

RIEMANN SURFACES, OPERATOR FIELDS, STRINGS ANALOGUES OF THE FOURIER-LAURENT BASES

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1. INTRODUCTION

I am going to tell about one of important applications of ideas of soliton theory to complex analysis of Riemann surfaces. It has been used, since several years ago, for the construction of operator fields in string theory by Krichever and myself. Since then other people have been working here. The idea is originated from the study of discrete systems in soliton theory. The most well-known discrete system is Toda lattice which was discovered by Toda in early '70's and discrete KdV equation. Both of them were integrated by using inverse scattering type methods in the period between '73 and '74. In fact, at least Lax pairs were discovered for both of them at that period by different people such as Flaschka, Manakov, etc. (Manakov found also discrete KdV equation in that period). Here a discrete system means that it is discrete in x -variable and is continuous in time variables.

The integrability theory of Toda lattice leads to the discrete Schrödinger operator which may be written in the form

$$(1.1) \quad \lambda\psi_n = C_{n-1}^{1/2}\psi_{n-1} + V_n\psi_n + C_n^{1/2}\psi_{n+1}.$$

The discrete KdV just corresponds to $V_n \equiv 0$. This equation has the name “Langmuir chain” in Manakov’s paper because it is originated from some approximation of Langmuir oscillations in plasma physics. The standard discretization of Schrödinger operator is such that $C_n \equiv 1$. By the inverse scattering method, we may, of course, construct all these operators. So we have general Schrödinger type operators of second order in discrete case and their specializations. The case $C_n \equiv 1$ is classical in computing mathematics and the case $V_n \equiv 0$ is originated from soliton theory. People in computing mathematics never used this approximation. In fact it is much more convenient. When we are doing scattering theory, we will have some spectral data. For example if we study the periodic problem for the discrete Schrödinger operator, you will find some Riemann surface and points on it which correspond to poles of a Bloch function. They are so-called algebraic geometrical spectral data. Here it is completely impossible to recognize which data corresponds to the standard discretization of the Schrödinger operator. On the contrary it was discovered in soliton theory that we could know them for the operator corresponding to the discrete KdV. In this sense, the conditions $C_n \equiv 1$ in computing mathematics are non-integrable, while the discrete KdV equation is integrable.

In a periodic case, from Toda lattice we will obtain any hyperelliptic Riemann surface as a spectral curve (see Novikov in [3], Tanaka and Date in [1]):

$$\mu^2 = P_{2g+2}(\lambda)$$

such that the infinite point in λ is not a branching point, where $P_{2g+2}(\lambda)$ is a polynomial in λ of degree $2g+2$ and g is the genus of the curve. The zero potential case corresponding to the discrete KdV also leads to a special class of hyperelliptic curve which are symmetric:

$$\begin{array}{c} \lambda\text{-plane} \text{-----} \times \text{-----} | \text{-----} \times \text{-----} \\ \lambda_{-k} \qquad \qquad 0 \qquad \qquad \lambda_k \\ \lambda_k: \text{branching point} \end{array}$$

that is, branching points are symmetric with respect to the reflection in the origin. The spectral data corresponding to the points on this Riemann surface has also symmetric properties (see Novikov in [3], Krichever in [2]).

There is also a corresponding theory like KP-hierarchy type. Such kind of equations are discrete in the variable n and have two continuous variables, so-called two dimensional variants. For example, two dimensionalized Toda lattice is a discrete analogue of KP-hierarchy. But the only thing which is important for us now is the functional construction connected with it. The theory of commuting operators in the discrete case is more or less the same as that in the continuum case in principle. But there are some technical difficulties. These technical differences are important for us now. Let us consider a commuting pair of difference operators P_k and Q_l of order k and l respectively. Here the order of a difference operators is the number of shifts it contains. We assume, for simplicity, that k and l are relatively prime. In other words we only consider the rank one case. Here the rank of a commuting pair (P, Q) is the dimension of the space of common eigenfunctions of P and Q . The theory of commuting operators of rank greater than one is essentially more complicated and now we do not need to consider it. Anyway in our rank one case the spectral data which determines operators (P_k, Q_l) are, in principle, Riemann surface Γ and the collection of poles (P_1, \dots, P_g) of a common eigenfunction of P_k and Q_l , that is, the solution of

