# HOMOTOPICALLY EQUIVALENT SMOOTH MANIFOLDS. I

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In this paper we introduce a method for the investigation of smooth simply connected manifolds of dimension $n \geq 5$ that permits a classification of them with exactness up to orientation-preserving diffeomorphisms. This method involves a detailed investigation of the properties of the so-called Thom complexes of normal bundles and is based on a theorem of Smale concerning the equivalence of the concepts of "$h$-cobordism" and "orientation-preserving diffeomorphism." In the last chapter we work out some simple examples. Appendices are given in which the results of this paper are applied to certain other problems.

## Introduction

This paper is devoted to a study of the following question: What are the invariants that define the property of two smooth oriented manifolds of being diffeomorphic to each other? It is clear that for manifolds to be diffeomorphic it is necessary that they be homotopically equivalent. A more refined necessary condition is given by the tangent bundle of a manifold. Speaking in modern terms, to die manifold $M^n$ corresponds an Atiyah–Hirzebruch–Grothendieck functor

$$K_R(M^n) = Z + \tilde{K}_R(M^n),$$

and by a tangent bundle we mean a certain distinguished element $\tau(M^n) \in \tilde{K}_R(M^n)$, the "stable tangent bundle" with the exception of its degree. Although the ring $\tilde{K}_R(M^n)$ itself is homotopically invariant, it is well known that the element $\tau(M^n)$ is not homotopically invariant, and what is more, it can have infinitely many values. For two manifolds $M_1^n$, and $M_2^n$ to be diffeomorphic it is necessary that there exist a homotopy equivalence $f\colon M_1^n \to M_2^n$ such that

$$f^*\tau(M_2^n) = \tau(M_1^n),$$

where $f^*\colon \tilde{K}_R(M_2^n) \to \tilde{K}_R(M_1^n)$. If this latter necessary condition is fulfilled, then the direct products $M_1^n \times R^N$ and $M_2^n \times R^N$ are diffeomorphic (Mazur). But this result of Mazur is of little help in determining whether or not $M_1^n$ and $M_2^n$ are diffeomorphic. Even for $n = 3$ there exist nondiffeomorphic manifolds that satisfy the indicated necessary conditions for manifolds to be diffeomorphic (lenses). To be sure, these manifolds are not simply connected. For simply connected manifolds the papers of Whitehead on simple homotopy type or the papers of Smale [17, 19] yield a stronger result, namely, that the direct products by a ball $M_1^n \times D^N$ and $M_2^n \times D^N$ are diffeomorphic. Nevertheless examples by Milnor [10] of differentiable structures on spheres show that, for simply connected manifolds combinatorially equivalent to a sphere, multiplication by a closed ball actually eliminates the existence of a finer distinction between smooth structures.

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In the papers by Milnor [9] and by Milnor and Kervaire [6] a more or less complete classification was finally given of homotopy spheres with exactness up to $h$-homology ($J$-equivalence) in terms of the standard homotopy groups of spheres.

The foundation for this classification was laid by papers of Smale [17, 19], who demonstrated that, for simply connected manifolds of dimension $n \geq 5$, the concepts of "$h$-homology" and "orientation-preserving diffeomorphism" coincide. In addition, Smale proposed a method that permits this classification and Wall gave a good classification of manifolds in certain simple examples (cf. [18, 27]).

In the present paper we investigate the class of smooth manifolds $\{M_i^n\}$ that are homotopically equivalent among themselves and such that for any pair $i, j$ there exists a homotopy equivalence $f_i \colon M_i^n \to M_j^n$ of degree $+1$, and also

$$f^* \tau(M_j^n) = \tau(M_i^n),$$

where $f^* \colon \tilde{K}_R(M_j^n) \to \tilde{K}_R(M_i^n)$ and $\tau(M^n)$ represents the stable tangent bundle. Thus we consider the class of smooth manifolds having the same homotopy type and tangent bundle. The basic problem is to give a classification of manifolds of the class $\{M_i^n\}$ for $n \geq 5$, assuming that $\pi_1(M_i^n) = 0$. The approach used in this paper is connected with a consideration of the Thom complex $T_N$ of the stable normal bundle for the manifold $M_0^n$ belonging to the class $\{M_i^n\}$. The complex $T_N = T_N(M_0^n)$ is obtained by a contraction of the boundary of the $\epsilon$-neighborhood $U_\epsilon^{N+n}$ of the manifold $M_0^n$ in the space $R^{N+n}$ into a point, i.e.,

$$T_N = U^{N+n} / \partial U^{N+n},$$

and it is easily shown that the complex $T_N$ of dimension $n+N$ is a pseudomanifold with fundamental cycle $[T_N]$, belonging to a form of the homomorphism of Hurewicz

$$H \colon \pi_{n+N}(T_N) \to H_{n+N}(T_N).$$

The finite set $A = H^{-1}([T_N])$ is examined. The group $\pi(M_0^n, SO_N)$ acts on this set, and on the set of orbits $A/\pi(M_0^n, SO_N)$ there acts the group $\pi^+(M_0^n, M_0^n)$ of homotopy classes of mappings $f \colon M_0^n \to M_0^n$ of degree $+1$ and such that

$$f^* \tau(M_0^n) = \tau(M_0^n).$$

A proof of the following assertion is the main objective of Chapter I.

**Classification Theorem.** *There exists a natural mapping of sets* $\{M_i^n\} \to (A/\pi(M_0^n, SO_N))/\pi^+(M_0^n, M_0^n)$, *possessing the following properties*:

a) *if under this mapping two manifolds* $M_1^n$ *and* $M_2^n$ *go into one and the same element, then one can find a Milnor sphere* $\tilde{S}^n \in \theta(\partial\pi)$ *such that* $M_1^n = M_2^n \# \tilde{S}^n$;

b) *conversely, if* $M_1^n = M_2^n \# \tilde{S}^n$, *then under this mapping they go into one and the same element of the set* $(A/\pi(M_0^n, SO_N))/\pi^+(M_0^n, M_0^n)$, *where* $\tilde{S}^n \in \theta^n(\partial\pi)$;

c) *if* $n \neq 4k+2$, *then this mapping of sets is epimorphic.*

From this theorem one can immediately draw certain conclusions. For example, one can easily prove the following

**Corollary.** *The homotopy type and the rational classes of Pontrjagin determine a smooth simply connected manifold* $M^n$ *to within a finite number of possibilities for* $n \geq 5$. *If the groups* $H_{4i}(M^n)$ *are finite for* $0 < 4i < n$, *then there exists a finite number of orders of smoothness on the topological manifold* $M^n$ (*a result of the finiteness of the set* $A$).

In reality the solution of the problem obtained by the author is much more significant in homotopy terms than in the way it is formulated in the cited Classification Theorem. A series of geometric properties of manifolds admits a natural interpretation in terms of the homotopy properties of the space $T_N$. These properties are studied at the end of Chapter I (Theorems 6.9 and 6.10) and throughout Chapter II, which is also concerned with a development of the methods of numerical calculation. We mention here a number of problems that are studied at the end of Chapter I and in Chapter II.

1. The conditions under which a mapping $f\colon M^n \to M^n$ of degree +1 is homotopic to a diffeomorphism (Theorems 6.9 and 6.10).

2. A study of the action of the group $\pi^+(M_0^n, M_0^n)$ on the set $A/\pi(M_0^n, SO_N)$ (§7).

3. A determination of the obstructions $d_i(M_1^n, M_0^n) \in H_{n-i}(M_1^n, \pi_{N+i}(S^N))$ to the manifold $M_1^n \in \{M_i^n\}$ being diffeomorphic to the manifold $M_0^n$ (§8).

4. The connected sum of a manifold with a Milnor sphere and its homotopic meaning (§9).

5. The variation in smoothness on a $\pi$-manifold along a cycle of minimal dimension (§9).

6. Variation in smoothness and Morse's reconstruction (§10).

In Chapter III the results of Chapters I and II are applied to the working out of examples. The result of §14 was independently obtained by W. Browder [29].

In addition to the main text of the paper there is inserted at the end four appendices, written quite concisely and not very rigorously. The reader can regard these appendices (together with the results of §§10 and 12) as annotations of new results, the complete proofs of which will be published in later parts of this article. However, in these appendices and in §§10 and 12 we have sketched out the proofs with sufficient detail that a specialist might completely analyze them without waiting for the publication of later parts.

In Appendix 1 the results of §14 are expressed in the language, suitable for calculations, of the Atiyah–Grothendieck–Hirzebruch $K$- and $J$-functors, and there is indicated an application of these results to Pontrjagin's theory of classes.

Appendix 2 is devoted to (i) an extension of the results of the paper to combinatorial manifolds and (ii) an investigation of the relation between smooth and combinatorial manifolds.

Appendix 3 is devoted to a study of the action of the Milnor groups $\theta^{4k-1}(\partial\pi)$ on manifolds and to the problem of singling out the group $\theta^{4k-1}(\partial\pi)$ as a direct summand in the group $\theta^{4k-1}$.

In Appendix 4 we study the problem of determining the euclidean spaces in which a nontrivial Milnor sphere can be embedded in such a way that its normal bundle there is trivial.

# Chapter I
# The fundamental construction [1]

## § 1. Morse's reconstruction

The material of this section is largely borrowed from other papers (for example, [9] and [5]) and is essentially a somewhat generalized account of them in a terminology adapted to our present purposes.

Let $M^n \subset R^{n+N}$ be a smooth manifold with or without boundary, smoothly situated in a euclidean space $R^{n+M}$ of sufficiently large dimension. Let $S^i \times D^{n-i}_\epsilon \subset M^n$ be a smooth embedding of the direct product $S^i \times D^{n-i}_\epsilon$ in $M^n$, where $D^{n-i}_\epsilon$ is a ball in the space $R^{n-i}$ (of radius $\epsilon$) in the natural coordinate system. Consider the diffeomorphism

$$h\colon \partial D^{i+1} \times D^{n-i}_\epsilon \to S^i \times D^{n-1}_\epsilon \subset M^n$$

such that $h(x, y) = (x, h_x(y))$, where $h_x \in SO_{n-i}$. The set of maps $h_x$, $x \in S^i$, defines a smooth map $d(h)\colon S^i \to SO_{n-i}$, which completely defines the diffeomorphism $h$.

Let us put

$$(1)\qquad \begin{aligned} B^{n+1}(h) &= M^n \times I\left(0, \frac{1}{2}\right) \cup_h D^{i+1} \times D^{n-i}_\epsilon, \\ M^n(h) &= (M^n \setminus S^i \times D^{n-i}_\epsilon) \cup_h D^{i+1} \times \partial D^{n-i}_\epsilon. \end{aligned}$$

The operation of going from $M^n$ to $M^n(h)$ is called "Morse's reconstruction." It is well known that:

1. $\partial B^{n+1}(h) = M^n \cup (-M^n(h))$ if $M^n$ is closed.
2. The manifolds $B^{n+1}(h)$ and $M^n(h)$ can be defined as smooth orientable manifolds.
3. The subspace $(M^n \times 1/2) \cup_h D^{i+1} \times 0 \subset B^{n+1}(h)$ is a deformation retract of $B^{n+1}(h)$.
4. The manifold $B^{n+1}(h)$ is defined up to diffeomorphism by the homotopy class $\tilde{d}(h)$ of the smooth map $d(h)\colon S^i \to SO_{n-i}$; $\tilde{d}(h) \in \pi_i(SO_{n-i})$.
5. The manifold $B^{n+1}(h)$ can be so situated in the direct product $R^{n+N} \times I(0, 1)$ that

$$B^{n+1}(h) \cap R^{n+N} \times 1 = M^n(h),$$
$$B^{n+1}(h) \cap R^{n+N} \times 0 = M^n$$

and $B^{n+1}(h)$ approaches the boundaries $R^{n+N} \times 1$ and $R^{n+N} \times 0$ orthogonally.

On the tubular neighborhood $T_{2\epsilon}$ (of radius $2\epsilon$) of the sphere $S^i \subset M^n$, where $T_{2\epsilon} = S^i \times D^{n-i}_{2\epsilon}$, let there be given a frame field $\tau^N$ continuous on $T_{2\epsilon}$ and normal to the manifold in $R^{n+N}$. We have

**Lemma 1.1.** *Suppose the inclusion homomorphism $\pi_i(SO_{n-i}) \to \pi_i(SO_{N+n-i})$ is an epimorphism. Then the diffeomorphism*

$$h\colon \partial D^{i+1} \times D^{n-i}_\epsilon \to S^i \times D^{n-i}_\epsilon \subset M^n$$

*may be chosen in such a way that the frame field $\tau^N$, which is normal to $T_{2\epsilon}$ in $R^{N+n}$, can be extended to a frame field $\tilde{\tau}^N$ on $(T_{2\epsilon} \times I(0, 1/2)) \cup_h D^{i+1} \times D^{n-i}_\epsilon$ that is normal to $B^{n+1}(h)$ in the direct product $R^{n+N} \times I(0, 1)$.*

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[1] Chapter I is a detailed account of a note by the author [14].

