]),\ H^*_{\mathrm{st}}$$

which are K_κ- and K_κ^{st}-modules. On the homological level these two types of modules are practically the same (they have the same generators and all relations are inside the ring K_κ^{st}).

For $k=1$ the ring K is obviously the ring of principal ideals (euclidean ring). For $k>1$ the ring K_κ^{st}, as was observed later, is also euclidean iff $\mathrm{Irr}(\omega)=k$. (Sikorav, 1987).

As a consequence we may define a Betti number $b_i(M,[\omega])$ and a torsion number $q_i(M,[\omega])$ like the same quantities for ordinary $\mathbb{Z}$-modules, but with a different geometric meaning.

The Morse complex for any concrete form ω belongs to the large ring K_κ; its homology depends only on the cohomology class $[\omega]$. For the Poincaré-dual objects we have to replace $(-\infty)$ by $(+\infty)$ in the definition of the formal series and to use the function $(-S)$ instead of S. The Morse-type inequalities have the natural form (Novikov, 1981, for $k=1$):

$$m_i(\omega)\geqslant b_i([\omega])+q_i([\omega])+q_{i-1}([\omega]).$$

For $k=1$, $\dim M\geqslant 5$, the exactness of these inequalities was proven by Farber [2]. For $k>1$ some results of this type were obtained by Pazhitnov [3a, b]. This last author also proved that outside of a finite number of "jumping hyperplanes" in the space $H^1(M,\mathbb{R})$, the quantities $b_i([\omega]), q_i([\omega])$ are constant in each domain.

For the computation of the quantities $b_i([\omega])$ and $m_i([\omega])$ it is very useful to consider the family of 1-dimensional representations

$$\rho\colon \pi_1(M)\to\mathbb{C}^*$$

generating the groups $H^*_\rho(M,\mathbb{C})$ and $H_*^\rho(M,\mathbb{C})$. These groups coincide with the cohomology generated by the closed 1-form Ω and the operator d_Ω:

$$d_\Omega(u)=du+\Omega\wedge u$$

satisfying

$$d_\Omega^2=0.$$

The correspondence has the form

$$\rho(\gamma)=\exp\left[\oint_\gamma\Omega\right].$$

For all $\Omega\in H^1(M,\mathbb{R})$ the numbers $b_i^\rho=\operatorname{rank}H_i^\rho(M,\mathbb{C})$ may be considered as functions on the space of all representations $\rho\colon\pi_1\to\mathbb{C}^*$, with coordinates $(\rho_1,\dots,\rho_l)$, $l=b_1$, $\rho_j\neq 0$. There are a finite number of algebraic subvarieties $W=W_1\cup\dots\cup W_{L_i}$, defined over $\mathbb{Z}$, such that ([1d]):

(1) $b_i^\rho=b_i^\infty$ for $\rho\notin(W_1\cup\dots\cup W_{L_i})=W$ (the Betti number at a generic point).
(2) $b_i^\rho>b_i^\infty$ for $p\in W_j$. (The Betti number b_i^ρ will be even larger on the intersections $W_j\cap W_k$ and so on.)

For knots

$$S^1 \subset S^3$$

we have $H_1(S^3 \setminus S^1, \mathbb{Z}) = \mathbb{Z}$ and all the manifolds W_j are exacdy the roots of the Alexander polynomial, $i = 1$. This was deduced by the present author from an old observation of J. Milnor formulated in different terms. There is a theorem (Novikov [1d], Pazhitnov [3a]):

(a) The "generic" Betti number b_i^∞ is equal to the Betti number $b_i([\omega])$ mentioned above (see [3a]).

(b) The final result for the numbers b_i^∞ may be obtained in terms of "Massey products", as follows [1d]. Consider the spectral sequence starting from the ordinary cohomology

$$\begin{aligned} &E_1^i = H^i(M, \mathbb{C}), \quad E_{i+1}^* = H^*(E_i, d_i), \\ &d_1(u) = u \wedge \omega, \\ &d_2(u) = \{\{u, \omega\}, \omega\} \quad \text{(the Massey product)}, \\ &d_k(u) = \{\dots\{u, \omega\}, \dots, \omega\} \quad \text{(the } k\text{-fold Massey product)}. \end{aligned}$$

The groups E_∞^i have ranks equal to $b_i([\omega]) = b_i^\infty$ (Novikov [1d]; I have been informed that in the investigation of $\mathbb{Z}$-coverings a similar spectral sequence was found by Milnor who did not however compute its differentials.)