$$\begin{aligned} P_k \varphi &= \mu \varphi, \\ Q_l \varphi &= \lambda \varphi, \end{aligned}$$

which is unique up to constant multiplication by our assumption. More precisely we have to include, as data, two points $\{P_+, P_-\}$ on Γ and a set of local coordinates z_+, z_- at P_+, P_- respectively. To sum up, our moduli space of rank one commuting difference operators are $\{\Gamma, (P_1, \dots, P_g), P_\pm, z_\pm\}$, where Γ is a Riemann surface, (P_1, \dots, P_g) a divisor of degree g on Γ , P_\pm two different points on Γ , z_\pm local coordinates at P_\pm . There is an analogous theory of commuting matrix operators which leads to also some Riemann surface and more number of points. That is also important for us in construction of operator fields in string theory on a Riemann surface for multi-string diagrams. But here we consider only the scalar case.

Now I explain the important analytical quantity in the discrete case. That is just a discrete analogue of Baker–Akhiezer type function $\psi(n, P)$, where P moves on a Riemann surface. The analytical properties of $\psi(n, P)$ is the following, where n is any integer.

- (1) It is everywhere meromorphic on the Riemann surface.
- (2) It has poles of order one at each point of P_1, \dots, P_g .
- (3) It has poles of order $\pm n$ in the points P_\pm , where a pole of order $-n$ ($n > 0$) means, of course, a zero of order n .

There is a very good theta function formula for it. So it is a well elaborated object for us. For an application to conformal field theory we have to consider a special case of Baker–Akhiezer type functions. First, for example, if the genus g of a Riemann surface is even, we have to consider the case where the divisor of poles $D = P_1 + \cdots + P_g$ is such that

$$D = \frac{g}{2}P_+ + \frac{g}{2}P_-.$$

It is a limiting case of Baker–Akhiezer type functions when its poles go to $\{P_+, P_-\}$. So these two points $\{P_+, P_-\}$ are special points in our construction. Second $\psi(n, P)$ may not necessarily be a scalar function but a tensor. That is, we may consider a tensor $\psi^\lambda(n, z)$ with a tensor weight λ on a Riemann surface and a spinor, where, generalizing the notation, we use $\psi^{1/2}(n, z)$ for a spinor, etc. People who study soliton theory have never discussed that in a soliton literature. Because it is not very essential. In fact, if we consider the ratio $\psi^\lambda(n, z)/\psi^\lambda(0, z)$, then it is a scalar. And as far as commuting pairs are concerned, we can reduce all things to the scalar case. So it is not very important for us to consider the difference between scalar and tensor constructions in the theory of commuting operators. The only remark which I have to say is the following. When we are just doing the tensor Baker–Akhiezer construction, the Riemann–Roch theorem shows the following:

	scalar	tensor
weight	0	λ
degree of the divisor of poles	g	$s = g - 2\lambda(g - 1)$

(For details see [9].)

So analytical properties here will be changed. The important thing for us now is that we want to have poles only in two points $\{P_\pm\}$ for all tensor weights. This is a starting point of the consideration which I am going to present now.

2. ANALOGUE OF FOURIER–LAURENT BASES ON RIEMANN SURFACES

In this section Γ denotes a Riemann surface of genus g and P_\pm two points on Γ . Now let us consider a collection $\{f_n^\lambda(z)\}$ of tensors of weight λ in the variable z with the following properties:

- (1) It is holomorphic on $\Gamma \setminus \{P_\pm\}$.
- (2) It has the following asymptotics:

$$f_n^\lambda(z) \sim \varphi_{n\pm}^\lambda z_\pm^{\pm n - s} (1 + O(z_\pm)) \quad \text{near } P_\pm,$$

$$S = \frac{g}{2} - \lambda(g - 1), \quad \varphi_{n\pm}: \text{ a constant},$$

where z_\pm are local coordinates near P_\pm respectively. In fact, in the case $\lambda \neq 0, 1$ and $g \neq 1$, for each $n \in \mathbb{Z} - g/2$, we can prove that there exists such a unique tensor $f_n^\lambda(z)$ up to a constant factor. For the case of exceptional values of λ and g , see [7] and [8]. Before stating a Fourier–Laurent type theorem, we must introduce some elementary but important quantity on a Riemann surface. Consider data $\{\Gamma, P_\pm, z_\pm\}$, so-called 1-string diagram. For these, there exists one and only one differential form dk on Γ which has the following analytical properties:

- (1) $dk \sim \pm \frac{dz_\pm}{z_\pm} (1 + O(z_\pm))$ near P_\pm ,
- (2) $\oint_C dk \in i\mathbb{R}$ for any $C \in H_1(\Gamma)$.

Then $\tau = \operatorname{Re} k(z)$ is a well-defined single-valued function on Γ and means a time. C_τ denotes a level of time τ , that is

$$C_\tau = \{z \in \Gamma \mid \tau(z) = \tau \equiv \text{const}\}.$$

The points P_- and P_+ correspond to in and out strings, where times are infinite. Then the Fourier type formula in our case is the following.

Theorem. *Let $f^\lambda(\sigma)$ be a smooth tensor field of the weight λ on the contour C_τ . Then*

$$f^\lambda(\sigma) = \sum_n f_n^\lambda(\sigma) \frac{1}{2\pi i} \oint_{C_\tau} f^\lambda(\sigma') f_{-n}^{1-\lambda}(\sigma')$$

and

$$\frac{1}{2\pi i} \oint_{C_\tau} f_n^\lambda(\sigma) f_m^{1-\lambda}(\sigma) = \delta_{n+m}.$$

Here note that the product $f^\lambda \cdot f^{1-\lambda}$ is a differential 1-form and we can integrate it along a contour. The proof of this theorem is, of course, an important part. For the proof we use analytical properties in the variable n and therefore use results of soliton theory. We do not go into details now but remark the following thing. Our bases $\{f_n^\lambda\}$ have remarkable properties in the variable n . In fact these functions are quasi-periodic on the variable n after some normalizations.

As for a Laurent theorem, we have to consider holomorphic tensor fields between two contours C_{τ_1} and C_{τ_2} , Then the analogue of a Laurent type theorem will be true. This is a harmonic analysis so to speak. People in harmonic analysis usually deal with only linear properties of these bases. But here we have to work with their multiplicative properties also.

3. ALMOST GRADED STRUCTURE OF THE BASE $\{f_n^\lambda\}$

The origin of the facts that Virasoro and Kac–Moody algebras are graded algebras is the following elementary property of Fourier–Laurent functions in genus 0:

$$z^n \cdot z^m = z^{n+m} \quad \text{for any integers } n \text{ and } m.$$

Now I explain which multiplicative properties are in our Fourier–Laurent bases. Before going to that, I will introduce special notations for the following objects.

λ	notation	name
-1	e_n	vector field
0	A_n	scalar
1/2	φ_n	spinor
1	$d\omega_n$	differential 1-form
2	$d^2\Omega_n$	quadratic differential form.

Our general formula is

$$(3.1) \quad f_n^\lambda \cdot f_m^\mu = \sum_{|k| \leq (g/2)} C_{n,m,k}^{\lambda,\mu} f_{n+m-k}^{\lambda+\mu}.$$

This means some generalization of a grading. Further vector fields act on tensor fields as Lie derivative and the corresponding formula is the following.

$$[e_n, f_m^\lambda] = \sum_{|k| \leq (3g/2)} R_{n,m}^{\lambda,k} f_{n+m-k}^\lambda.$$

In particular, for $\lambda = -1$, we have

$$[e_n, e_m] = \sum_{|k| \leq (3g/2)} C_{nm}^k e_{n+m-k}.$$

So we have a Lie algebra. The well-known theorem of Kac and Moody about the uniqueness of their algebras was proved in a class of graded algebras of a corresponding growth. In our case such type of theorem is impossible because we have a lot of such algebras of type (3.2). In fact we have them as much as the number of points in our moduli space.