*Proof.* Let us choose on $D^{i+1} \times 0 \subset R^{n+N} \times I(0,1)$ some continuous frame field $\tau_0^{N+n-i}$, normal to $D^{i+1} \times 0$ in $R^{n+N} \times I(0,1)$, and let us consider its restriction to the boundary

$$S^i \times 0 \subset M^n \subset R^{n+N} \times 0,$$

which we shall also denote by $\tau_0^{N+n-i}$. Since the homomorphism $\pi_i(SO_{n-i}) \to \pi_i(SO_{N+n-i})$ is onto, we can cboose on the sphere $S^i \times 0 \subset M^n$ an $(n-i)$-frame field $\tau^{n-i}$, normal to the sphere $S^i \times 0$ in the manifold $M^n$ and such that the combined frame field $(\tau^N, \tau^{n-i})$, normal to the sphere $S^i \times 0 \in R^{N+n} \times 0$, is homotopic to the field $\tau_0^{N+n-i}$ which is induced by the $(N+n-i)$-frame field $\tau_0^{N+n-i}$ on the ball

$$D^{i+1} \times 0 \subset R^{N+n} \times I(0,1).$$

Hence the field $(\tau^N, \tau^{n-i})$ may be extended onto the ball

$$D^{i+1} \times 0 \subset R^{N+n} \times I(0,1).$$

We shall denote this extension by $(\tilde{\tau}^N, \tilde{\tau}^{n-i})$, where $\tilde{\tau}^N$ is the extension of the first $N$-frame and $\tilde{\tau}^{n-i}$ is the extension of the last $(n-i)$-frame. Let us now “inflate” the ball

$$D^{i+1} \times 0 \subset R^{N+n} \times I(0,1).$$

by the last $n-i$ vectors of the frame $\tilde{\tau}^{n-i}$, more exactly, by the linear space of dimension $n-i$ defined by these $n-i$ vectors at each point of the ball. We shall denote this inflation by $Q$. The vectors of the frame $\tilde{\tau}^N$ will be normal to the inflation $Q$ and define an extension of the equipment of $\tau^N$ onto this inflation. The frame field $\tau^{n-i}$, which is normal to the sphere $S^i \times 0 \subset M^n$, is different from the original frame field on the sphere $S^i \times 0$ that was defined by the original coordinate system in the direct product $S^i \times D_\epsilon^{n-i} \subset M^n$. This difference is measured by the “discriminating” map $S^i \to SO_{n-i}$, which also defines the element $\tilde{d}(h) \in \pi_i(SO_{n-i})$ needed by us and the diffeomorphism

$$h \colon \partial D^{i+1} \times D_\epsilon^{n-i} \to M^n.$$

It is easy to see from (1) that

$$B^{n+1}(h) = \left[(M \setminus T_{2\epsilon}) \times I\left(0, \frac{1}{2}\right)\right] \cup \left[\left(T_{2\epsilon} \times I\left(0, \frac{1}{2}\right)\right) \cup_h Q\right],$$

and that the $N$-frame field is extended onto $Q$. But

$$Q \approx D^{i+1} \times D_\epsilon^{n-i},$$

where the sign $\approx$ means a diffeomorphism.

The lemma is proved. □

For convenience in applications of Lemma 1.1 we formulate the following statement.

**Lemma 1.2.** a) *Suppose $i < n-i$. Then the map*

$$\pi_i(SO_{n-i}) \to \pi_i(SO_{N+n-i})$$

*is always an epimorphism.*

b) *Suppose $i = 2k$ and $i = n-i$. Then the map*

$$\pi_{2k}(SO_{2k}) \to \pi_{2k}(SO_{N+2k})$$

*is also always an epimorphism.*

c) *Suppose* $i = 2k+1$, $i = n-i$. *In this case the map is epimorphic if and only if* $i \neq 1,3,7$. *If* $i = 1,3,7$, *then the factor group* $\pi_i(SO_{N+n-i})/\pi_i(SO_i)$, $i = n-i$, *has two elements.*

The proof of a) and b) is contained in [20], and that of c) is in [1].

## § 2. Relative π-manifolds

Let $M^n$ be a smooth manifold, either closed or with boundary, and let $W^i \subset M^n$ be a submanifold of it. We shall denote by $\nu^N(M^n)$ the normal bundle of the manifold $M^n \subset R^{N+n}$ and by $\nu^{n-i}(W^i, M^n)$ the normal bundle of the manifold $W^i$ in $M^n$.

**Definition 2.1.** Let $f\colon M_1^k \to M_2^n$ be a smooth map. We shall call $M_1^k$ an $(f,\pi)$-manifold mod $M_2^n$ if

$$f^*\nu^N(M_2^n) = \nu^N(M_1^k).$$

**Lemma 2.2.** *Suppose a sphere* $S^i \subset M_1^k$, *that is smoothly situated in* $M_1^k$, *is such that the map* $f|S^i \to M_2^n$ *is homotopic to zero. Then the bundle* $\nu^{k-i}(S^i, M_1^k)$ *has the following properties*:

1) *for* $i < k-i$ *the bundle* $\nu^{k-i}(S^i, M_1^k)$ *is trivial*;

2) *for* $i = k-i$, $i = 2s$ *the bundle* $\nu^{k-i}(S^i, M_1^k)$ *is trivial if and only if the self-intersection number* $S^i \cdot S^i$ *is* 0;

3) *for* $i = k-i$, $i = 1,3,7$ *the bundle* $\nu^{k-i}(S^i, M_1^k)$ *trivial*;

4) *for* $i = k-i$, $i = 2s+1$, $i \neq 1,3,7$ *the normal bundle is completely defined by the invariant* $\phi(S^i) \in Z_2$.

*If* $x \in \operatorname{Ker} f_* \subset \pi_i(M_1^k)$, *where* $x$ *is the homotopy class of the embedding* $S^i \subset M_1^k$, *and the group* $\pi_1(M_1^k) = 0$, *then* $\phi$ *defines the single-valued map*

$$\phi\colon \operatorname{Ker} f^* \to Z_2$$

*and*

$$\phi(x+y) = \phi(x) + \phi(y) + [H(x)\cdot H(y)]_{\bmod 2}, \tag{2}$$

*where* $H\colon \pi_i(M_1^k) \to H_i(M_1^k)$ *is the Hurewicz homomorphism.*

*Proof.* Let us consider the tubular neighborhood $T$ of the sphere $S^i$ in the manifold $M_1^k$, which is the space of an $SO_{k-i}$-bundle with base $S^i$. The map $f \circ j\colon T \to M_2^n$ is homotopic to zero and, by assumption,

$$j^*f^*\nu^N(M_2^n) = \nu^N(T),$$

where $j$ is an embedding of $S^i \subset M_1^k$. Hence $\nu^N(T)$ is trivial. Since the manifold $T$ is not closed the triviality of the bundle $\nu^N(T)$ implies that $T$ is parallelizable. Hence the normal bundle of a sphere $S^i$ in a manifold is completely defined by an element $\alpha \in \operatorname{Ker} p_*$, where

$$p\colon SO_{k-i} \to SO_\infty$$

and

$$p_*\colon \pi_{i-1}(SO_{k-i}) \to \pi_{i-1}(SO_\infty)$$

is a homomorphism of the natural embedding $p$. For $i < k-i$ the map $p$ is isomorphic, and this implies property 1).

If $i = k-i$, $i = 2s$, then

$$\operatorname{Ker} p_* = Z \subset \pi_{2s-1}(SO_{2s})$$

and, as is well known, the bundles over the sphere $S^{2s}$ that are defined by elements $\alpha \in \operatorname{Ker} p_* \subset \pi_{2s-1}(SO_{2s})$ are completely determined by the Euler class $\chi(\alpha)$, where $\chi(\alpha) \equiv 0 \pmod 2$. But the Euler class of a bundle is equal to the self-intersection number $S^i \cdot S^i$, and this implies property 2).

For $i = 1, 3, 7$, $i = k - i$ the kernel $\operatorname{Ker} p_* = 0$, and this implies property 3).

For $i \neq 1, 3, 7$, $i = 2s + 1$ the group $\operatorname{Ker} p_* = Z_2$ (cf. [1]). Hence the normal bundle $\nu^{k-i}(S^i, T)$ is defined by the invariant $\phi(S^i) \subset Z_2$.

Now $\pi_1(M_1^k) = 0$. Hence by Whitney's results two spheres $S_1^i, S_2^i \subset M_1^k$ which define one and the same element $x \in \pi_i(M_1^k)$, $i = k - i$, are regularly homotopic (cf. [25]). Hence

$$\phi(S_1^i) = \phi(S_2^i).$$

Thus the map

$$\phi\colon \operatorname{Ker} f_* \to Z_2$$

is defined since each element $x \in \operatorname{Ker} f_*$ may be realized by an embedded smooth sphere $S^i \subset M_i^k$ (cf. [9]). Let us now prove (2). Suppose there exist two cycles $x, y \in \operatorname{Ker} f_*$. We realize them by the spheres $S_1^i, S_2^i \subset M^n$, the number of points of intersection of which is equal to the intersection number $|H(x) \cdot H(y)|$ (cf. [25]). We form tubular neighborhoods $T_1$ and $T_2$ of the spheres $S_1^i$ and $S_2^i$ respectively in the manifold $M_1^k$ and we denote by

$$T(x, y) = T_1 \cup T_2$$

a smooth neighborhood of the union $S_1^i \cup S_2^i$. The manifold $T(x, y)$ is obviously parallelizable, and

$$H_i(T(x, y)) = Z + Z.$$

If the spheres do not intersect our statement is obvious. Let us assume $|H(x) \cdot H(y)| = 1$. Then

$$\pi_1(T(x, y)) = 0, \qquad H_j(T(x, y)) = 0, \quad j \neq i,$$

and the boundary $\partial T(x, y)$ is a homotopy sphere (cf. [8]).

Kervaire proved [4] that in the manifold $T(x, y)$

$$\phi(x + y) = \phi(x) + \phi(y) + [H(x) \cdot H(y)]_{\bmod 2},$$

which must also hold in $M_1^k \supset T(x, y)$ since the sphere $S^i$ realizing $x + y$ is in $T(x, y)$ and $\phi$ is an invariant of the normal bundle. If $|H(x) \cdot H(y)| > 1$, then the group

$$\pi_1|T(x, y)| = \pi_1(\partial T(x, y))$$

is free and the number of its generators is $|H(x) \cdot H(y)| - 1$; hence our argument does not go through. But by the reconstructions of Morse described in §1 it is possible to "seal up" the group $\pi_1(T(x, y)) = \pi_1(\partial T(x, y))$ and pass to a simply connected manifold $\tilde{T}(x, y) \subset M_1^k$ such that:

a) $\tilde{T}(x, y) = T(x, y) \cup_{h_1} D^2 \times D^{k-2} \cup_{h_2} \cdots \cup_{h_i} D^2 \times D^{k-2}$, where

$$t = |H(x) \cdot H(y)| - 1$$

and

$$h_q\colon \partial D^2 \times D^{k-2} \to \partial T(x, y);$$

b) $\tilde{T}(x, y)$ is parallelizable;

c) $H_i(\tilde{T}(x, y)) = Z + Z$, $H_j(\tilde{T}(x, y)) = 0$, $i \neq j$;

d) the spheres $S_1^i, S_2^i \subset T(x, y)$ generate the group $H_i(\tilde{T}(x, y))$ .

For this we must carry out the reconstructions of Morse in the interior of the manifold $M_1^k$, which is possible if $k \geq 6$. Then to the manifold $\tilde{T}(x,y)$ we may apply Kervaire's results [4] and get the equality (2):

$$\phi(x+y) = \phi(x) + \phi(y) + [H(x) \cdot H(y)]_{\text{(mod 2)}}.$$

(Concerning the reconstructions of Morse cf. papers [2] and [9].) Thus the lemma is proved. We note that our description of the behavior of the normal bundles of a sphere in a parallelizable manifold is not original and is contained in papers [9, 4] and others. □

**Definition 2.3.** If the map $f\colon M_1^n \to M_2^n$ has degree $+1$, then we shall say that the manifold $M_1^n$ is greater than or equal to $M_2^n$, and write $M_1^n \overset{f}{\geq} M_2^n$.[2]

**Lemma 2.4.** *If $M_1^n \overset{f}{\geq} M_2^n$, then the map $f^*\colon H^*(M_2^k, K) \to H^*(M_1^n, K)$ is a monomorphism for any field $K$.*

*Proof.* Let $x \in H^i(M_2^n, K)$, $x \neq 0$; then there exists a $y \in H^{n-1}(M_2^n, K)$ such that $(xy, [M_2^n]) = 1$. Since

$$(f^*(xy), [M_1^n]) = (f^*xf^*y, [M_1^n]) = (xy, f_*[M_1^n]) = (xy, [M_2^n]) = 1,$$

it follows that $f^*xf^*y \neq 0$ and therefore $f^*x \neq 0$.

The lemma is proved. □

**Lemma 2.5.** *If $\pi_1(M_1^n) = \pi_1(M_2^n) = 0$ and $M_1^n \overset{f}{\geq} M_2^n$, $M_2^n \overset{g}{\geq} M_1^n$, then the maps $f$ and $g$ are homotopy equivalences.*

*Proof.* The maps $f \circ g\colon M_2^n \to M_2^n$ and $g \circ f\colon M_1^n \to M_1^n$ are onto of degree $+1$. Hence by Lemma 2.4 they induce an isomorphism of the cohomologies over an arbitrary field $K$ and hence an isomorphism of the integral cohomologies and homologies. Whitehead's theorem enables us to complete the proof. □

**Remark 2.6.** Lemma 2.5 can also be stated as follows: *if $\pi_1(M_1^n) = \pi_1(M_2^n) = 0$, the homologies of the manifolds $M_1^n$ and $M_2^n$ are isomorphic, and $M_1^n \overset{f}{\geq} M_2^n$, then they are homotopically equivalent.*

## § 3. The general construction

Let $M^n$ be a smooth closed simply connected oriented manifold and $\nu^N(M^n)$ its stable normal bundle, the fiber of which is a closed ball $D^N$, and let us suppose that this bundle is oriented, i.e., the structural group is reduced to $SO_N$. We contract the boundary $\partial\nu^N(M^n)$ to a point and denote by $T_N(M^n)$ the obtained space, which is the Thom space of the bundle (cf. [22, 7]). We have

$$T_N(M^n) = \nu^N(M^n)/\partial\nu^N(M^n). \tag{3}$$

The Thom isomorphism

$$\phi\colon H_i(M^n) \to H_{N+i}(T_N(M^n)) \tag{4}$$

is well known.

As usual, we denote by $[M^n]$ the fundamental cycle of the manifold $M^n$ in the selected orientation.

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[2]It is also assumed that $M_2^n$ is an $(f,\pi)$-manifold modulo $M_1^n$.

