TWO-DIMENSIONAL SCHRÖDINGER OPERATOR: INVERSE SCATTERING TRANSFORM AND EVOLUTIONAL EQUATIONS

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ABSTRACT. The inverse problem for the two-dimensional Schrödinger operator on the data from one energy level is solved in a special class of "finite-zone" or "algebraic" operators. This class seems to be dense among all smooth periodic Schrödinger operators. Evolutional equations associated with this problem are constructed.

1. INTRODUCTION

The general nonrelativistic scalar Schrödinger operator in an external time-independent electromagnetic field F_{ij} has the form

$$\hat{H} = \sum_{\alpha=1}^{n} (\partial_\alpha - \mathrm{i}eA_\alpha)^2 + u(x), \quad x = (x_1, \dots, x_n), \tag{1}$$
$$i, j = 0, 1, \dots, n, \quad \alpha = 1, \dots, n, \quad \partial_\alpha = \partial/\partial x_\alpha.$$

By definition, we have electric and magnetic fields:

$$F_{ij} = -F_{ji}, \quad F_{0\alpha} = E_\alpha = \partial_\alpha u, \quad F_{\alpha\beta} = H_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{2}$$

"Gauge" transformations preserve the equation $\hat{H}\psi = \varepsilon\psi$:

$$A_\alpha \to A_\alpha + \partial_\alpha\varphi, \quad \psi \to \psi \exp(-\mathrm{i}e\varphi), \quad u \to u, \quad \varepsilon \to \varepsilon. \tag{3}$$

Using (3), we may reduce the operator $\hat{H}$ for $n = 1, 2$ to the canonical form

$$n = 1\colon \quad \hat{H} = \partial^2 + u(x), \quad \partial = \partial/\partial x, \tag{4}$$

$$n = 2\colon \quad \hat{H} = \partial\bar{\partial} + A\bar{\partial} + V(x, \bar{z}), \tag{5}$$
$$z = x + \mathrm{i}y, \quad \bar{z} = x - \mathrm{i}y, \quad \partial = \partial/\partial z, \quad \bar{\partial} = \partial/\partial\bar{z}.$$

It is well known that the remarkable one-parametric families of linear operators (4) are very important in the soliton theory [1][1]:

$$\frac{\mathrm{d}\hat{H}}{\mathrm{d}t} = [\hat{H}, B_m] \quad (\text{"Higher KdV's"}). \tag{6}$$

Here

$$B_m = \partial_x^{2m+1} + \sum_{j=1}^{2m} b_j(u, u_x, \dots)\, \partial_x^{2m-j}$$

[1]This discovery is the result of the papers of Kruskal and Zabusky (1965), Gardner, Green, Kruskal and Miura (1967) and Lax (1968).

is such a linear differential operator, that eq. (6) is equivalent to the nonlinear P.D.E.

$$
\begin{aligned}
&u_t = \phi_m(u, u_x, \dots, u_x^{(2m+1)}), \\
&B_1 = \partial_x^3 + \frac{3}{4}(u\partial_x + \partial_x u) \leftrightarrow \phi_1 = u_{xxx} + 6uu_x, \\
&B_0 = \partial_x \leftrightarrow \phi_0 = u_x.
\end{aligned} \tag{7}
$$

Direct generalization of (6) to $n = 2$ is possible *only* in parabolic case (8)

$$\hat{H} = \hat{P} = \sigma\partial_y + \partial_x^2 + u, \quad \sigma \in \mathbb{C}. \tag{8}$$

For example, the well-known KP-equation

$$W_x = U_y, \quad 3\sigma^2 W_y = U_t - 6uu_x - u_{xxx}, \quad \sigma^2 = \pm 1, \tag{9}$$

is equivalent to the "Lax-like" equation

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}t} = [\hat{P}, B], \quad B = \partial_x^3 + \frac{3}{2}u\partial_x + W$$

(Y. S. Driuma, A. B. Shabat and V. E. Zakharov 1974).