Now the analogue of the Heisenberg algebra on a Riemann surface is the algebra generated by $\{a_n, t\}$ with the following relations:

$$\begin{aligned} [a_n, a_m] &= \gamma_{n,m} \cdot t, \\ \gamma_{nm} &= \oint_{C_\tau} A_n dA_m, \\ t &\text{ is a central element,} \end{aligned}$$

where we fix a contour C_τ . And the analogue of Virasoro algebra is the central extension of the algebra of vector fields $\{e_n\}$ by the following commutation relations:

$$\begin{aligned} [e_n, e_m] &= \sum_{|k| \leq (3g/2)} C_{mn}^k e_{n+m-k} + \chi_{nm} \cdot t, \\ \chi_{nm} &= \frac{1}{48\pi i} \oint_{C_\tau} \{(e_n''' e_m - e_m''' e_n) - 2(e_n' e_m - e_m' e_n)R\} dz, \\ t &\text{ is a central element.} \end{aligned}$$

Here e_n', e_n'' , etc. mean that, expressing $e_n = F_n(\partial/\partial z)$ locally, then $e_n' = (\partial F_n/\partial z)(\partial/\partial z)$, etc. Further $R(z)$ is a projective connection which has, by definition, the following transformation property for a coordinate change $w = w(z)$:

$$R(w) = R(z)(w')^2 + \left(\frac{w'''}{w'} - \frac{3}{2} \left(\frac{w''}{w'} \right)^2 \right), \quad w' = \frac{dw}{dz}.$$

In this terminology the integrand in the definition of χ_{nm} is a well-defined 1-form. We remark that, in our case, there are more cohomology classes $\{\chi_{mn}\}$ than the ordinary case on a circle. In the definition of χ_{nm} we can integrate along any contour on a Riemann surface not passing two points $\{P_\pm\}$. We conjecture that the second cohomology group $H^2(\mathcal{G})$, which is the group of central extensions of our Lie algebra $\mathcal{G} = \{e_n\}$, is exactly equal to the first homology group $H_1(\Gamma \setminus \{P_\pm\})$ by the correspondence mentioned above. That is

Conjecture.

$$\begin{aligned} H^2(\mathcal{G}) &\cong H_1(\Gamma \setminus \{P_\pm\}), \\ \mathcal{G} &= \text{the Lie algebra } \{e_n\}. \end{aligned}$$

Instead of the conjecture, we can prove the following theorem.

Theorem. *There is one and only one central extension of the algebra $\{e_n\}$ such that the extended algebra is also almost graded. (For the exact definition of the almost gradedness, see [11].) Roughly speaking it is an algebra of the type (3.2) with $(3g/2)$ (the range of k) changed.*

4. BRIEF COMMENTS ON THE REPRESENTATION THEORY

Let $X^\mu(Q)$ and $P^\mu(Q)$, $Q \in \Gamma$ be an operator valued function and 1-form on Γ which satisfy the following commutation relations:

$$[X^\mu(Q), P^\mu(Q')] = \begin{cases} -i\xi^{\mu\nu} \Delta_\tau(Q, Q'), & \text{if } \tau(Q) = \tau(Q') = \tau, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\Delta_\tau(Q, Q')$ is a delta function on C_τ , that is, for any smooth function $f(Q)$ on C_τ

$$f(Q) = \frac{1}{2\pi i} \oint_{C_\tau} f(Q') \Delta_\tau(Q, Q').$$

And $\xi^{\mu\nu} = \xi^\mu \delta_\mu^\nu$ is a metric of a physical space. By using these quantities, let us define a 1-form $J(\sigma)$ on Γ by

$$J^\mu(\sigma) = \partial_\sigma X^\mu d\sigma + \pi P^\mu(\sigma).$$

Consider the decomposition of this quantity by our base $\{\omega_n\}$:

$$J^\mu(\sigma) = \sum_n \alpha_n^\mu \omega_n(\sigma).$$

Then it can be shown that $\{\alpha_n^\mu\}$ satisfy the following commutation relations of the Heisenberg type algebra:

$$[\alpha_n^\mu, \alpha_m^\nu] = \xi^{\mu\nu} \gamma_{nm}.$$

Starting from this, we can define the analogue of the energy momentum tensor, can do a Sugawara type construction of a representation of the analogue of the Virasoro algebra $\{e_n, t\}$ and so on. In fact, all things in the case $g = 0$ go analogously in our case. For example we can construct an analogue of the Verma module by using semi-infinite forms of the base $\{f_n^\lambda\}$ which is similar to the constructions by Feigin and Fuchs [5] (for details, see [7]). It is not surprising but quite natural. Because our algebra $\{e_n\}$ is a subalgebra of the algebra of vector fields on a circle. Our Fourier–Laurent theorem shows that the topological closure of our algebra coincides with the algebra of vector fields on a circle.