**Lemma 3.1.** *The homology class $\phi[M^n]$ belongs to the image of the Hurewicz homomorphism $H\colon \pi_{N+n}(T_N(M^n)) \to H_{N+n}(T_N(M^n))$.*

*Proof.* Let us construct an element $x \in \pi_{N+n}(T_N(M^n))$ such that $H(x) = \phi[M^n]$. Let the manifold $M^n$ be smoothly situated in the sphere $S^{N+n}$. Its closed tubular neighborhood $T \in S^{N+n}$ is diffeomorphic to the space of the bundle $\nu^N(M^n)$ in the natural way, since $T$ is canonically fibered by normal balls $D^N$. We effect the natural diffeomorphism $T \to \nu^N(M^n)$ and consider the composition

$$T \to \nu^N(M^n) \to T_N(M^n);$$

the map $T \to T_N(M^n)$ transforms the boundary $\partial T$ into a point and is therefore extended to the map $S^{N+n} \to T_N(M^n)$ that transforms all of the exterior $S^{N+n} \setminus T$ into the same point. This map obviously represents the needed element $x \in \pi_{N+n}(T_N(M^n))$. The lemma is proved. □

In what follows an important role will be played by the set

$$H^{-1}\phi[M^n] \subset \pi_{N+n}(T_N(M^n)),$$

which we shall always denote by $A(M^n)$. We consider an arbitrary element $\alpha \in A(M^n)$ and the map

$$\tilde{f}_\alpha\colon S^{N+n} \to T_N(M^n)$$

representing it.

From the paper of Thom [22] there easily follows

**Lemma 3.2.** *There exists a homotopic smooth map*

$$f_\alpha\colon S^{N+n} \to T_N(M^n)$$

*such that:*

a) *the inverse image $f_\alpha^{-1}(M^n)$ is a smooth manifold $M_\alpha^n$, smoothly situated in the sphere $S^{N+n}$;*

b) *for every point $x \in M_\alpha^n$ the map $f_\alpha$ transforms the $\epsilon$-ball $D_x^N$, normal to $M_\alpha^n$ in $S^{N+n}$, into the $\epsilon$-ball $D_{f_\alpha(x)}^N$, normal to $M^n$ in $T_N(M^n)$, and the map $f_\alpha\colon D_x^N \to D_{f_\alpha(x)}^N$ is a linear nonde generate map for all $x \in M_\alpha^n$;*

c) *the maps $f_\alpha|M_\alpha^n \to M^n$ and $f_\alpha|D_x^N \to D_{f_\alpha(x)}^N$ have degree $+1$ for all $x \in M_\alpha^n$.*

*Proof.* Points a) and b) are taken from Thom's paper [22]. For the proof of point c) we observe that the map $\tilde{f}_\alpha\colon S^{N+n} \to T_N(M^n)$ and hence $f_\alpha$ have degree $+1$ (this makes sense because $T_N(M^n)$ is a pseudomanifold with fundamental cycle $[T_N] = \phi[M^n]$). Hence the map $f_\alpha$ must have degree $+1$ in the tubular neighborhood of $M_\alpha^n = f_\alpha^{-1}(M^n)$. We reduce the structural group of the bundle $\nu^n(M_\alpha^n)$ to $SO_N$ so that all maps $f_\alpha\colon D_x^N \to D_{f_\alpha(x)}^N$ have determinants $> 0$. Then on the manifold $M_\alpha^n$ there is uniquely defined an orientation which is induced by the orientations of the sphere $S^{N+n}$ and the fiber $D_x^n$. In this orientation the map $f_\alpha\colon M_\alpha^n \to M^n$ has degree $+1$ since the degree of the bundle map

$$\nu^N(M_\alpha^n) \to \nu^N(M^n) \to T_N(M^n)$$

is $+1$ and is equal to the product of the degrees of the map of the base $M_\alpha^n$ and fiber $D_x^N$, $x \in M^n$; on the fiber $D_x^N$, as a result of the choice of its orientation, this degree is equal to $+1$, from which follows the desired statement. The lemma is proved. □

**Corollary 3.3.** *The manifold* $M^n_\alpha \overset{f}{\geq} M^n$.

*Proof.* The map $f_\alpha$ has degree $+1$ and is clearly such that

$$f^*_\alpha \nu^N(M^n) = \nu^N(M^n_\alpha). \qquad \square$$

**Corollary 3.4.** *If* $\pi_1(M^n_\alpha) = 0$ *and* $H_i(M^n_\alpha) = H_i(M^n)$, $i = 0, 1, 2, \ldots, n$, *then the map* $f_\alpha \colon M^n_\alpha \to M^n$ *is a homotopy equivalence.*

The proof follows from Corollary 3.3, Lemma 2.3 and Remark 1 on page 8.

We denote by $\bar{A}(M^n) \subset A(M^n)$ the subset consisting of those elements $\alpha \in \bar{A}(M^n)$ for which there exist representatives $f_\alpha \colon S^{N+n} \to T_N(M^n)$ satisfying Lemma 3.2 and such that the inverse image $f^{-1}_\alpha(M^n) = M^n_\alpha$ is a manifold homotopically equivalent to $M^n$. The set $\bar{A}(M^n)$ will be of interest to us below. In studying it the three following important questions are appropriate:

1. What place is taken by the submanifold $\bar{A}(M^n)$ in $A(M^n)$, i.e., in which classes $\alpha \in A(M^n) \in \pi_{N+n}(T_N(M^n))$ are there representatives $f_\alpha \colon S^{N+n} \to T_N(M^n)$ for which the manifold

$$M^n_\alpha = f^{-1}_\alpha(M^n)$$

is homotopically equivalent to $M^n$ (in which classes $\alpha \in A(M^n)$ are there found manifolds of the same homotopy type as $M^n$)?

2. Suppose two manifolds $M^n_{\alpha,1}$, and $M^n_{\alpha,2}$ are found in one and the same class $\alpha \in \bar{A}(M^n)$ and both are homotopically equivalent to $M^n$. This means that there are two homotopic maps of a sphere

$$f_{\alpha,i} \colon S^{N+n} \to T_N(M^n)$$

such that

$$f^{-1}_{\alpha,i}(M^n) = M^n_{\alpha,i}, \qquad i = 1, 2.$$

How are the manifolds $M^n_{\alpha,1}$ and $M^n_{\alpha,2}$ connected?

3. In which classes $\alpha_i \in \bar{A}(M^n)$ can one and the same manifold $M^n_1$ be found that is homotopically equivalent to $M^n$?

The following three sections will be devoted to the solution of these questions.

## § 4. A realization of the classes

This section is devoted to a study of the question, in which classes $\alpha \in A(M^n)$ are to be found the manifolds that are homotopically equivalent to $M^n$. First we prove a number of easy lemmas of algebraic character. We consider two arbitrary finite complexes $X, Y$ and a map $f \colon X \to Y$. Let $K$ be an arbitrary field. We will assume that

$$\pi_1(X) = \pi_1(Y) = 0.$$

**Lemma 4.1.** *Suppose for any* $K$ *the map* $f_* \colon H_i(X, K) \to H_i(Y, K)$ *is epimorphic for* $i \leq j+1$ *and isomorphic for* $i \leq j$. *Then* $f_* \colon H_i(X, Z) \to H_i(Y, Z)$ *is epimorphic for* $i \leq j + 1$ *and isomorphic for* $i \leq j$.

*Proof.* We consider the cylinder $C_f = X \times I(0, 1) \cup_f Y$, which is homotopically equivalent to $Y$, and the exact sequence of the pair $(C_f, X)$

$$H_i(X) \xrightarrow{f_*} H_i(Y) \to H_i(C_f, X) \xrightarrow{\partial} H_{i-1}(X) \xrightarrow{f_*} H_{i-1}(Y) \tag{5}$$

for $i \leq j+1$. From the sequence (5) it follows that $H_i(C_f, X, K) = 0$ for $i \leq j+1$. Therefore

$$H_i(C_f, X, Z) = 0, \qquad i \leq j+1.$$

Returning to the exact sequence (5) (in the homologies over $Z$) we obtain all of the statements of the lemma. The lemma is proved. □

**Lemma 4.2.** *Suppose the map $f\colon X \to Y$ is such that the map $f_*\colon H_i(X, Z) \to H_i(Y, Z)$ is an epimorphism for $i \leq j+1$ and an isomorphism for $i \leq j$. Then the map $f_*\colon \pi(X) \to \pi_i(Y)$ is an isomorphism for $i \leq j$ and an epimorphism for $i \leq j+1$, and conversely.*

*Proof.* We consider two exact sequences which together with the Hurewicz homomorphism form the commutative diagram

$$\begin{array}{ccccccc} H_i(X,Z) & \xrightarrow{f_*} & H_i(Y,Z) & \longrightarrow & H_I(C_f,X,Z) & \xrightarrow{\partial} & H_{i-1}(X,Z) \\ \uparrow{\scriptstyle H} & & \uparrow{\scriptstyle H} & & \uparrow{\scriptstyle H} & & \uparrow{\scriptstyle H} \\ \pi_i(X) & \xrightarrow{f_*} & \pi_i(Y) & \longrightarrow & \pi_i(C_f,X) & \xrightarrow{\partial} & \pi_{i-1}(X) \end{array} \tag{6}$$

for $i \leq j+1$. It is easy to see that

$$H_i(C_f, X, Z) = 0, \qquad i \leq j+1.$$

Since $\pi_1(X) = \pi_1(Y) = 0$, we have

$$\pi_i(C_f, X) = 0, \qquad i \leq j+1,$$

from which follows Lemma 4.2 (in the direct sense). The converse statement is proved analogously. The lemma is proved. □

For definiteness, in the sequel we will always denote the homomorphisms $\pi_i(X) \to \pi_i(Y)$ and $H_i(X) \to H_i(Y)$, corresponding to the map $f\colon X \to Y$, by $f_*^{(\pi_i)}$ and $f_*^{(H_i)}$.

**Lemma 4.3.** *Under the same conditions as in Lemma 4.2 the homomorphism*

$$H\colon \operatorname{Ker} f_*^{(\pi_{j+1})} \to \operatorname{Ker} f_*^{(H_{j+1})}$$

*is an epimorphism.*

*Proof.* The following diagram is commutative and its lines are exact:

$$\begin{array}{ccccc} \pi_{j+2}(C_f, X) & \longrightarrow & \operatorname{Ker} f_*^{(\pi_{j+1})} & \longrightarrow & 0 \\ \approx \downarrow {\scriptstyle H} & & \downarrow & & \\ H_{j+2}(C_f, X, Z) & \longrightarrow & \operatorname{Ker} f_*^{(H_{j+1})} & \longrightarrow & 0. \end{array} \tag{7}$$

From the proof of Lemma 4.2 we know that

$$H_i(C_f, X) = \pi_i(C_f, X) = 0, \qquad i \leq j+1.$$

Therefore

$$\pi_{j+2}(C_f, X) \approx H_{j+2}(C_f, X).$$

The simple diagram completes the proof. □

We now consider a map of manifolds $f\colon M_1^n \to M_2^n$ having degree $+1$. We will be interested in the case when the kernels $\operatorname{Ker} f_*^{(\pi_i)}$ are trivial for $i < [n/2]$. We consider separately the cases of even and odd $n$. The following two lemmas hold.

**Lemma 4.4.** *Suppose $n = 2s$ and the groups $\operatorname{Ker} f_*^{(\pi_i)}$ are trivial for $i < s$. Then the group $\operatorname{Ker} f_*^{(H_s)}$ is free abelian, is singled out as a direct summand in the group $H_s(M_1^n, Z)$, and the matrix of intersections of the base cycles of the group $\operatorname{Ker} f_*^{(H_s)}$ is unimodular.*

**Lemma 4.4′.** *Suppose $n = 2s+1$ and the groups $\operatorname{Ker} f_*^{(\pi_i)}$ are trivial for $i < s$. Then the group $\operatorname{Ker} f_*^{(H_s+1)}$ is free abelian, and both $\operatorname{Ker} f_*^{(H_s)}$ and $\operatorname{Ker} f_*^{(H_{s+1})}$ are singled out as direct summands in the groups $H_s(M_1^n, Z)$ and $H_{s+1}(M_1^n, Z)$ respectively. The finite part $\operatorname{Tor}\operatorname{Ker} f_*^{(H_s)}$ of the group $\operatorname{Ker} f_*^{(H_s)}$ is closed under the duality of Alexander, i.e., the matrix of linkages of the generating elements of order $p^i$ in a certain primary decomposition is unimodular $\operatorname{mod} p^i$ for a fixed value of the numbers $p, i$. The matrix of intersections of the groups $\operatorname{Ker} f_*^{(H_{s+1})}$ and $\operatorname{Ker} f_*^{(H_s)} / \operatorname{Tor}\operatorname{Ker} f_*^{(H_s)}$ is also unimodular.*

We will conduct the proof of both lemmas simultaneously, starting from the identity

$$f_*(f^*x \cap y) = x \cap f_*y, \tag{8}$$

which holds for any continuous map $f$. In our case $f_*[M^n] = [M_2^n]$ and the operation $\cap[M_1^n]$ coincides with the isomorphism $D$ of the duality of Poincaré. In this way we get

$$f_*Df^* = D,$$

from which follow the direct sums

$$H_i(M^n) = \operatorname{Ker} f_*^{(H_i)} + Df^*H^{n-i}(M_2^n) \tag{9}$$

over any coefficient domain and for any values of $i$. Consequently, the singling out as a direct summand is proved in all cases. The absence of torsion in the groups $\operatorname{Ker} f_*^{(H_s)}$ for $n = 2s$ and $\operatorname{Ker} f_*^{(H_{s+1})}$ for $n = 2s+1$ follows from the fact that the groups $\operatorname{Ker} f_*^{(H_{s-1})}$ are trivial in both cases, and from the principle of duality of Alexander, connecting the torsions of the groups $H_{s-1}(M_1^n)$ and $H_{n-s}(M_1^n)$ for both values of $n$. It remains to prove the unimodularity of the corresponding matrices of intersections or linkages. We show that the groups $\operatorname{Ker} f_*^{(H_i)}$ and $Df^*H^{n-i}(M_2^n)$ are orthogonal to each other with respect to the operation of intersection of cycles for any values of $i$ and over any group of coefficients. In fact, let $x \in H^{n-i}(M_2^n)$ and $y \in \operatorname{Ker} f_*^{(H_i)}$. Then

$$(f^*x \cap [M_1^n]) \cdot y = (f^*x, y) = (x, f_*y) = 0 \tag{10}$$

and any element of the group $Df^*H^{n-i}(M_2^n)$ has the form

$$f^*x \cap [M_1^n].$$

Thus the groups $\operatorname{Ker} f_*^{(H_i)}$ and $Df^*D^{-1}H_i(M_2^n)$ are orthogonal. Applying this orthogonality, we obtain the unimodularity of the matrices of intersections in all the necessary cases. The statement concerning the matrices of linkages follows from the fact that the linkages can be defined in terms of the intersections of cycles modulo $p^i$. Thus Lemmas 4.4 and 4.4′ are proved.