There is an elementary *theorem*: any Lax-like deformation (6) of the class of all smooth two-dimensional Schrödinger operators (1) is trivial for $n \geq 2$.

A nontrivial two-dimensional generalization of eq. (6) was found by S. V. Manakov in ref. 2:

$$\frac{\mathrm{d}\hat{H}}{\mathrm{d}t} = [\hat{H}, B] + C\hat{H}. \tag{10}$$

The deformation (10) for certain linear P.D.O.'s B, C and the Schrödinger operator (5) is equivalent to the system of nontrivial nonlinear P.D.E.'s

$$
\begin{aligned}
V_t &= \phi_1(V, A, V_x, A_x, V_y, A_y, \dots), \\
A_t &= \phi_2(V, A, V_x, A_x, V_y, A_y, \dots)
\end{aligned} \tag{11}
$$

for all smooth (complex) coefficients $V(x, y, t)$, $A(x, y, t)$.

Eq. (10) looks like a Lax equation on the set of all solutions of (12),

$$\hat{H}\psi \leftrightarrow \left(\frac{\mathrm{d}\hat{H}}{\mathrm{d}t} - [\hat{H}, B]\right)\psi = 0. \tag{12}$$

The periodic inverse spectral problem for the two-dimensional Schrödinger operator (1), based on the spectral data corresponding to one fixed energy level $\varepsilon = \varepsilon_0$, was posed and considered by B. A. Dubrovin, I. M. Krichever and S. P. Novikov in refs. 3, 4. It was solved in [3] for a certain class of "algebraic" operators—the two-dimensional analog of the well-known "finite-zone" operator on the given level, $\varepsilon = \varepsilon_0$—see section 2. Some nontrivial sufficient "reality" conditions (such that $\hat{H}$ is self-adjoint but $A \neq 0$) were noneffectively found by I. V. Cherednic in [5].

Problem. Which spectral data in [3] provide the real "purely potential" operators? This class is most important (see the end of section 2):

$$A \equiv 0, \quad \hat{H} = \partial\bar{\partial} + V(x, y), \quad V \in \mathbb{R}. \tag{13}$$

This problem was partially solved in the recent papers of S. P. Novikov and A. P. Veselov [6, 7] in terms of Riemann surfaces with some group of involutions and corresponding Prym θ-functions—see section 3.

P. G. Grinevitch and R. G. Novikov have recently found the analog of these results for some class of decreasing potentials $V \to 0$, $x^2+y^2 \to \infty$, using the technique of S. V. Manakov ("nonlocal Riemann problem"—see ref. 12).

The deformations (10) preserving a class of the purely potential self-adjoint operators (13) were found and studied in [6]. The simplest and important example is

$$\begin{gathered}\frac{\mathrm{d}\hat{H}}{\mathrm{d}t} = [\hat{H}, B] + f\hat{H},\\ \hat{H} = \partial\bar{\partial} + V, \quad B = \mathcal{D} + \bar{\mathcal{D}}, \quad \mathcal{D} = \partial^3 + u\partial,\\ V_t = (\partial^3 + \bar{\partial}^3)V + \partial(uV) + \bar{\partial}(\bar{u}V),\\ \bar{\partial}u = 3\,\partial V, \quad f = \partial u + \bar{\partial}\bar{u}.\end{gathered} \tag{14}$$

It is possible to exploit (14) for the effectivisation of θ-functional formulas for $\hat{H}$ (and also for recognition of Prym functions, like in [8] for the Jacobian varieties. This program was developed recently by I. A. Taimanov. For some result see section 4).

The difference analog of this theory was constructed recently by I. M. Krichever [11].

Conjectures (S. P. Novikov). 1) The algebraic (rank $l = 1$) operators $\hat{H}$ corresponding to one energy level generate a dense family among all smooth real, purely potential periodic operators for $n = 2$.