For knot spaces we have: the torsion number $q_1([\omega])$ is different from zero iff the first (constant) coefficient in the Alexander polynomial is different from ± 1 (Lasarev, Moscow University, 1989). The paper [6b] contains some results for links.

Remark. A very interesting investigation has been carried out recently in [5d] for the "2-bridge knots"; the "jumping manifolds" W_j were computed for the 2-dimensional representations of the group $\pi_1(S^3 \setminus S^1)$. The structure of the moduli space of representations $\rho\colon \pi_1 \to \mathrm{GL}(n, \mathbb{C})$, $n > 1$ is also nontrivial. It would be very interesting to compare the Jones-type polynomials of knots to these structures (i.e., the moduli space of representations plus the set of jumping submanifolds).

For the complex line $(\lambda[\omega]) \in H^1(M, \mathbb{R})$ we have a family $(\rho(\lambda))\colon \pi \to \mathbb{C}^*$. This family represents an algebraic curve iff the line is rational ($\mathrm{Irr}(\omega) = 1$). In that case the number of the jumping points $W \cap (\rho(\lambda))$ for b_i (these are the roots of an Alexander-type polynomial) is finite for all i.

There are two homotopy-invariant real numbers:

$$\begin{aligned} r_1(M, [\omega]) &= \min |W \cap \rho(\lambda)| \leqslant 1, \\ r_2(M, [\omega]) &= \max |W \cap \rho(\lambda)| \geqslant 1. \end{aligned}$$

The Morse complex for the quantized forms (up to a nonzero factor) will have coefficients belonging to the ring K. We have only one variable t here.

Conjecture. *For any closed quantized analytic* 1*-form co the boundary operator* ∂ *in the Morse complex has all coefficients* $a_{pq}^{(i)} \in K$ *with positive part convergent in some region* $|t| \leqslant r_1^*(\omega) \leqslant r_1$. *(If we replace* S *by* $(-S)$ *and* t *by* t^{-1} *we have to replace* r_1 *by* r_2^{-1}.*) In particular,* $r_1^* \neq 0$ *and there will be coefficients* $a_{pq}^{(i)}$ *with radius of convergence not greater than* $r_1(M, [\omega])$ *if at least one jumping point really exists.*

This conjecture has been investigated by some of my students during the last several years but it is still unsolved. Recently Arnol'd proved a theorem, about the growth of topological complexity for everywhere-defined mappings of closed manifolds, inspired by this problem, but so far his result is not enough. In our case we do not have a mapping, but a cobordism with two equal boundaries generating the $\mathbb{Z}$-covering over the compact manifold; the function S is constant on the boundaries. For $k > 1$, I am not able at present to formulate any acceptable and simple conjecture about the analytical properties of the Morse complex.

Problem. Consider the hypersurfaces in the torus T^n given by a trigonometric polynomial equation of order N. How to estimate the homological invariants discussed above in terms of the degree N?

2. Quasiperiodic structures on leaves

For closed 1-forms without critical points on closed manifolds the topology of the leaves is very simple. In that case the manifold itself is obviously a fibre bundle over the circle

$$p\colon M \to S^1$$

and a basic quantized form is exactly the form $p^*(d\varphi) = \omega_0$.

The form ω_0 admits a lot of small (and not small) nonquantized perturbations without any critical points, such that the topology of the leaves is not so obvious. All the leaves are the same but each of them may be nontrivial. It is not very hard to prove that all the leaves are diffeomorphic to a $\mathbb{Z}^{k-1}$-covering of a compact manifold (here k is equal to $\mathrm{Irr}(\omega)$). I will call them *periodic* manifolds with $k-1$ periods.

The situation for the nonsingular leaves of Morse forms with nontrivial critical points is much more complicated and leads to *quasiperiodicity.*

Consider a finite collection of manifolds V_j with $(k-1)$-stratified boundary. This means exactly that we are given mappings:

$$\psi\colon \partial V_j \to \partial I^{k-1}$$

onto the boundary of the unit cube, transversal along smaller cubes of all dimensions. Our notation for the inverse images of faces, in ∂V_j, will use the same indices as in the boundary of the cube I^{k-1}.