We note a useful supplement to Lemma 4.4.

**Lemma 4.5.** *The map* $H\colon \operatorname{Ker} f_*^{(\pi_s)} \to \operatorname{Ker} f_*^{(H_s)}$ *for* $n = 2s$ *is an isomorphism if the groups* $\operatorname{Ker} f_*^{(\pi_i)} = 0$ *for* $i < s$.

*Proof.* We consider, as in the proof of Lemma 4.3, the commutative diagram

$$\begin{array}{ccccccccc} H_{s+1}(M_1^n) & \xrightarrow{\approx} & H_{s+1}(M_2^n) & \longrightarrow & H_{s+1}(C_f, M_1^n) & \xrightarrow{\partial} & \operatorname{Ker} f_*^{(H_s)} & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & & & \\ \pi_{s+1}(M_1^n) & \longrightarrow & \pi_{s+1}(M_2^n) & \longrightarrow & \pi_{s+1}(C_f, M_1^n) & \xrightarrow{\partial} & \operatorname{Ker} f_*^{(\pi_s)} & \longrightarrow & 0. \end{array} \tag{11}$$

Since the maps $f_*^{(H_i)}$ for $i < s$ are isomorphisms, the map $f_*^{(H_{s+1})}$ is also an isomorphism. From the exactness of the sequences we conclude that the map

$$\partial\colon H_{s+1}(C_f, M_1^n) \to \operatorname{Ker} f_*^{(H_s)}$$

is an isomorphism. Therefore the map

$$\partial H = H\partial\colon \pi_{s+1}(C_f, M_1^n) \to \operatorname{Ker} f_*^{(H_s)}$$

is an isomorphism and the map

$$H\colon \operatorname{Ker} f_*^{(\pi_s)} \to \operatorname{Ker} f_*^{(H_s)}$$

is also an isomorphism. The lemma is proved. □

We now investigate an arbitrary element $\alpha \in A(M^n)$. We have the following

**Lemma 4.6.** *For every element* $\alpha \in A(M^n)$ *there exists a map* $f_\alpha\colon S^{N+n} \to T_N(M^n)$ *satisfying Lemma* 3.2 *and such that the inverse image* $M_\alpha^n = f_\alpha^{-1}(M^n) \subset S^{N+n}$ *possesses the following properties*:

a) $\pi_1(M_\alpha^n) = 0$;

b) *the maps* $f_*^{(H_s)}\colon H_s(M_\alpha^n) \to H_s(M^n)$ *are isomorphisms for* $s < [n/2]$.

*Proof.* We will by induction construct the maps

$$_sf_\alpha\colon S^{N+n} \to T_N(M^n),$$

satisfying Lemma 3.2, for which the groups

$$H_i(M_{\alpha,s}^n), \qquad M_{\alpha,s}^n = {}_\alpha f_s^{-1}(M^n)$$

will be isomorphic to the groups $H_i(M^n)$, $i < s$. Since the maps $_sf_\alpha\colon M_{\alpha,s}^n \to M^n$ have degree $+1$, this isomorphism is established by the map $_sf_{\alpha*}^{(H_i)}$. From Lemmas 4.1–4.3 it follows that the map $_sf_\alpha^{(H_s)}$ is an epimorphism and all of the base cycles $x_1, \dots, x_l \in \operatorname{Ker} {}_sf_{\alpha*}^{(H_s)}$ can be realized by a system of smoothly embedded disjoint spheres $S_1^s, \dots, S_l^s \subset M_{\alpha,s}^n$, on which the map $_sf_\alpha|S_j^s$ is homotopic to zero. We assume that the maps $_if_\alpha$ are already constructed for $i \le s$ and we construct the map $_{s+1}f_\alpha$ by reconstructing the map $_sf_\alpha$.

**Step 1.** We deform the map $_sf_\alpha$ to the map $_s\tilde{f}_\alpha$ such that

$$_s\tilde{f}_\alpha(T(S_1^s)) = g_0 \in M^n$$

where $g_0$ is a point in the manifold $M^n$. The deformation is assumed to be smooth, and $T(S_1^s) \subset M_{\alpha,s}^n$ denotes a smooth tubular neighborhood of the sphere $S_1^s \subset M_{\alpha,s}^n$. In the fiber $D_{g_0}^N \subset \nu^N(M^n)$ we take the frame $\tau_0^N$, which determines the orientation of the fiber $D_{g_0}^N$. The inverse image $_s\tilde{f}_\alpha^*\tau_0^N$ represents a continuous $N$-frame field $\tau^N$

on $T(S_1^s)$ that is normal to $T(S_1^s) \subset S^{N+n}$, since the map ${}_s\tilde{f}_\alpha$, satisfies Lemma 3.2. The arbitrariness in the choice of the frame $\tau_0^N$ is immaterial for our purposes.

**Step 2.** According to Lemma 2.1 the tube $T(S_1^s)$ is diffeomorphic to $S_1^s \times D_\epsilon^{n-s}$, where $\epsilon > 0$ is a small number. We assign in $T(S_1^s)$ the coordinates $(x, y)$, $x \in S_1^s$, $y \in D_\epsilon^{n-s}$. As a result of Step 1, on the tube $T(S_1^s)$ there is constructed the field $\tau^N$. We consider the direct product $S^{N+1} \times I(0,1)$. We will assume that

$$ {}_s\tilde{f}_\alpha \colon S^{N+n} \times 0 \to T_N(M^n), \qquad M^n_{\alpha,s} \subset S^{N+n} \times 0. $$

We construct a membrane $B^{n+1}(h) \in S^{N+n} \times I(0,1)$, orthogonally approaching the boundaries, such that the field $\tau^N$ can be extended to a certain field $\tilde{\tau}^N$ that is normal to

$$ B^{n+1}(h) \setminus \left[ (M^n_{\alpha,s} \setminus T(S_1^N)) \times I\left(0, \frac{1}{2}\right) \right] $$

in the direct product $S^{N+n} \times I(0,1)$, where

$$ h \colon \partial D^{s+1} \times D_\epsilon^{n-s} \to T(S_1^s), \qquad h(x,y) = (x, d(h)_x(y)), $$
$$ d(h) \colon S_1^s \to SO_{n-s}. $$

Such a choice of the membrane $B^{n+1}(h)$ is possible according to Lemmas 1.1 and 1.2.

**Step 3.** We extend the map ${}_s\tilde{f}_\alpha \colon M^n_{\alpha,s} \to M^n$ to a smooth map ${}_sF_\alpha \colon B^{n+1}(h) \to M^n$, putting

$$ (12) \qquad \begin{gathered} {}_sF_\alpha = {}_sf_\alpha | B^{n+1}(h) \cap S^{N+n} \times 0, \\ {}_sF_\alpha(D^{s+1} \times D_\epsilon^{n-s}) = g_0 = {}_sf_\alpha(T(S_1^s)). \end{gathered} $$

We extend the map

$$ {}_sF_\alpha \colon B^{n+1}(h) \to M^n $$

to the map

$$ {}_sF_\alpha \colon T(B^{n+1}(h)) \to T_N(M^n), $$

where $T(B^{n+1}(h))$ is a tubular neighborhood of $B^{n+1}(h)$ in $S^{N+1} \times I(0,1)$, according to the frame field $\tilde{\tau}^N$ that is normal to the part of $B^{n+1}(h)$ in $S^{N+n} \times I(0,1)$ which is diffeomorphic to $D^{s+1} \times D_\epsilon^{n-s} \subset B^{n+1}(h)$. On the remaining part

$$ B^{n+1}(h) \setminus D^{s+1} \times D_\epsilon^{n-s} = M^n_{\alpha,s} \times I\left(0, \frac{1}{2}\right) $$

the extension of the map is trivial. In their intersection

$$ M^n_{\alpha,s} \times I\left(0, \frac{1}{2}\right) \cap D^{s+1} \times D_\epsilon^{n-s} = T(S_1^s) $$

these extensions are compatible with the general frame field $\tau^N$. Further, by the well-known method of Thom, we extend the map ${}_sF_\alpha$ onto the entire product $S^{N+n} \times I(0,1)$.

Now we put

$$ {}_sf_\alpha^{(1)} = {}_sF_\alpha | S^{N+n} \times 1 $$

Clearly, the map ${}_sf^{(1)}$ satisfies Lemma 3.2 and

$$ {}_sf_\alpha^{(1)^{-1}}(M^n) = M^n_{\alpha,s}(h). $$

Since $2s + 1 < n$, we conclude that

$$ \operatorname{Ker} {}_sf^{(1)}_{\alpha*} = \operatorname{Ker} {}_sf_{\alpha*}/(x_1). $$

Putting, iterating the construction,

$$_{s+1}f_\alpha = {}_s f_\alpha^{(l)},$$

we obtain the statement of the lemma. □

An analysis of the case $s = [n/2]$ is more difficult and will be broken down into the following cases:

1) $n = 4k$, $s = 2k$, $k \geq 2$;
2) $n = 4k+2$, $s = 2k+1$, $k \geq 1$, $k \neq 1, 3$;
3) $n = 4k+2$, $s = 2k+1$, $k = 1, 3$;
4) $n = 4k+1$, $s = 2k$, $k \geq 1$;
5) $n = 4k+3$, $s = 2k+1$, $k \geq 1$.

**Lemma 4.7.** *Let $n = 4k$. For every element $\alpha \in A(M^n)$ there exists a map $f_\alpha \colon S^{N+n} \to T_N(M^n)$, satisfying Lemma 3.2, such that the inverse image $M^n_\alpha = f_\alpha^{-1}(M^n)$ is homotopically equivalent to $M^n$.*

*Proof.* Applying Lemma 4.6, we can construct a map $_{2k}f_\alpha \colon S^{N+n} \to T_N(M^n)$ such that

$$\operatorname{Ker} {}_{2k}f_{\alpha *}^{(H_i)} = 0, \qquad i < 2k,$$

where

$$_{2k}f_\alpha \colon M^n_{\alpha,k} = {}_{2k}f_\alpha^{-1}(M^n) \to M^n.$$

According to Lemma 4.4 the group

$$\operatorname{Ker} {}_{2k}f_{\alpha *}^{(H_{2k})} = L_{2k} \subset H_{2k}(M^n_{\alpha,2k})$$

is free abelian, singled out as a direct summand in the group $H_{2k}(M^n_{\alpha,2k})$, and the matrix of intersections of the base cycles $l_1, \dots, l_m \subset L_{2k}$ is unimodular. We select in the group $H_{2k}(M^n_{\alpha,2k})/\operatorname{Tor}$ a base $l_1, \dots, l_m, q_1, \dots, q_p$ such that

$$q_i \circ l_j = 0, \quad i = 1, \dots, p, \quad j = 1, \dots, m;$$

this can be done in view of the unimodularity of the matrix

$$(l_j \circ l_t), \quad j, t = 1, \dots, m.$$

The matrix $(q_i \circ q_j)$ is equivalent to the matrix of intersections of the base cycles of the group $H_{2k}(M^n)/\operatorname{Tor}$ and, moreover,

$$({}_{2k}f_{\alpha *} q_i) \circ ({}_{2k}f_{\alpha *} q_j) = q_i \circ q_j.$$

Since

$$_{2k}f_\alpha^* \nu^N(M^n) = \nu^N(M^n_{\alpha,k})$$

and the map $_{2k}f_\alpha$ has degree $+1$ it follows from a formula of Hirzebruch [3] that the indices (signatures) of the manifolds $M^n_{\alpha,2k}$ and $M^n$ are equal to each other. Therefore the signature of the matrix $(l_i \circ l_j)$, $i, j = 1, \dots, m$, is equal to zero (the matrix of intersections of the manifold $M^n_{\alpha,2k}$ splits, by virtue of what has been said above, into two matrices, one of which is identical to the matrix of intersections of the manifold $M^n$, and the other of which is the matrix $(l_i \circ l_j)$, $i, j = 1, \dots, m$). On the other hand, the self-intersection indices $l_i \circ l_j$ are even. For a proof of the evenness of the numbers $l_i \circ l_j$ we realize the cycle $l_i$ by a smooth sphere $S^{2k}_i \subset M^n_{\alpha,2k}$ according to Whitney [25] and Lemma 4.3. Then we consider the tubular neighborhood of the sphere, $T(S^{2k}_i) \subset M^n_{\alpha,2k}$, which is a parallelizable manifold (cf. point 1) in the proof of Lemma 2.2). The self-intersection index of a sphere in a parallelizable manifold is always even, from which we obtain the desired statement. Thus the signature of the matrix $(l_i \circ l_j)$ is equal to zero and

$$l_i \circ l_j \equiv 0 \pmod 2.$$

According to [9] one can find a base $l'_1, \dots, l'_m$, $m = 2m'$, such that

a) $l'_i \circ l'_i = 0$, $1 \le i \le m$;

b) $l'_{2i+1} \circ l'_{2i+2} = 1$, $i = 0, 1, \dots, m' - 1$;

c) $l'_k \circ l'_j = 0$ otherwise, i.e., the matrix can be reduced to the form

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ & & 0 & 1 \\ 0 & & 1 & 0 \end{pmatrix} \tag{13}$$

We realize the cycles $l_i$, $i = 1, \dots, m$, by smoothly embedded spheres $S_i^{2k} \subset M^n_{\alpha,2k}$ in such a way that their geometric intersections correspond to the algebraic intersection indices (the number of points of an intersection $S_i^{2k} \cap S_j^{2k}$ is equal to the index $|S_i^{2k} \circ S_j^{2k}|$; this can be done for $k > 1$; cf. [26, 9]). According to Lemma 2.2 the normal bundles $\nu^{2k}(S_i^{2k}, M^n_{\alpha,2k})$ are trivial. Then we exactly repeat Steps 1, 2, 3 of the proof of Lemma 4.6, using Lemma 1.2. As a result of a reconstruction of Morse, the manifold $M^n_{\alpha,2k}$ is simplified (one reconstruction of Morse over the sphere $S_i^{2k}$ obliterates the integral square $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; cf. [9]). Iterating the operation, we arrive at a map

$$f_\alpha \colon S^{N+n} \to T_N(M^n)$$

such that $\operatorname{Ker} f_{\alpha *}^{(H_j)} = 0$, $i \le 2k$, and $\pi_1(M^n_\alpha) = 0$. According to the duality of Poincaré,

$$\operatorname{Ker} f_{\alpha *}^{(H_j)} = 0, \qquad j > 2k,$$

and the groups $H_i(M^n_\alpha)$ and $H_i(M^n)$ are isomorphic. According to Lemma 2.4 and Remark 1 on page 8, the manifold $M^n_\alpha$, is homotopically equivalent to the manifold $M^n$. The lemma is proved. □

Now let $n = 4k + 2$, $k \ne 1, 3$, $k > 1$.