2) All such algebraic operators have the spectral data described in [7] (the analogous problem is not solved for KP either).

2. Two-dimensional algebraic operators. Spectral data and inverse problem

First recall some definitions from [3].

Definition 1. A two-dimensional P.D.O. L_1 is algebraic if there are nontrivial P.D.O.'s L_2, L_3, B_{ij} such that (15) is true:

$$[L_i, L_j] = B_{ij}L_1, \quad i,j = 1,2,3. \tag{15}$$

General properties of algebraic operators:

1) There is a polynomial $P(\lambda,\mu)$ such that

$$L_1\psi = 0 \implies P(L_2, L_3)\psi = 0. \tag{16}$$

2) The common eigenfunction $\psi(x,y,\lambda,\mu)$

$$L_1\psi = 0, \quad L_2\psi = \lambda\psi, \quad L_3\psi = \mu\psi \tag{17}$$

is meromorphic on the Riemann surface Γ; see its analytical properties below:

$$\begin{gathered}P(\lambda,\mu) = 0, \quad (\lambda,\mu) = Q \in \Gamma,\\ \psi(x,y,\lambda,\mu) = \psi(x,y,Q).\end{gathered} \tag{18}$$

Definition 2. The rank of an algebraic operator L_1 is the dimension of the space $\psi(x,y,Q)$ in a "general" point $Q \in \Gamma$.

We shall discuss in this work only the algebraic operator of rank $l = 1$. See the general theory for $l > 1$ in [9].

Suppose that Γ is nonsingular and has rank $l = 1$. The analytical properties of the algebraic Schrödinger operator (5) were described in [3]:

1) The common normalised eigenfunction $\psi(x, y, Q)$, $\hat{H}\psi = 0$, $\psi(0, 0, Q) \equiv 1$, $Q \in \Gamma$ is meromorphic on $\Gamma \setminus (P_1 \cup P_2)$; the points $P_1 \neq P_2 \in \Gamma$ are some "infinite" points with local parameters $k_1^{-1} = w_1$, $k_2^{-1} = w_2$, $k_i(Q) \to \infty$, $Q \to P_i$, $i = 1, 2$.

2) In general ψ has g different poles $Q_1, \dots, Q_g$, whose position are independent of (x, y); $g = g(\Gamma)$ is the genus of Γ.

3) ψ has the asymptotic

$$
\begin{gathered}
Q \to P_1, \psi(x, y, Q) = C_1(x, y)\, \mathrm{e}^{k_1 z} \Big(1 + \sum_{i \geq 1} \eta_i w_1^i\Big), \\
Q \to P_2, \psi(x, y, Q) = C_2(x, y)\, \mathrm{e}^{k_2 \bar{z}} \Big(1 + \sum_{i \geq 1} \xi_i w_2^i\Big), \\
z = x + \mathrm{i}y, \quad \bar{z} = x - \mathrm{i}y.
\end{gathered}
\tag{19}
$$

Definition 3. The quantities $(\Gamma, P_1, P_2, k_1, k_2, Q_1, \dots, Q_g)$ with the properties 1–3 mentioned above are "spectral data" for the generic algebraic Schrödinger operator $L_1 = \hat{H}$ of the general form (1) for $n = 2$ and rank $l = 1$.

Any spectral data $(\Gamma, P_1, P_2, k_1, k_2, Q_1, \dots, Q_g)$ for the nonsingular surface Γ with genus $g(\Gamma) = g$, generic divisor $\mathcal{D} = Q_1 + \dots + Q_g$, any two points $P_1 \neq P_2$ and k_1, k_2 local parameters determine the unique function $\psi(x, y, Q)$ and the Schrödinger operator $\hat{H}$ such that

$$
\begin{gathered}
C_1 \equiv 1, \quad \hat{H} = \partial\bar{\partial} + A\bar{\partial} + V, \\
\hat{H}\psi \equiv 0, \quad A = -\partial \ln C_2, \quad V = -\frac{\partial \eta_1}{\partial \bar{z}}.
\end{gathered}
$$