Definitions. (a) The sequence $j(n_1, \dots, n_{k-1})$ for all $n_j \in \mathbb{Z}$ is *admissible* iff we can glue the manifolds $V_{j(n_1,\dots,n_{k-1})}$ along all the neighboring faces of the $(k-1)$-lattice. This means that the neighboring faces are canonically diffeomorphic.

(b) The sequence is *quasiperiodic* iff the corresponding function has only a finite number of quasiperiods (or basic frequencies).

(c) The sequence is *special-quasiperiodic* iff there is a finite number of open nonself-intersecting domains $U_1, \dots, U_s$ in the torus T^{k-1}, with $U_p \cap U_q = \varnothing$ but with $(k-2)$-dimensional common boundary, satisfying

$$\bigcup_q \bar{U}_q = T^{k-1},$$

an initial vector $v_0 \in T^{k-1}$ and basic vectors $u_1, \dots, u_{k-1} \in \mathbb{R}^{k-1}$ such that:

(1) The points $v_0 + \sum n_j u_j$ never belong to the boundary of the domains U_p for all $n_j \in \mathbb{Z}$.

(2) For $v_0 + \sum n_j u_j \in U_p$ we put:

$$j(n_1, \dots, n_{k-1}) = p.$$

The author conjectured in [2] that all nonsingular leaves are quasiperiodic. This conjecture was recently proved in the collection of papers [4b, 5b, 6a] (the crucial result was obtained in [5b]). The theorem is that the nonsingular leaves are in fact special-quasiperiodic manifolds.

Suppose that $\operatorname{Irr} \omega = k$ and

$$\omega = \sum_{j=1}^{k} \kappa_j \omega_j.$$

Here

$$[\omega_1], \dots, [\omega_l] \in H^1(M, \mathbb{Z})$$

is the basis such that

$$(\omega_i, \gamma_j) = 0, \quad \text{for } j \geqslant k+1,\ i \leqslant k.$$

The forms generate a mapping $f\colon \bar{M} \to \mathbb{R}^{k-1}$ by the formulas

$$f(x) = 2\pi \left(\int_{x_0}^{x} \omega_1, \dots, \int_{x_0}^{x} \omega_{k-1} \right).$$

Here $\bar{M}$ is the minimal covering which resolves the multivaluedness of the form ω:

$$p^* \omega = dS.$$

Consider now the mapping f of the nonsingular leaf

$$f\colon V \to \mathbb{R}^{k-1}$$

and the family of cubes:

$$I_0^{k-1} + v_0 + \sum_{j=1}^{k-1} n_j e_j \subset \mathbb{R}^{k-1}.$$

Here $v_0 \in \mathbb{R}^{k-1}$ is a generic point, e_j are basis vectors, I_0^{k-1} is constructed on the vectors $e_1, \dots, e_{k-1}$. The inverse images

$$f^{-1}(I_0^{k-1} + v_0 + \sum n_j v_j)$$

present us the collection of manifolds which we need:

In this collection there is only a finite number of different objects; they generate the special quasiperiodic structure on the generic leaf.

Conjecture. *The generic common level of any number of closed 1-forms is quasiperiodic.*

The paper [4a] studies the special case of the semiclassical motion of an electron in the lattice generated by the constant magnetic field (i.e., in the space of "quasimomenta" T^3 with given "dispersion function" g, whose levels are the "Fermi-surfaces" $M^2 \subset T^3$). The motion is orthogonal to the magnetic field; the corresponding curves are exactly the levels of the constant 1-form $\omega = \sum a_j \, d\varphi_j$ in $\mathbb{R}^3$ restricted to the Fermi-surface M^2.

Theorem. *Suppose the form ω is a small perturbation of a rational form. Any open trajectory of ω belongs to some finite strip on the plane orthogonal to the magnetic field.*

The main problems in this example are:

(1) Scattering. Which asymptotic behavior may the open trajectories have in the space $\mathbb{R}^3$? I think that generically only the scattering for the angle 0 is possible. The "shifts" will characterize this process asymptotically.

(2) The quasiperiodic structure of the leaves. The process is basically quantum; there are quantum jumps from one component into another. Therefore the leaf should be considered as one global object (forgetting that it is topologically not connected). Which kind of "quasicrystals" on the plane will this object produce?

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