**Lemma 4.8.** *For every element $\alpha \in A(M^n)$ there exists a map $f_\alpha \colon S^{N+n} \to T_N(M^n)$ satisfying Lemma 3.2 such that the inverse image $M^n_\alpha = f_\alpha^{-1}(M^n)$ possesses the following properties*:

a) $\pi_1(M^n_\alpha) = 0$;

b) $H_i(M^n_\alpha) = H_i(M^n)$, $i \ne 2k + 1$;

c) $\operatorname{Ker} f_{\alpha *}^{(H_{2k+1})} = Z + Z$ *or* $0$;

d) *denoting the base cycles of the group* $\operatorname{Ker} f_{\alpha *}^{(H_{2k+1})}$ *by* $x, y$, $x \circ y = 1$, *if* $\operatorname{Ker} f_{\alpha *}^{(H_{2k+1})} = Z + Z$, $\phi(x) = \phi(y) = 1$.

*Proof.* Using the results of Lemma 4.6, we consider the map

$${}_{2k+1}f_\alpha \colon S^{N+n} \to T_N(M^n)$$

satisfying Lemma 3.2 and such that

$$H_i(M^n_{\alpha,2k+1}) = H_i(M^n), \qquad i < 2k+1,$$
$$\operatorname{Ker}{}_{2k+1}f^{(H_{2k+1})}_{\alpha *} = Z + \dots + Z;$$

the matrix of intersections of the base cycles of the group $\operatorname{Ker}{}_{2k+1}f^{(H_{2k+1})}_{\alpha *}$ is skew-symmetric and unimodular. It can therefore be reduced to the base $x_1, \dots, x_{2l} \in \operatorname{Ker}{}_{2k+1}f^{(H_{2k+1})}_{\alpha *}$, the matrix of intersections of which has the form

$$\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ \dots & \dots & \dots & \dots \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}. \tag{14}$$

Thus we determine, the invariant $\phi(x) \in Z_2$, $x \in \operatorname{Ker}{}_{2k+1}f^{(H_{2k+1})}_{\alpha *}$ , which is such that

$$\phi(x+y) = \phi(x) + \phi(y) + (x \circ y)_{\bmod 2}$$

according to Lemmas 2.2 and 4.4. We put

$$\phi({}_{2k+1}f_\alpha) = \sum_{i=1}^{l} \phi(x_{2i-1})\phi(x_{2i}).$$

If $\phi({}_{2k+1}f_\alpha) = 0$, then it is possible to choose a base $x'_1, \dots, x'_{2l}$ such that

$$\phi(x'_i) = 0, \qquad i = 1, \dots, 2l.$$

If $\phi({}_{2k+1}f_\alpha) = 1$, then one can find a base $x'_1, \dots, x'_{2l}$ such that

$$\phi(x'_1) = \phi(x'_2) = 1$$

and

$$\phi(x'_i) = 0, \qquad i > 2$$

(cf. [4]). We realize the cycles by smoothly embedded spheres $S^{2k+1}_i \subset M^n_{\alpha,2k}$, that intersect each other if and only if their intersection indices are different from zero, and at not more than one point (cf. [9¿ 25]). If $\phi({}_{2k+1}f_\alpha) = 0$, then the normal bundles $\nu^{2k+1}(S^{2k+1}_i, M^n_{\alpha,2k+1})$ are trivial. If $\phi({}_{2k+1}f_\alpha) = 1$, then the bundles $\nu^{2k+1}(S^{2k+1}_i, M^n_{\alpha,2k+1})$ are trivial only for $i > 2$. Repeating Steps 1, 2, 3 of Lemma 4.6 and using Lemmas 1.2 and 4.7, we employ the reconstructions of Morse to seal the spheres $S^{2k+1}_{2i-1}$, $i \geq 2$, each time killing the square $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If $\phi({}_{2k+1}f_\alpha) = 0$, then we also seal the sphere $S^{2k+1}_1$, since its normal bundle in the manifold $M^n_{\alpha,2k+1}$ is trivial in this case. As a result we arrive at the map

$$f_\alpha \colon S^{N+n} \to T_N(M^n),$$

possessing the properties a)–d).

Thus the lemma is proved. □

We now investigate the case $n = 6, 14 = 4k+2$, $k = 1, 3$.

**Lemma 4.8′.** *For every element* $\alpha \in A(M^n)$ *there exists a map* $f_\alpha\colon S^{N+n} \to T_N(M^n)$ *such that*:

1) $\pi_1(M^n_\alpha) = 0$;
2) $H_i(M^n_\alpha) = H_i(M^n)$, $i \neq 2k+1$;
3) $\operatorname{Ker} f^{(H_{2k+1})}_{\alpha *} = Z + Z$ *or* $0$.

Although the formulations of Lemmas 4.8 and 4.8′ are analogous, it will be seen from the proof that these cases are essentially distinct. As above, we construct the map

$$_{2k+1}f_\alpha\colon S^{N+n} \to T_N(M^n).$$

We have

$$\operatorname{Ker} {}_{2k+1}f^{(H_i)}_{\alpha *} = 0, \qquad i < 2k+1,$$

and the group $\operatorname{Ker} {}_{2k+1}f^{(H_{2k+1})}_{\alpha *}$ is free abelian; in this last group we select base cycles $x_1, \dots, x_{2l}$, the matrix of intersections of which has the form (14). We realize these cycles by the spheres $S^{2k+1}_i \subset M^n_{\alpha,2k+1}$. It is possible to compute a map $_{2k+1}f_\alpha$ such that

$$_{2k+1}f_\alpha(S^{2k+1}_{2i-1}) = {}_{2k+1}f_\alpha(S^{2k+1}_{2i}) = g_0 \in M^n, \qquad i = 1, \dots, l,$$

where $g_0$ is a point in the manifold $M^n$. On the spheres $S^{2k+1}_{2i-1}$ and $S^{2k+1}_{2i}$ there appear the frame fields $\tau^N_{2i-1}$ and $\tau^N_{2i}$, which are normal to $M^n_{\alpha,2k+1}$. The maps

$$j_*\colon \pi_3(SO_3) \to \pi_3(SO_{N+3})$$

and

$$j_*\colon \pi_7(SO_7) \to \pi_7(SO_{N+7})$$

are not epimorphic. In fact,

$$\operatorname{Coker} j_* = Z_2.$$

We select arbitrary frame fields $\tau^{2k+1}_{2i-1}, \tau^{2k+1}_{2i}$ that are normal to $S^{2k+1}_{2i-1}$ and $S^{2k+1}_{2i}$ in $M^n_{\alpha,2k+1}$ (we recall that in this case the normal bundles $\nu^{2k+1}(S^{2k+1}_{2i-1}, M^n_{\alpha,2k+1})$ and $\nu^{2k+1}(S^{2k+1}_{2i}, M^n_{\alpha,2k+1})$ are trivial). Under an arbitrary variation of the fields $\tau^{2k+1}_{2i-1}$ and $\tau^{2k+1}_{2i}$, the combined frame fields $(\tau^N_{2i}, \tau^{2k+1}_{2i-1})$ and $(\tau^N_{2i}, \tau^{2k+1}_{2i})$ that are normal to the spheres $S^{2k+1}_{2i-1}$ and $S^{2k+1}_{2i}$ in $S^{N+n}$, distinguish the elements $\psi_{2i-1}, \psi_{2i} \in \operatorname{Coker} j_*$. If $\psi_{2i-1} \neq 0$ and $\psi_{2i} \neq 0$, then an equipment cannot be extended onto the balls $D^{2k+2}_{2i-1}, D^{2k+2}_{2i} \subset S^{N+n} \times I(0,1)$. There therefore appears an obstruction to a carrying over of the equipments of $\tau^N_{2i-1}$ and $\tau^N_{2i}$ under a reconstruction of Morse (depending on $\tau^{2k+1}_{2i-1}$ or $\tau^{2k+1}_{2i}$) with value in the group $\operatorname{Coker} j_*$, equal to

$$\psi_{2i-1} = \psi_{2i-1}(S^{2k+1}_{2i-1})$$

and

$$\psi_{2i} = \psi_{2i}(S^{2k+1}_{2i})$$

It is easy to see that the invariants $\psi$ depend only on the cycle $x_s \in \operatorname{Ker} {}_{2k+1}f^{(H_{2k+1})}_{\alpha *}$ and not on the sphere $S^{2k+1}_s$ realizing the cycle $x_s$, since

$$\operatorname{Ker} {}_{2k+1}f^{(H_{2k+1})}_{\alpha *} = \operatorname{Ker} {}_{2k+1}f^{(\pi_{2k+1})}_{\alpha *}$$

according to Lemma 4.5, and the homotopy spheres of dimension $2k+1$ in $M^n_{2k+1,\alpha}$ are regularly homotopic (cf. [25]). Thus we determine the invariant

$$\psi(x) \in Z_2, \qquad x \in \operatorname{Ker} {}_{2k+1}f^{(H_{2k+1})}_{\alpha *}.$$

We note further that by analogy with the invariant $\phi$ it is possible to find a base $x'_1, \dots, x'_{2l}$, such that $\psi(x'_s) = 0$, $s > 2$ (cf. [15]). It is therefore possible, following the previous proofs, to seal the cycles $x'_s$, $s \geq 3$, by means of the reconstructions of Morse. If $\psi(x'_1) \neq 0$ and $\psi(x'_2) \neq 0$, then it is not possible to carry out any further resealing (the obstruction to a carrying over of the equipment is different from zero). But if $\psi(x'_s) = 0$, $s = 1$ or 2, then it is possible to reseal the cycle $x'_s$, killing the square $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. As a result, in both cases we arrive at the statement of the lemma. The lemma is proved. □

**Remark 4.9.** A detailed analysis of the invariant $\psi$ and the reconstructions of Morse in this case (for $k = 0$) is contained in a paper by L. S. Pontrjagin [15].

It remains for us to investigate the case of odd values of $n$. We note to begin with that in this case the reconstructions of Morse, the carrying over of the equipments (of frame fields) onto a membrane and the carrying over of maps does not meet with any difficulties; but it is not clear that a manifold is simplified as the result of a reconstruction of Morse (this question is resolved trivially in all remaining cases). If $n = 2i + 1$, then under a reconstruction of Morse over a cycle (sphere) of dimension $i$ there is formed as a result a new cycle of the same dimension $i$, that was previously homologous to zero. We consider an arbitrary closed simply connected manifold $Q^N$. Suppose the group $H_i(Q^n)$ has a torsion $\operatorname{Tor} H_i(Q^n) \neq 0$. We select in the group $\operatorname{Tor} H_i(Q^n)$ a minimal system of generators $x_1, \dots, x_l$ of orders $q_1, \dots, q_l$ respectively. As is well known, for two cycles $x, y \in \operatorname{Tor} H_i(Q^n)$ there is defined a "linking coefficient" $\operatorname{Lk}(x, y) \in Z_{d(q,q')}$, where $q$ and $q'$ are the orders of the elements $x$ and $y$ and $d(q, q')$ is their greatest common divisor. Namely,

$$\operatorname{Lk}(x, y) = \partial^{-1}(qx) \circ y \equiv x \circ \partial^{-1}(q'y) \bmod d(q, q'). \tag{15}$$

We formulate the Poincaré–Alexander duality.[3]

Suppose $x_1, \dots, x_l \in TorH_i(Q^n)$ is a minimal system of $p$-primary generators of orders $q_1, \dots, q_l$ respectively. Then there exists a minimal system of generators $y_1, \dots, y_l \in \operatorname{Tor} H_i(Q^n)$ of orders $q_1, \dots, q_l$ such that

$$\operatorname{Lk}(x_m, y_t) = \delta_{mt} \bmod d(q_m, q_t). \tag{16}$$

We assume that the cycle $x_1$ is realized by the sphere $S^i_1 \subset Q^n$, and the bundle $\nu^{i+1}(S^i_1, Q^n)$ is trivial.

The tubular neighborhood $T(S^i_1)$ of the sphere $S^i_1$ in $Q^n$ is diffeomorphic to $S^i_1 \times D^{i+1}_\epsilon$, $\epsilon > 0$ being a small number.

We divide the reconstruction of Morse into two steps.

**Step 1.** $Q^n \to Q^n \setminus S^i_1 \times D^{i+1}_\epsilon = \bar{Q}^n$.

**Step 2.** $\bar{Q}^n \to \bar{Q}^n \cup_h D^{i+1} \times S^i_\epsilon = Q^n(h)$, where $h\colon \partial D^{i+1} \times D^{i+1}_\epsilon \to Q^n$ (cf. §1).