The coefficients of $\hat{H}$ are complex, periodic or quasi-periodic (with $2g$ quasi-periods) functions on (x, y):

$$
\begin{gathered}
V = \partial\bar{\partial} \ln \theta(U_1 z + U_2 \bar{z} + \zeta_0 + A(P_1)), \\
A = -\partial \ln \frac{\theta(U_1 z + U_2 \bar{z} + \zeta_0 + A(P_2))}{\theta(U_1 z + U_2 \bar{z} + \zeta_0 + A(P_1))},
\end{gathered}
\tag{20}
$$

$$
\psi = \frac{\theta(A(P) + zU_1 + \bar{z}U_2 + \zeta_0)\, \theta(A(P_1) + \zeta_0)}{\theta(A(P) + \zeta_0)\, \theta(A(P_1) + zU_1 + \bar{z}U_2 + \zeta_0)} \cdot \exp\bigg(z\Big(\int_{P_0}^{P} \Omega_1 - \alpha\Big) + \bar{z} \int_{P_1}^{P} \Omega_2\bigg).
$$

Changes of the local parameters $w_1 = aw_1' + \cdots$, $w_2 = bw_2' + \cdots$ lead only to the linear transformation

$$
\begin{gathered}
\hat{H} \to \hat{H}' = a^{-1}b^{-1}(\partial'\bar{\partial}' + A'\bar{\partial}' + V'(z', \bar{z}')), \\
z = az', \quad \bar{z} = b\bar{z}', \quad A' = aA(az', b\bar{z}'), \quad V' = abV(az', b\bar{z}').
\end{gathered}
$$

For the self-adjoint operators $\hat{H}$ with periodic coefficients A, V a function ψ is a Bloch function corresponding to the zero-energy level.

Here (a_j, b_j) is the canonic basis of $H_1(\Gamma, \mathbb{Z})$, $\hat{\omega}_1, \dots, \hat{\omega}_g$ is the basis of holomorphic forms on Γ and Ω_1, Ω_2 are the meromorphic forms with the poles only in P_1, P_2

respectively such that

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij};$$

$$\oint_{a_k} \hat{\omega}_j = 2\pi \mathrm{i} \delta_{jk}, \quad \oint_{b_\mu} \hat{\omega}_\nu = \hat{B}_{\mu\nu} = \hat{B}_{\nu\mu}, \quad \oint_{a_k} \Omega_\alpha = 0,$$

$$\Omega_\alpha = -w_\alpha^{-2}\, \mathrm{d}w_\alpha (1 + \text{reg.}), \quad \alpha = 1,2, \quad U_\alpha^j = \oint_{b_j} \Omega_\alpha, \tag{21}$$

$$\hat{\theta}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\eta_1, \dots, \eta_g) = \sum_{N \in \mathbb{Z}^g} \exp\left\{\frac{1}{2}\langle \hat{B}(N+\alpha), N+\alpha\rangle + \langle \eta + 2\pi \mathrm{i}\beta, N+\alpha\rangle\right\}.$$

$A\colon \Gamma \to J(\Gamma)$ is the "Abel map"

$$A(P)^i = \int_{P_0}^{P} \hat{\omega}_i \quad (i = 1, \dots, g),$$

α is constant, which is determined from the property

$$\left(\int_{P_0}^{P} \Omega_1 - \alpha\right) = w_1^{-1} + \mathcal{O}(w_1).$$

The important problem is: for which spectral data the corresponding operators have $A \equiv 0$ and V real and smooth. It will be discussed in section 3.