5. FINAL REMARKS

Up to now we only consider 1-string diagram $\{\Gamma, P_\pm, z_\pm\}$. Of course we can consider a multi-string diagram which is, by definition, the set of data $\{\Gamma, P_{\pm\alpha}, z_{\pm\alpha} \mid 1 \leq \alpha \leq l\}$, where Γ is a Riemann surface, l is some natural number, $P_{\pm\alpha}$ are points on Γ in a general position and $z_{\pm\alpha}$ are local coordinates near $P_{\pm\alpha}$ respectively. In this case we can also construct a Fourier–Laurent type bases and can prove a Fourier–Laurent type theorem. In that proof we use a vector analogue of discrete Baker–Akhiezer functions in the theory of commutative difference operators with matrix coefficients. Second we can define a natural pairing between left and right (or in and our) Fock spaces which are, by definition, analogues of Verma module in the form of semi-infinite exterior product of our bases $\{f_n^\lambda\}$. So we may compute expectation values of arbitrary operators. But after computations we shall have some questions. That is, how do we obtain answers which really coincide with a string theory. About these, it was not written in our first papers in 1987. It was observed only one year ago by Krichever. If we fix a spinor structure, we may obtain some Fermionic theory. Our computation shows that it exactly coincides, for example, with the results of papers by Eguchi and Ooguri. But here how do we

obtain a real bosonic string theory? For that, we have to introduce all, so-called Bloch spinors corresponding to 1-dimensional representation of the fundamental group of a Riemann surface Γ :

$$\rho: \pi_1(\Gamma) \rightarrow C^* \quad \text{s.t. } |\rho| = 1.$$

To define a Bloch spinor we have to fix a canonical basis $\{a_i, b_j\}$ of cycles, that is $a_i \cdot a_j = b_i \cdot b_j = 0$, $a_i \cdot b_j = \delta_{ij}$ for $1 \leq i, j \leq g$ (g is the genus of Γ), other than ρ . The observation is that, to obtain the correct answer, we have to integrate the quantity over all representations $\{\rho\}$. Finally as to all the contents here, in particular, for exact definitions of all things, we refer [11] and references cited therein.

REFERENCES

- [1] E. Date and S. Tanaka, Prog. Theor. Phys. **55** (1976), 457.
- [2] B. A. Dubrovin, I. M. Krichever and S. P. Novikov, *Integrable Systems I. Sovremennye Problemy Matematiki. Fund. Napr., Dynamical Systems-4* (translated to English by Springer-Verlag, 1989).
- [3] B. A. Dubrovin, V. B. Matveev and S. P. Novikov, Usp. Math. Nauk **31** (1976), 55.
- [4] T. Eguchi and H. Ooguri, Nucl. Phys. **B282** (1987), 308.
- [5] B. L. Feigin and D. B. Fuchs, Funct. analys. i pril. **16** (1982), 47.
- [6] S. V. Manakov, Sov. Phys. JETP **40** (1974), No. 2.
- [7] I. M. Krichever and S. P. Novikov, Funct. analys. i pril. **21** (1987), 46.
- [8] I. M. Krichever and S. P. Novikov, Funct. analys. i pril. **21** (1987), 47.
- [9] I. M. Krichever and S. P. Novikov, Funct. analys. i pril. **23** (1989), 24.
- [10] I. M. Krichever and S. P. Novikov, Journal of Geometry and Physics, 1989 (to appear).
- [11] I. M. Krichever and S. P. Novikov, "Riemann surfaces, operator fields, strings, analogue of the Fourier-Laurent bases" (Preprint IHES, 1989).

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