We consider the cycle

$$b(x_1) = g_0 \times \partial D^{i+1}_\epsilon \subset \bar{Q}^n, \qquad g_0 \in S^i_1.$$

**Lemma 4.10.** *$H_s(\bar{Q}^n) = H_s(Q^n)$ for $s < i$. There is defined an epimorphism $x_{1*}\colon H_i(\bar{Q}^n) \to H_i(Q^n)$, the kernel of which is generated by the cycle $b(x_1)$. In the group $H_i(\bar{Q}^n)$ it is possible to select generators $\tilde{y}_j = x^{-1}_{1*} y_j$, $j = 1, \dots, l$, such that*

$$b(x_1) = q_1 \tilde{y}_1. \tag{17}$$

[3]The meaning of linking coefficients and duality is not restricted to a system of $p$-generators.

*Proof.* $H_s(\bar{Q}^n) = H_s(Q^n)$, $s < i$, as long as $n = 2i+1 > 2s+1$, and therefore all $s$-dimensional cycles and $(s+1)$-dimensional membranes can be assumed to be nonintersecting with $S_1^i$. For $s = 1$ we can assume that the $s$-dimensional cycles do not intersect with $S_1^i$. Therefore an embedding induces the epimorphism

$$x_{1*}\colon H_i(\bar{Q}^n) \to H_i(Q^n).$$

But the membranes have dimension $i+1$ and intersect with $\bar{S}_1^i$ at isolated points. Consequently, two cycles that are homologous in $Q^n$ will be homologous in $\bar{Q}^n$ modulo $b(x_1)$. Hence

$$\operatorname{Ker} x_{1*} = (b(x_1)).$$

In the homology class $y_1 \in H_i(Q^n, Z)$ one can find a cycle $\bar{y}_1$ and a membrane $\partial^{-1}(q\bar{y}_1)$ such that the intersection index

$$\partial^{-1}(q\bar{y}_1) \circ x_1 = 1,$$

from which it follows that the cycle $b(x_1)$ is homologous to $q\bar{y}_1$. Thus the lemma is proved. □

It is well known that the linkages $\operatorname{Lk}(x, y)$ are bilinear, symmetric for odd $i$ and antisymmetric for even $i$. We select in the group $\operatorname{Tor} H_i(Q^n, Z)$ a $p$-primary system of subgroups

$$H(p, s_p) \supset H(p, s-1) \supset \cdots \supset H(p, 1),$$

where

$$\operatorname{Tor} H_i(Q^n, Z) = \sum_{p,k} H(p,k)/H(p,k-1).^4$$

Thus, to the group $H(p, s_p)$ are referred all elements of a group having orders of the form $p^j$, and to $H(p,k)/H(p,k-1)$ are referred all $p$-primary generators of order $p^k$ and $H(p,k)/H(p,k-1)$ represents the subgroup $\tilde{H}(p,k) \subset H_1(Q^n, Z)$ spanned by them.

**Lemma 4.11.** a) *A decomposition of the group* $\operatorname{Tor} H_i(Q^n, Z)$ *into the direct sum of groups* $H(p,k)/H(p,k-1)$ *can be performed by a suitable choice of p-primary generators such that* $\operatorname{Lk}(x, y) = 0$ *if* $x \in \tilde{H}(p, k-1)$, $y \in \tilde{H}(p, k_2)$, $k_1 \neq k_2$;

b) *in each group* $\tilde{H}(p,k)$ *one can choose a system of p-primary generators* $x_1, \dots, x_l, y_1, \dots, y_{2m} \in \tilde{H}(p,k)$ *such that*:

$$\left.\begin{aligned} &\operatorname{Lk}(x_s, y_l) = 0, && 1 \le s \le t,\ 1 \le l \le 2m, \\ &\operatorname{Lk}(x_{s_1}, x_{s_2}) = 0, && s_1 \neq s_2, \\ &\operatorname{Lk}(y_{l_1}, y_{l_2}) = 0, && |l_1 - l_2| > 1, \\ &\operatorname{Lk}(y_{l_1}, y_{l_1}) = 0, && l_1 + l_2 \equiv 1 \pmod 4, \end{aligned}\right\} \tag{18}$$

$$\operatorname{Lk}(x_s, x_s) \not\equiv 0 \pmod p, \quad 1 \le s \le t, \tag{19}$$

$$\left.\begin{aligned} &\operatorname{Lk}(y_l, y_l) \equiv 0 \pmod p, && 1 \le l \le 2m, \\ &\operatorname{Lk}(y_{2l-1}, y_{2l}) \equiv 0 \pmod{p^k}, && 1 \le l \le m. \end{aligned}\right\} \tag{20}$$

[4] The choice is such that $H(p,k) = H(p,k)/H(p,k-1) + H(p,k-1)$, $\tilde{H}(p,k) = H(p,k)/H(p,k-1)$.

*Proof.* It is easily seen that for any choice of a system of $p$-primary generators in the group $H(p, s_p)$ the matrix of linking coefficients for the generating elements of order $p^{s_p}$ (considered mod $p^{s_p}$) has a determinant that is relatively prime with $p$. We put $k = s_p$ and consider the subgroup $H(p, s_p - 1)$ such that

$$\mathrm{Lk}(x, y) = 0,$$

where $x \in H(p, s_p - 1)$ and $y$ is a generator of order $p^{s_p}$. Now one can choose a new system of $p$-primary generators in which all generators of orders less than $p^{s_p}$ belong to the subgroup $H(p, s_p - 1)$. We presuppose by induction that in the group $H(p, s_p)$ there are chosen subgroups $H(p, k)$ and a system of $p$-primary generators such that:

a) all generators of order not greater than $p^k$ belong to $H(p, k)$;

b) $\mathrm{Lk}(x, y) = 0$, $x \in H(p, k)$, $y$ being a generator of order $> p^k$.

We construct the group $H(p, k - 1)$. We consider the subgroup $H(p, k)$ and assume that $H(p, k - 1)$ consists of all elements $x \in H(p, k - 1)$ such that

$$\mathrm{Lk}(x, y) = 0,$$

where $x \in H(p, k - 1)$ and $y$ is a generator of order $p^k$. Since the matrix of linking coefficients for the base cycles of order $p^k$ of the group $H(p, k)$ (the linking coefficients are assumed at this step to be determined mod $p^k$) has a determinant that is relatively prime with $p$ it follows that the group $H(p, k - 1)$ constructed by us possesses all the necessary properties. Thus we have decomposed the group $H(p, s_p)$ into the direct sum of the groups

$$\tilde{H}(p, k) = H(p, k)/H(p, k - 1)$$

so that

$$\mathrm{Lk}(\tilde{H}(p, k_1), \tilde{H}(p, k_2)) = 0, \qquad k_1 \neq k_2.$$

Point a) of the lemma is completely proved. For the proof of point b) we note that each group $\tilde{H}(p, k)$ represents a linear space over the ring $Z_{p^k}$ with scalar product $\mathrm{Lk}(x, y)$ having a determinant that is relatively prime with $p$. Consequently, either

1) in the original base one finds a generator $\tilde{x}_1$ such that $\mathrm{Lk}(\tilde{x}_1, \tilde{x}_2) \not\equiv 0 \pmod{p}$, or

2) one finds a pair of generators $\tilde{y}_1, \tilde{y}_2$ such that

$$\mathrm{Lk}(\tilde{y}_1, \tilde{y}_1) \equiv 0 \pmod{p}, \qquad \mathrm{Lk}(\tilde{y}_2, \tilde{y}_2) \equiv 0 \pmod{p},$$
$$\mathrm{Lk}(\tilde{y}_1, \tilde{y}_2) \not\equiv 0 \pmod{p}.$$

If case 1) holds, then one must select a base $(\tilde{x}_1, x_2, \dots, x_t, y_1, \dots, y_s)$ such that

$$\mathrm{Lk}(x_j, \tilde{x}_1) = \mathrm{Lk}(y_j, \tilde{x}_1) = 0, \qquad j \geq 2.$$

If case 2) holds, then

$$\begin{vmatrix} \mathrm{Lk}(\tilde{y}_1, \tilde{y}_1) & \mathrm{Lk}(\tilde{y}_1, \tilde{y}_2) \\ \pm \mathrm{Lk}(\tilde{y}_1, \tilde{y}_1) & \mathrm{Lk}(\tilde{y}_2, \tilde{y}_2) \end{vmatrix} = \begin{vmatrix} pa_{11} & a_{12} \\ \pm a_{12} & pa_{22} \end{vmatrix} \not\equiv 0 \pmod{p};$$

we select a new base $\{x_j, \tilde{y}_\epsilon, y_l\}$, $l \geq 2$, such that

$$\mathrm{Lk}(x_j, \tilde{y}_\epsilon) = \mathrm{Lk}(y_l, \tilde{y}_\epsilon) = 0, \qquad \epsilon = 1, 2.$$

In the second case we put

$$y_1 = \tilde{y}_1, \qquad y_2 = \frac{1}{\mathrm{Lk}(\tilde{y}_1, \tilde{y}_2)} \tilde{y}_2.$$

Then in both cases we select the other required generators in subgroups that are orthogonal to $\tilde{x}_1$ (in the first case) and orthogonal to $\tilde{y}_1, \tilde{y}_2$ (in the second case) in such a way that the relations (18)–(20) are fulfilled. The lemma is proved. □

In the sequel we will always compose a minimal system of generators of the group $\operatorname{Tor} H_i(Q^n, Z)$ from the $p$-primary generators constructed in Lemma 4.11. We will select a minimal (with respect to the number of generators) system, and the generating element $x$ of aggregate order

$$q = \prod_{p \in I} p^{k_p}$$

will in a canonical manner be decomposed into a sum of primary generators $x = \sum_p x(p)$ of orders $p^{k_p}$. We divide the set of indices $J$ into two parts: in the first part $J_1$ we put all $p$ for which the elements $x(p)$ satisfy condition (19), and in the second part we put all $p$ for which the $x(p)$ satisfy condition (20). Putting

$$\bar{x} = \sum_{p \in J_1} x(p), \qquad \bar{\bar{x}} = \sum_{p \in J_2} x(p),$$

we get that for $\bar{\bar{x}}$ there exists a base element $\bar{\bar{y}}$, independent of $\bar{\bar{x}}$, such that the number $\operatorname{Lk}(\bar{\bar{x}}, \bar{\bar{y}})$ is relatively prime to the order of $\bar{\bar{x}}$ (equal to the order of $\bar{\bar{y}}$).

**Lemma 4.12.** *If $n = 2i + 1$ and $i$ is even, then the order of the element $\bar{\bar{x}}$ is equal to 2 (if $\bar{x} \neq 0$).*

It is evident that the proof of the lemma immediately follows from the antisymmetry $\operatorname{Lk}(\bar{x}, \bar{x}) = -\operatorname{Lk}(\bar{x}, \bar{x})$ that must be relatively prime to the order of $\bar{x}$. The lemma is proved. □

Suppose the cycle $\bar{x}_1$ is realized by the sphere $S_1^i \subset Q^n$, $i$ even. According to Lemma 4.10, to the element $\bar{x}_1 \in H_i(Q^n)$ corresponds an element $\tilde{\bar{x}}_1 \in H_i(\bar{Q}^n)$ such that $b(\bar{x}_1) = 2\tilde{\bar{x}}_1$. One can assume that $\tilde{\bar{x}}_1$ lies on the boundary of the tubular neighborhood

$$T(S^i) \subset Q^n, \qquad T(S^i) = S^i \times D^{i+1}.$$

**Lemma 4.13.** *The kernel of the homomorphism*

$$H_i(\bar{Q}^n) \to H_i(Q^n(h)),$$

*for any $h \colon \partial D^{i+1} \times D_\epsilon^{n-i} \to T(S_1^i)$ such that*

$$h(x, y) = (x, h_x(y)), \qquad h_x \in SO_{i+1},$$

*is generated by the element $(1 + 2\lambda(h))\tilde{\bar{x}}_1$, where $\lambda(h)$ is a certain integer.*

*Proof.* We consider the map $d(h) \colon S_1^i \to SO_{i+1}$, defining a reconstruction of Morse, and we denote by $y(h)$ the homology class of the cycle $\tilde{y}(h) \in \partial T(S_1^i)$, defined by the first vector of the frame field $d(h)$, normal to $S_1^i$ in $Q^n$, $y(h) \in H_i(\bar{Q}^n)$. There exists a number $\lambda(h)$ such that

$$y(h) = \tilde{\bar{x}}_1 + \lambda(h) b(\bar{x}_1)$$

or

$$y(h) = (1 + 2\lambda(h))\tilde{\bar{x}}_1.$$

Clearly, under the inclusion homomorphism $H_i(\bar{Q}^n) \to H_i(Q^n)$ the kernel is generated by the element

$$y(h) = (1 + 2\lambda(h))\tilde{\bar{x}}_1.$$

The lemma is proved. □

Thus we have eliminated the element $\bar{x}_1$ of order 2. Therefore in the group $H_i(Q^n(h))$ of generators not satisfying condition (20) there will be one less (for $i$ even), since all such generators have order 2 according to Lemma 4.12.