There is an important class of two-dimensional operators in the external periodic (or constant) magnetic field $H(x,y)$ and the electric lattice potential $V(x,y)$:

$$\begin{gathered} \hat{H} = \partial\bar{\partial} + A\partial + V, \quad V(x+T_1, y) = V(x, T_2 + y) = V(x,y), \\ H(x,y) = \bar{\partial}A(x,y) = H(x+T_1, y) = H(x, y+T_2). \end{gathered} \tag{22}$$

For the operators (22) we have periodic fields, but nonperiodic operators. This class is not contained in our theory. Its mathematical theory is quite different, see ref. 10. Our theory considers only the case in which the average magnetic field $\bar{H}$ (or the magnetic "flux") is trivial—"topoligically trivial" magnetic fields as the cohomology classes on the torus T^2. In this case "physical" magnetic fields are usually identically zero in the real crystals. So the most important case in our theory is $A \equiv 0$ (section 3).

3. Schrödinger operators with zero magnetic field. Prym θ-functions

The simplest examples of the algebraic purely potential operators are

$$\hat{H} = \partial\bar{\partial} + V(x,y), \quad V(x,y) = V_1(x) + V_2(y) \tag{23}$$

(here the operators $H_1 = \partial_x^2 + V_1(x)$ and $H_2 = \partial_y^2 + V_2(y)$ are "finite-zoned" or "finite-gap" 1-dimensional operators). The operators (23) are algebraic, corresponding to *any* level; the last property of (23) makes an exception, see section 4.

Any spectral data $(\Gamma, P_1, P_2, k_1, k_2, Q_1, \dots, Q_g)$, satisfying the following conditions a), b), give purely potential Schrödinger operators $\hat{H} = \partial\bar{\partial} + V(x,y)$:

a) the nonsingular surface Γ has an involution

$$\sigma\colon \Gamma \to \Gamma, \quad \sigma^2 = 1$$

such that

$$\sigma(P_1) = P_1, \quad \sigma(P_2) = P_2, \quad \sigma(k_\alpha) = -k_\alpha \quad (\alpha = 1,2). \tag{24}$$

b) The divisor of poles $\mathcal{D} = Q_1 + \dots + Q_g$ satisfies the relationship

$$\mathcal{D} + \sigma(\mathcal{D}) \cong K + P_1 + P_2. \tag{25}$$

Here K is the canonical divisor (a divisor of differential forms) and $\cong$ means the so-called "linear equivalence" of the divisors.

The potential V is real if spectral data have the following properties:

c) There is an anti-involution τ

$$\tau \colon \Gamma \to \Gamma$$

such that the pair (σ, τ) generates the group $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$\tau^2 = 1, \quad \tau\sigma = \sigma\tau, \quad \tau(P_1) = P_2, \quad \tau(k_1) = \bar{k}_2 \tag{26}$$

and the divisor $\mathcal{D}$ is τ-invariant:

$$\tau(\mathcal{D}) = \mathcal{D}. \tag{27}$$

Remark. I. Shafarevitch and V. Shockurov explained to us that (25) is solvable iff the involution σ has exactly 2 fixed points P_1, P_2, see ref. 7.

Choose the canonical basis (21) $a_j, b_j \in H_1(\Gamma)$, $j = 1, \dots, g = 2g_0$ the basis of holomorphic differential 1-forms $\hat{\omega}_j$ and the meromorphic differentials Ω_α, $\alpha = 1, 2$ with the properties (23), (28):

$$\sigma(a_i) = a_{i+g_0}, \quad \sigma(b_i) = b_{i+g_0}, \quad i = 1, \dots. \tag{28}$$

Definition 4. The Prym differentials ω are meromorphic differentials on Γ such that

$$\sigma^* \omega = -\omega.$$

We can construct the basis of the holomorphic Prym differentials from (28)

$$\begin{gathered} \omega_1, \dots, \omega_{g_0}, \quad \omega_i = \hat{\omega}_i - \hat{\omega}_{i+g_0}, \\ \oint_{a_k} \omega_j = \delta_{kj}, \quad B_{kj} = \oint_{b_k} \omega_j = B_{jk}. \end{gathered} \tag{29}$$