Suppose $i$ is arbitrary (either even or odd) and that $\bar{\bar{x}}_1$ is a generating cycle $\bar{\bar{x}}_1 \in H_i(Q^n)$, satisfying condition (20) and realized by the sphere $S_1^i \subset Q^n$ with trivial normal bundle $\nu^{i+1}(S_1^i, Q^n)$. Suppose also that the cycle $\bar{\bar{x}}_2 \in H_i(Q^n)$ is such that $\mathrm{Lk}(\bar{\bar{x}}_1, \bar{\bar{x}}_2) = 1$. We denote, as in Lemma 4.10, the generators corresponding to them by $\tilde{\bar{\bar{x}}}_1, \tilde{\bar{\bar{x}}}_2 \in H_i(\bar{Q}^n)$, where $b(\bar{\bar{x}}_1) = q_1\tilde{\bar{\bar{x}}}_2$, $q_1$ is the order of the generators $\bar{\bar{x}}_1, \bar{\bar{x}}_2 \in H_i(Q^n)$ and $\tilde{\bar{\bar{x}}}_1$ is the homology class in $H_i(Q^n)$ of the cycle $\bar{\bar{x}}_1(h)$ defined by the first vector of an $(i+1)$-frame field $h\colon S^i \to SO_{i+1}$ on the boundary $\partial T(S_1^i)$ for a fixed $h$. Then we have

**Lemma 4.14.** *The kernel of the inclusion homomorphism $H_i(\bar{Q}^n) \to H_i(Q^n(h))$ is generated by the element $\tilde{\bar{\bar{x}}}_1$, and the group $H_i(Q^n(h))$ has one generator less than the group $H_i(Q^n)$.*

The proof of the lemma follows from the definition of Morse's reconstruction and the relation $b(\bar{\bar{x}}_1) = q_1\tilde{\bar{\bar{x}}}_2$.

**Remark.** The element $\tilde{\bar{\bar{x}}}_2 \in H_i(Q^n(h))$ has order $\lambda q_1$, where

$$\lambda \equiv \mathrm{Lk}(\bar{\bar{x}}_1, \bar{\bar{x}}_1) \pmod{q_1},$$

and the number $\mathrm{Lk}(\tilde{\bar{\bar{x}}}_2, \tilde{\bar{\bar{x}}}_2)$ is relatively prime to $\lambda q_1$ if $\lambda \neq 0$ (i.e., the element $\tilde{\bar{\bar{x}}}_2$ satisfies condition (19) in the manifold $Q^n(h)$).

Let $i$ be odd. We consider the element $\bar{x}_1 \in H_i(Q^n)$ realized by the sphere $S_1^i \subset Q^n$ with trivial normal bundle $\nu^{i+1}(S_1^i, Q^n)$. The self-linking coefficient

$$\mathrm{Lk}(\bar{x}_1, \bar{x}_1) = \lambda \pmod{q},$$

where $q$ is the order of $\bar{x}_1$ and $\lambda$ is relatively prime to $q$. It follows from Lemma 4.10 that on the boundary $\partial T(S_1^i)$ one can find a cycle $\tilde{\bar{x}}_1$ such that the relation

$$\lambda b(\bar{x}_1) = q\tilde{\bar{x}}_1$$

will be fulfilled in the homology group $H_i(\bar{Q}^n)$.

We consider the map $h\colon S_1^i \to SO_{i+1}$ and the kernel of the embedding

$$j_*\colon \pi_i(SO_{i+1}) \to \pi_i(SO_\infty),$$

which is isomorphic to the group $Z$ for $i$ odd, $\operatorname{Ker} j_* = Z$.

We denote by $y(h)$ the homology class in $\bar{Q}^n$ of the cycle defined on $\partial(S_1^i) = S_1^i \times S_\epsilon^i$ ($b(\bar{x}_1) = g_0 \times S_\epsilon^i$, $g_0 \in S_1^i$, as above) by the first vector of the frame field $h$. Let $\mu \in \operatorname{Ker} j_* = Z$ ($\mu$ a number).

**Lemma 4.15.** *The kernel of the inclusion homomorphism*

$$H_i(\bar{Q}^n) \to H_i(Q^n(h))$$

*is generated by the element $y(h) = \tilde{\bar{x}}_1 + \gamma b(\bar{x}_1)$. The kernel of the inclusion homomorphism*

$$H_i(\bar{Q}^n) \to H_i(Q^n(h+\mu)), \qquad \mu \in \operatorname{Ker} j_* = Z,$$

*is generated by the element $y(h+\mu) = y(h) + 2\mu b(\bar{x}_1)$.*

The proof of Lemma 4.15 immediately follows from the definition of Morse's reconstruction and the structure of the homomorphism $\operatorname{Ker} j_* \to H_i(S^i)$ induced by the map $SO_{i+1} \to S^i$ (projection) under which the generator of the group $\operatorname{Ker} j_*$ goes over into the cycle $2[S^i]$. Therefore

$$y(h+\mu) = y(h) + 2\mu b(\bar{x}_1).$$

Let us prove that

$$y(h) = \tilde{\bar{x}}_1 + \gamma b(\bar{x}_1).$$

To do this we consider the intersection index

$$[\partial^{-1} q_1 y(h)] \cdot \bar{x}_1 = \lambda \bmod q_1 = \lambda + \gamma q_1.$$

On the other hand,

$$[\partial^{-1} b(\bar{x}_1)] \cdot \bar{x}_1 = 1.$$

Therefore

$$[\partial^{-1}(q_1 y(h) - q_1\gamma b(\bar{x}_1))] \cdot \bar{x}_1 = \lambda,$$

from which it follows that one can put $\tilde{\bar{x}}_1 = y(h) - \gamma b(x)$. The lemma is proved. □

**Lemma 4.16.** *One can choose a number $\mu$ such that in the group $H_i(Q^n(h+\mu))$ we will have*:

a) $\tilde{\bar{x}}_1 = 0$, $\lambda b(\bar{x}_1) = 0$ *($\gamma$ even)*,

b) $\tilde{\bar{x}}_1 = b(\bar{x}_1)$, $(\lambda_1 - q_1)\tilde{\bar{x}}_1 = 0$ *($\gamma$ odd)*,

*where in both cases the order of the "new" element $b(\bar{x}_1)$ is less then $q_1$; the number* $\operatorname{Lk}(b(\bar{x}_1), b(\bar{x}_1))$ *is relatively prime to the order of the element $b(\bar{x}_1)$.*

*Proof.* Since $\lambda b(\bar{x}_1) = q_1\tilde{\bar{x}}_1$ in $\bar{Q}^n$ and $\tilde{\bar{x}}_1 = y(h) - \gamma b(\bar{x}_1)$, we have

$$y(h+\mu) = y(h) + 2\mu b(\bar{x}_1) = \tilde{\bar{x}} + \gamma b(\bar{x}_1) + 2\mu b(\bar{x}_1).$$

In passing to $Q^n(h+\mu)$ the relation $y(h+\mu) = 0$ is imposed. Therefore

$$\tilde{\bar{x}}_1 = -(\gamma + 2\mu) b(\bar{x}_1) \quad (\text{in } Q^n(h+\mu)),$$
$$\lambda b(\bar{x}_1) = q_1 \tilde{\bar{x}}_1 \quad (\text{in } \bar{Q}^n),$$

from which follows the possibility of making such a choice of $\mu$ ($\mu = -\gamma/2$ for $\gamma$ even and $2\mu - 1 = \gamma$ for $\gamma$ odd).

Clearly, by virtue of Lemma 4.11, the element $b(\bar{x}_1)$ does not link with the other base cycles.

Thus the assertion is proved. □

Now we apply the proved lemmas to a study of the maps

$$f_\alpha \colon S^{N+n} \to T_N(M^n),$$

where $n = 2i+1$.

**Lemma 4.17.** *Let $\alpha \in A(M^n)$. There exists a map*

$$f_\alpha \colon S^{N+n} \to T_N(M^n)$$

*such that the inverse image $f_\alpha^{-1}(M^n) = M_\alpha^n$ is homotopically equivalent to $M^n$.*

*Proof.* As above, we consider the map

$$_if_\alpha \colon S^{N+n} \to T_N(M^n),$$

constructed according to Lemma 4.6, and the preimage

$$M^n_{\alpha,i}|_if_\alpha^{-1}(M^n),$$

on which the groups $H_s(M^n_{\alpha,i})$ are isomorphic to the groups $H_s(M^n)$ for $s < i$ and $\pi_1(M^n_{\alpha,i}) = 0$. The group $\operatorname{Ker}{}_if^{(H_i)}_{\alpha*}$ is singled out in $H_i(M^n_{\alpha,i})$, $n = 2i + 1$, as a direct summand according to Lemma 4.4′. The group $\operatorname{Ker}{}_if^{(H_{i+1})}_{\alpha*}$ is free abelian, according to Lemma 4.5. First we attempt by means of Morse's reconstructions to kill the group $\operatorname{Tor}\operatorname{Ker}{}_if^{(H_i)}_{\alpha*}$, using the Poincaré–Alexander duality. If $i$ is even, then, on the basis of Lemmas 4.12 and 4.13, we kill all elements not satisfying condition (20), without increasing the number of generators, and next, by Lemma 4.14, we kill the elements satisfying condition (20), decreasing the number of generators by 1 with each reconstruction. If $i$ is odd, then by means of Morse's reconstructions we successively kill the generators satisfying condition (20), each time decreasing the number of generators by 1 (by Lemma 4.14), and next, by Lemmas 4.15–4.16, we commence to decrease the order of some generator satisfying condition (19), each time not increasing the number of generators yet reducing the order of this generator (varying the reconstruction mod $\operatorname{Ker} j_* \subset \pi_i(SO_{i+1})$), which preserves the possibility of carrying over the frame fields (cf. the proofs of Lemmas 1.1 and 1.2). Thus, as a result we kill the group $\operatorname{Tor}\operatorname{Ker}{}_if^{(H_i)}_{\alpha*}$. Then, according to the results in [4], we easily kill the elements of infinite order and so arrive at the needed manifold $M^n_\alpha$ and the map

$$f_\alpha \colon S^{N+n} \to T_N(M^n)$$

by analogy with Lemmas 4.7, 4.8 and 4.9. The lemma is proved. □

We collect the results of the lemmas into the following theorem.

**Theorem 4.18.** *If $n = 4k$, $k \geq 2$, or $n = 2k + 1$, then each element*

$$\alpha \in A(M^n) \subset \pi_{N+n}(T_N(M^n)), \qquad A(M^n) = H^{-1}\phi[M^n]$$

*is represented by a map $f_\alpha \colon S^{N+n} \to T_N(M^n)$ that is t-regular and such that*

$$\pi_1(M^n_\alpha) = 0, \qquad H_i(M^n_\alpha) = H_i(M^n)$$

*for $i = 2, \dots, n-2$, where $M^n_\alpha = f_\alpha^{-1}(M^n)$. Thus the manifold $M^n_\alpha$ is homotopically equivalent to $M^n$ with degree $+1$ and $\nu^N(M^n_\alpha) = f^*_\alpha \nu^N(M^n)$. If $n = 4k + 2$, $k \geq 1$, then for any element $\alpha \in A(M^n)$ one can choose a map $f_\alpha \colon S^{N+n} \to T_N(M^n)$ of the same homotopy class as $\alpha$ such that*

$$\pi_1(M^n_\alpha) = 0, \qquad H_i(M^n_\alpha) = H_i(M^n)$$

*for $i \geq 2k$, where $M^n_\alpha = f_\alpha^{-1}(M^n)$; moreover,*

$$\operatorname{Ker} f^{(H_{2k+1})}_{\alpha*} = Z + Z,$$

*and there is defined an invariant $\phi(\alpha) \in Z_2$ for $n = 4k+2$, $k \neq 1, 3$, and $\psi(\alpha) \in Z_2$ for $n = 6, 14$, the equality of these invariants to zero being a sufficient condition for reseating the groups $\operatorname{Ker} f^{(H_{2k+1})}_{\alpha*} = Z + Z$ by means of the reconstructions of Morse.*

The theorem is a formal unification of the preceding lemmas.

## § 5. The manifolds in one class

For any element $\alpha \in \bar{A}(M^n) \subset A(M^n)$ the map representing it

$$f_\alpha\colon S^{N+n} \to T_N(M^n)$$

is called admissible if it satisfies Lemma 3.2 and if the inverse image

$$f_\alpha^{-1}(M^n) = M^n_\alpha$$

is homotopically equivalent to $M^n$.

**Theorem 5.1.** *Let $f_{\alpha,i}\colon S^{N+n} \to T_N(M^n)$, $i = 1, 2$, be two admissible homotopic maps and $M^n_{\alpha,i} = f^{-1}_{\alpha,i}(M^n)$. If $n$ is even, then the manifolds $M^n_{\alpha,i}$ are diffeomorphic with degree $+1$. If $n$ is odd, then there exists a Milnor sphere $\tilde{S}^n \in \theta^n(\partial\pi)$, which is the boundary of a $\pi$-manifold, such that the manifolds $M^n_{\alpha,1}$ and $M^n_{\alpha,2} \# \tilde{S}^n$ are diffeomorphic with degree $+1$.*

*Proof.* We consider the homotopy

$$F\colon S^{N+n} \times I \to T_N(M^n),$$

where $F|S^{N+n} \times 0 = f_{\alpha,1}$ and $F|S^{N+n} \times 1 = f_{\alpha,2}$. We divide the proof into a number of steps.

**Step 1.** We $t$-regularize the homotopy $F$. Then we consider the inverse image

$$F^{-1}(M^n) \subset S^{N+n} \times I(0,1),$$

which is a manifold $N^{n+1}$ with boundary

$$\partial N^{n+1} = M^n_{\alpha,1} \cup (-M^n_{\alpha,2}),$$

and

$$\nu^N(N^{n+1}) = F^*\nu^N(M^n).$$

Thus there is defined a map $F|N^{n+1} \to M^n$, which is a homotopy equivalence of degree $+1$ on each of the boundaries. The manifold $N^{n+1}$ is an $(F,\pi)$-manifold mod $M^n$.

**Step 2.** We consider the decompositions into direct sums

$$\left.\begin{aligned} H_j(N^{n+1}) &= H_j(M^n_{\alpha,i}) + \operatorname{Ker} F_*^{(H_j)}, \quad i = 1, 2,\\ \pi_j(N^{n+1}) &= \pi_j(M^n_{\alpha,i}) + \operatorname{Ker} F_*^{(\pi_j)}, \quad i = 1, 2,\\ H^j(N^{n+1}) &= H^j(M^n_{\alpha,i}) + \operatorname{Coker} F_*, \quad i = 1, 2, \end{aligned}\right\} \tag{21}$$

that arise from the natural retractions of a membrane onto each of the boundaries:

$$(f_{\alpha,i})^{-1} \cdot F\colon N^{n+1} \to M^n_{\alpha,i}, \tag{22}$$

where the maps $(f_{\alpha,i})^{-1} \cdot f_{\alpha,i}\colon M^n_{\alpha,i} \to M^n_{\alpha,i}$ are homotopic to the identity maps.