The lattice (29) determines some abelian variety $P(\Gamma, \sigma)$ ("Prym variety") and the θ-functions (30), which depend on g_0 variables: $\theta(\eta_1, \dots, \eta_{g_0}) = \theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](\eta_1, \dots, \eta_{g_0})$,

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\eta_1, \dots, \eta_{g_0}) = \sum_{N \in \mathbb{Z}^{g_0}} \exp\left\{\frac{1}{2}\langle B(N+\alpha), N+\alpha\rangle + \langle \eta + 2\pi \mathrm{i}\beta, N+\alpha\rangle\right\}. \tag{30}$$

Both the meromorphic differentials Ω_α are Prym differentials,

$$\sigma^* \Omega_\alpha = -\Omega_\alpha.$$

Any meromorphic differential form $\Omega_\alpha^{(k)}$ which has only one pole P_α and property (31) is the Prym differential

$$\begin{gathered} \oint_{\alpha_j} \Omega_\alpha^{(k)} = 0, \quad j = 1, \dots, 2g_0, \\ \Omega_\alpha(k) = w_\alpha^{2k}\, \mathrm{d}w_\alpha(1 + \text{reg.}), \quad \sigma^* \Omega_\alpha^{(k)} = -\Omega_\alpha^{(k)}, \\ \Omega_\alpha^{(1)} = \Omega_\alpha, \quad \sigma^* w_\alpha = -w_\alpha. \end{gathered} \tag{31}$$

We have a collection of g_0-vectors $\boldsymbol{U}_\alpha^{(k)}$:

$$
\begin{aligned}
&U_{\alpha_j}^{(k)} = \oint_{b_j} \Omega_\alpha^{(k)}, \quad \alpha = 1, 2; \quad k = 1, 2, \ldots, \quad j = 1, 2, \ldots, g_0, \\
&\boldsymbol{U}_\alpha^{(1)} = \boldsymbol{U}_\alpha.
\end{aligned} \tag{32}
$$

Coefficients and eigenfunctions of the Schrödinger operators $H = \partial\bar{\partial} + V$ may be written by the following formulas in Prym θ-functions:

$$
\begin{aligned}
V &= 2\,\partial\bar{\partial} \ln \theta(\boldsymbol{U}_1 z + \boldsymbol{U}_2 \bar{z} + \zeta_0) + c(\Gamma, \sigma), \\
\psi(x, y, P) &= \frac{\theta(\eta(P) + zU_1 + \bar{z}U_2 + \zeta_0)\,\theta(\zeta_0)}{\theta(\eta(P) + \zeta_0)\,\theta(zU_1 + \bar{z}U_2 + \zeta_0)} \\
&\qquad \times \exp\biggl(z\biggl(\int_{P_0}^{P} \Omega_1 - \alpha\biggr) + \bar{z} \int_{P_1}^{P} \Omega_2\biggr), \\
\eta(P)^i &= \int_{P_1}^{P} \omega_i \quad (i = 1, \ldots, g_0), \quad \hat{H}\psi = 0.
\end{aligned} \tag{33}
$$

The constant g_0-vector ζ_0 depends only on the divisor $\mathcal{D}$. For the real potentials $V(x, y)$ we have a factor-surface Γ_0 with the anti-involution τ_0 induced by τ:

$$
\Gamma_0 = \Gamma/\sigma, \quad \tau_0 \colon \Gamma_0 \to \Gamma_0, \quad \tau_0^2 = 1, \quad \tau_0(P_1) = P_2.
$$

The genus $g(\Gamma_0)$ is equal to $g_0 = g/2$. In the general case the anti-involution τ_0 has q smooth fixed ovals S_i:

$$
S_1, \ldots, S_q \subset \Gamma_0, \quad \tau|_{S_j} \equiv 1, \quad j = 1, \ldots, q \le g_0 + 1, \quad S_i \cap S_j = \phi.
$$

By definition, the so-called "M-curves" (Γ_0, τ_0) have exactly the maximal possible number of ovals, $q = g_0 + 1$.

Conjecture. Formula (33) gives the real smooth algebraic potential $V(x, y)$ only if θ is the θ-function of some Prym variety $P(\Gamma, \sigma)$; $\boldsymbol{U}_1, \boldsymbol{U}_2$ are the vectors of the b-period of the corresponding meromorphic Prym differentials (31) and ζ_0 is some admissible constant vector; the set of all admissible constant vectors $P^0_{\mathbb{R}}(\Gamma, \sigma) \subset P(\Gamma, \sigma)$ is always connected and non-empty iff (Γ_0, τ_0) is the M-curve. The corresponding operators are positive only if the conditions (35) (below) are satisfied.

1) If $Q \in \Gamma$ is such that $\sigma\tau(Q) = Q$ the Bloch's function $\psi(x, y, Q)$ is bounded for all real $x, y \in \mathbb{R}$

$$
|\psi(x, y, Q)| < \text{const} < \infty, \quad \sigma\tau(Q) = Q. \tag{34}
$$

(The fixed ovals of anti-involution $\sigma\tau$ give a "real Fermi-curve" on the level $\varepsilon = 0$.)

2) Suppose that the pair (Γ_0, τ_0) is M-curve, $\sigma\tau$ has exactly $2g_0 + 1$ ovals $(a'_1, \ldots, a'_{g_0}, a''_1, \ldots, a''_{g_0}, b)$ such that for $\mathcal{D} = Q_1 + \cdots + Q_g$, $g = 2g_0$, we have

$$
\sigma\tau(a'_j) = a''_j, \quad \sigma\tau(b) = b, \quad Q_j \in a'_j, \quad Q_{j+g_0} \in a''_j, \quad j = 1, \ldots, g_0. \tag{35}
$$

In this case the operator $\hat{H} = \partial\bar{\partial} + V$ is positive $\hat{H} > 0$.

Conjecture. Suppose, that $\sigma\tau$ has no fixed points and τ has exactly $d + 2l$ ovals $(b_1, \ldots, b_d, a'_1, a''_1, \ldots, a'_l, a''_l)$ such that $\sigma\tau(b_j) = b_j$, $\sigma\tau(a'_q) = a''_q$. The number of different dispersion relations $\varepsilon_j(P_1, P_2)$ (less than zero) is at least S:

$$
d \equiv 1 \pmod 2, \quad \varepsilon_j(P_1, P_2) < 0, \quad S \ge (d - 1)/2, \quad j = 1, \ldots, S.
$$

Of special interest is the degenerate case pt. 2). Suppose that we have a family of data $(\Gamma(\lambda),\dots)$ such that:

$$\Gamma(\lambda)\to\Gamma(\lambda_0)=\bar\Gamma,\quad \lambda\to\lambda_0,\quad b\to \text{point } Q_0\in\bar\Gamma.$$

In this case we obtain the so-called "ground state" $\varepsilon=0$,

$$(\hat H\varphi,\varphi)>0,\quad \varphi\in\mathcal{L}_2(\mathbb{R}^2),\quad \hat H\psi(x,y,Q_0)=0.$$

The Prym variety of the limiting singular curve $\bar\Gamma$ with an involution is nonsingular; the corresponding Prym θ-functions are also nonsingular. Adequate formulas for the ground-state eigenfunction ψ may be found in [6].

4. Nonlinear equations as the deformations of the two-dimensional Schrödinger operator

General Schrödinger operator (1) for $n=2$ has a number of deformations (10). The first examples were found in [3]. The "hierarchy" of all such deformations with multiparametric Ψ function may be easily deduced from (14′).

The function $\psi=\psi(x,y,t_2',t_2'',\dots,t_i',t_i'')$ has the analytical properties like in section 2, but pt. 1) is replaced by 1′:

1) ψ has the asymptotic

$$(14') \qquad \begin{aligned} &Q\to P_1,\quad \psi=C_1(x,y)\,\mathrm{e}^{k_1z+\sum_{i\ge2}k_1^it_i'}\Big(1+\sum_{i\ge1}\eta_iw_1^i\Big),\\ &Q\to P_2,\quad \psi=C_2(x,y)\,\mathrm{e}^{k_2z+\sum_{i\ge2}k_2^it_i''}\Big(1+\sum_{i\ge1}\xi_iw_2^i\Big),\end{aligned}$$

General formulas for $A(x,y,t',t'')$ and $V(x,y,t',t'')$ may be obtained trivially from (20) by putting additional terms in the argument of (20); these terms are linearly dependent on all t_i',t.

The deformations of purely potential operators were first considered in [6]:

Any deformation (14′) such that $t_{2i}'=t_{2i}''=0$ preserves the class of purely potential operators ($C_1=1$, C_2 constant). The deformation preserves the "reality" property if $t_{2j+1}'=t_{2j+1}''\in\mathbb{R}$. The latter deformations have the form

$$(36) \qquad \begin{gathered} \frac{\partial\hat H}{\partial t_{2j+1}}=[\hat H,a_j\mathcal{D}_j+\bar a_j\bar{\mathcal{D}}_j]+C_j\hat H,\\ a_jt_{2j+1}=t_{2j+1}'=\bar t_{2j+1}'',\quad \hat H=\partial\bar\partial+V,\\ \mathcal{D}_j=\partial^{2j+1}+u_1^{(j)}\partial^{2j-1}+\cdots,\quad a_j\in\mathbb{C},\\ \mathcal{D}_1=\partial^3+u_1\partial,\quad C_1=a_1\partial u_1+\bar a_1\bar\partial u_1,\\ \mathcal{D}_0=\partial,\quad C_0=0.\end{gathered}$$

According to the natural variant of the so-called "Novikov conjecture" formula (37) satisfies (36) for $j=1$ iff it corresponds to some pair (Γ,σ) (it corresponds to some triple (Γ,σ,τ) in the real case, see also section 3):

$$(37) \qquad \begin{gathered} V(x,y,t)=2\,\partial\bar\partial\ln\theta(U_1z+U_2\bar z+Wt+\zeta_0)+c,\\ c=\text{const},\quad a_1=1,\quad t_1=t,\quad W=U_1^{(2)}+U_2^{(2)}.\end{gathered}$$

Definition 5. We call $g_0 \times g_0$ matrix $B_{\mu\nu}$ "generic" if the rank of the matrix

$$(\tilde\theta_{11}[n], \tilde\theta_{12}[n], \dots, \tilde\theta_{g_0 g_0}[n], \tilde\theta[n])$$

is equal to $g_0(g_0+1)/2 + 1$.

Here $\tilde\theta_{ij}[n] = \partial_i \partial_j \tilde\theta\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right](w)|_{w=0}$, $n \in \mathbb{Z}_2^{g_0}$ and $\tilde\theta$ is the θ-function corresponding to the Riemann matrix $2B$.

Theorem (I. A. Taimanov)**.** *Suppose that matrix $B_{\mu\nu}$ is generic and the g_0-vectors U_1, U_2 are linearly independent. If formula* (37) *satisfies the equation* (36) *for $j = 1$, then vector W and constant C may be calculated as the functions of $U_1, U_2, B_{\mu\nu}$. For $g_0 = 2$ any generic matrix $B_{\mu\nu}$ and independent vectors U_1, U_2 give the algebraic purely potential Schrödinger operator and the solution of* (36) *for $j = 1$, $a_j = 1$, using the formulas for c, W.*

The structure of exact formulas for $c(U_1, U_2, B_{\mu\nu})$ contains very interesting information on some identities between the θ-constants.

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