It is evident that

$$\left.\begin{aligned} H_j(N^{n+1}, M^n_{\alpha,i}) &= \operatorname{Ker} F_*^{(H_j)}, \quad i = 1, 2,\\ \pi_j(N^{n+1}, M^n_{\alpha,i}) &= \operatorname{Ker} F_*^{(\pi_j)}, \quad i = 1, 2,\\ H^j(N^{n+1}, M^n_{\alpha,i}) &= \operatorname{Coker} F_*, \quad i = 1, 2. \end{aligned}\right\} \tag{23}$$

We have the following

**Lemma 5.2.** *Among the groups* $\operatorname{Ker} F_*^{(H_j)}/\operatorname{Tor}$ *and* $\operatorname{Ker} F_*^{(H_{n+1-j})}/\operatorname{Tor}$ *there is defined by means of the intersection index a nonsingular wimodular scalar product. Among the groups* $\operatorname{Tor}\operatorname{Ker} F_*^{(H_j)}$ *and* $\operatorname{Tor}\operatorname{Ker} F_*^{(H_{n-j})}$ *there is defined the Alexander duality: for every minimal system of generators* $x_1, \dots, x_l \in \operatorname{Tor}\operatorname{Ker} F_*^{(H_j)}$ *there exists a minimal system of generators* $y_1, \dots, y_l \in \operatorname{Tor}\operatorname{Ker} F_*^{(H_{n-j})}$ *such that the order of* $y_i$ *is equal to the order of* $x_i$, $i = 1, \dots, l$, *and* $\operatorname{Lk}(x_i, y_j) = \delta_{ij}$.

*Proof.* Lemma 5.3 is an immediate consequence of the decompositions into direct sums (21), the isomorphisms (23) between the various groups mod $M^n_{\alpha,i}$ and the groups $\operatorname{Ker} F_*^{(H_k)}$, and the Poincaré–Alexander duality $D$:

$$D\colon H_j(N^{n+1}, M^n_{\alpha,1}) \xrightarrow{\approx} H^{n+1-j}(N^{n+1}, M^n_{\alpha,2}),$$
$$\operatorname{Tor} H_j(N^{n+1}, M^n_{\alpha,1}) \approx \operatorname{Tor} H_{n-j}(N^{n+1}, M^n_{\alpha,2}). \tag{24}$$

The lemma is proved. □

**Step 3.** By means of Morse's reconstructions we successively kill the groups $\pi_1(N^{n+1}), \operatorname{Ker} F_*^{(H_2)}, \dots,$ etc., reconstructing the map $F$ onto the reconstructed manifold and using in this regard all of the techniques proved in §4.

**Case 1.** If $n$ is even, then $n+1$ is odd and the successive reconstructions of the groups $\operatorname{Ker} F_*^{(H_j)}$ up to $j = n/2$ do not encounter obstructions. While if $\operatorname{Ker} F_*^{(H_j)} = 0$ for $j \le n/2$, then, by Lemma 5.3, $\operatorname{Ker} F_*^{(H_{n+1-j})} = 0$ (and $\pi_1 = 0$). Therefore the membrane $N^{n+1}$ contracts onto each of its boundaries, effecting by the same token a $J$-equivalence ($h$-cobordism) of them. According to a theorem of Smale [19] the manifolds $M^n_{\alpha,1}$ and $M^n_{\alpha,2}$ are diffeomorphic.

**Case 2.** If $n = 4k-1$, then $n+1 = 4k$. By analogy with the preceding case one can obtain the result that $\operatorname{Ker} F_*^{(H_j)} = 0$ for $j < 2k$ and $\operatorname{Ker} F_*^{(H_j)} = 0$ for $j > 2k$. The matrix of intersections of the free abelian group $\operatorname{Ker} F_*^{(H_{2k})}$ will be unimodular and will have even numbers on its diagonal (in exact analogy with Lemma 4.7), but its signature, in contrast to the situation in Lemma 4.7, is not necessarily equal to zero, since the formula of Hirzebruch [3] is applicable only for closed manifolds. We denote this matrix of intersections by $B = (b_{ij})$, where $b_{ij} = x_j \cdot x_j$ and $x_1, \dots, x_s$ is a base of the group $\operatorname{Ker} F_*^{(H_{2k})}$. We denote the signature of the matrix $B$ by $\tau(B)$. It is known (cf. [8]) that $\tau(B) \equiv 0$ (modulo 8) since $\det B = \pm 1$ and $b_{ii} \equiv 0$ (mod 2).

We construct, following Milnor [8], a $\pi$-manifold $M^{n+1}(B)$ such that:

a) $\pi_1(M^{n+1}(B)) = 0$;
b) $H_j(M^{n+1}(B)) = 0$, $j \ne 0, 2k$;
c) $\partial M^{n+1}(B)$ is a homotopy sphere

$$\tilde{S}^n = \partial M^{n+1}(B) \in \theta^n(\partial\pi);$$

d) the matrix of intersections of the base cycles of the group $H_{2k}(M^{n+1}(B))$ is such that its signature

$$\tau(M^{n+1}(B)) = -\tau(B).$$

We now consider the manifold

$$N^{n+1} \cup_{f_0} D^n_\epsilon \times I(0,1) \cup_{f_1} M^{n+1}(B) = N^{n+1}(B), \tag{25}$$

where

$$f_0\colon D^n_\epsilon \times 0 \to M^n_{\alpha,2},$$
$$f_1\colon D^n_\epsilon \times 1 \to \partial M^{n+1}(B)$$

($f_0, f_1$ are diffeomorphisms having the necessary degree, equal to $\pm 1$). Clearly,

$$\partial N^{n+1}(B) = M^n_{\alpha,1} \cup (-M^n_{\alpha,2} \# \tilde{S}^n).$$

In addition there are defined the retractions

$$F_1\colon N^{n+1}(B) \to M^n_{\alpha,1}, \qquad F_1\colon N^{n+1}(B) \to M^n_{\alpha,2} \# \tilde{S}^n, \tag{26}$$

induced by the retractions $(f_{\alpha,1})^{-1} \cdot F$ and $(f_{\alpha,2})^{-1} \cdot F$. Since $M^{n+1}(B)$ is a $\pi$-manifold, it is easy to see that

$$F_i^* \nu^N(M^n_{\alpha,i}) = \nu^N(N^{n+1}(B)), \qquad i = 1, 2.$$

By construction, the signature of the matrix of intersections of the base cycles of the group $\operatorname{Ker} F_{i*}^{(H_{2k})}$, $i = 1, 2$, is equal to the sum of signatures

$$\tau(B) + \tau(M^{n+1}(B)) = 0.$$

Further, we repeat completely the arguments of Lemma 4.7, we reconstruct by the same method the group $\operatorname{Ker} F_{i*}^{(H_{2k})}$, killing it, and we apply the theorem of Smale (cf. Case 1). In this way. Case 2 is investigated.

**Case 3.** $n = 4k + 1$, $n + 1 = 4k + 2$. By analogy with Cases 1 and 2 and the proofs of Lemmas 4.8 and 4.9 we will assume that the membrane $N^{n+1}$ is such that:

a) $\operatorname{Ker} F_*^{(H_j)} = 0$, $j < 2k + 1$,

b) $\pi_1(N^{n+1}) = 0$,

c) $\operatorname{Ker} F_*^{(H_{2k+1})} = Z + Z$ or 0, depending on which of the invariants $\phi$ (for $k \neq 1, 3$) or $\psi$ (for $k = 1, 3$) is obstructing a reconstruction of Morse.

In the first place, the invariant $\psi$ (for the cases $k = 1, 3$) did not obstruct the reconstructions of Morse but only the carrying over of the frame fields (cf. Lemma 4.9), which is of no concern to us at this point. Therefore we carry out these reconstructions (without being concerned about the fields) and get that

$$\operatorname{Ker} F_*^{(H_{2k+1})} = 0, \qquad k = 1, 3.$$

Thus the membrane contracts onto each of its boundaries and is therefore (cf. [19]) diffeomorphic to $M^n_{\alpha,1} \times I$.

If $k \neq 1, 3$, then on the base cycles $x, y \in \operatorname{Ker} F_*^{(H_{2k+1})}$ is defined the invariant $\phi(x), \phi(y)$.

If $\phi(x) = 0$ or $\phi(y) = 0$, then we carry out a reconstruction of Morse, recalling the meaning of $\phi$ (an invariant normal to the bundle of an embedded sphere $S^{2k+1} \subset N^{4k+2}$). Suppose $\phi(x) \neq 0$ and $\phi(y) \neq 0$. We construct, according to Kervaire [4], a $\pi$-manifold $M^{4k+2}(\phi)$ such that:

a) the boundary $\partial M^{4k+2}(\phi)$ is a homotopy sphere;

b) $\pi_1(M^{4k+2}(\phi)) = H_j(M^{4k+2}(\phi)) = 0$, $j \neq 0, 2k + 1$;

c) $H_{2k+1}(M^{4k+2}(\phi)) = Z + Z$; and denoting the base cycles by $\bar{x}, \bar{y}$,

d) $\phi(\bar{x}) = \phi(\bar{y}) = 1$.

As in Case 2 we pat

$$N^{4n+2}(\phi) = N^{4k+2} \cup_{f_0} D^{4k+1}_\epsilon \times I(0,1) \cup_{f_1} M^{4k+2}(\phi), \tag{27}$$

where

$$f_0\colon D_\epsilon^{4k+1}\times 0\to M_{\alpha,1}^{4k+1},$$
$$f_1\colon D_\epsilon^{4k+1}\times 1\to M_{\alpha,2}^{4k+2}$$

are diffeomorphisms having the necessary degree, equal to +1. Then

$$\partial N^{4k+2}(\phi)=M_{\alpha,1}^{4k+1}\cup(-M_{\alpha,2}^{4k+1}\,\#\,\partial M^{4k+2}(\phi)).$$

Using next the relation

$$\phi(z+t)=\phi(z)+\phi(t)+Z\cdot t|_{\bmod 2},$$

we find a new base $x_1, x_2, x_3, x_4\in\operatorname{Ker} F_{1*}^{(H_{2k+1})}$, where

$$F_1\colon N^{4k+2}(\phi)\to M_{\alpha,1}^{4k+1}$$

is a natural retraction (here $\phi(x_i)=0$, $i=1,2,3,4$), and we seal cycles by means of Morse's reconstructions. Then the theorem of Smale (cf. Case 1) is again applied. The theorem is proved. □

## § 6. One manifold in different classes

We will consider only maps

$$f_\alpha\colon S^{N+n}\to T_N(M^n)$$

that are admissible in the sense of §5.

**Lemma 6.1.** *The homotopy class of an admissible map*

$$f_\alpha\colon S^{N+n}\to T_N(M^n)$$

*is completely defined by*:

a) *a manifold $M_\alpha^n$ that is homotopically equivalent to the manifold $M^n$ with degree +1 and such that $M_\alpha^n\geq M^n$*;

b) *some (arbitrary) embedding of $M_\alpha^n\in S^{N+n}$*;

c) *some (arbitrary up to homotopy) smooth map $\tilde f_\alpha\colon M_\alpha^n\to M^n$ of degree +1, for which $\tilde f_\alpha^*\nu^N(M^n)=\nu^N(M_\alpha^n)$*;

d) *some (arbitrary up to homotopy) smooth map of $SO_N$-bundles*

$$\tilde{\tilde f}_\alpha\colon \nu^N(M_\alpha^N)\to\nu^N(M^n)$$

*that covers the smooth map $\tilde f_\alpha\colon M_\alpha^n\to M^n$.*

*Proof.* If we are given a manifold $M_\alpha^n$, an embedding of $M_\alpha^n\subset S^{N+n}$, a map $\tilde f_\alpha\colon M_\alpha^n\to M^n$ and a map of bundles $\tilde{\tilde f}_n\colon\nu^N(M_\alpha^n)\to\nu^N(M^n)$ covering $\tilde f_\alpha$, then the map $f_\alpha$ is completely defined on the tubular neighborhood $T(M_\alpha^n)\subset S^{N+n}$, since the tube $T(M_\alpha^n)$ is the space of the normal bundle $\nu^N(M_\alpha^n)$. In the construction of the Thom complex $T_N(M^n)$ an extension of $f_\alpha$ onto the rest of the sphere $S^{N+n}$ is carried out trivially (in a neighborhood of a critical point of the Thom complex) and uniquely to within homotopy. We now assume that the embedding $M^n\subset S^{N+n}$ is subjected to isotopies, and the maps $\tilde f_\alpha$ and $\tilde{\tilde f}_\alpha$ are subjected to homotopies, where all of the isotopies and homotopies are smooth, and a homotopy of the map $\tilde{\tilde f}_\alpha$ is a homotopy of maps of $SO_N$-bundles that covers a homotopy of $\tilde f_\alpha$. These isotopies and homotopies simultaneously define an embedding of

$$M_\alpha^n\times I(0,1)\subset S^{N+n}\times I(0,1)$$

and a map $F$ of the tubular neighborhood

$$T(M^n_\alpha \times I(0,1)) \subset S^{N+n} \times I$$

($T(M^n_\alpha \times I(0,1))$ is diffeomorphic to $\nu^N(M^n_\alpha) \times I(0,1)$) into the space $T_N(M^n)$, where $F(M^n_\alpha \times I) \subset M^n$. Further, the map

$$F\colon T(M^n_\alpha \times I) \to T_N(M^n)$$

is extended in a well-known manner to the map

$$F\colon S^{N+n} \times I \to T_N(M^n),$$

where $F|S^{N+n} \times 0 = f_\alpha$. Consequently, the homotopy class $\alpha$ of the map $f_\alpha$ does not depend on the arbitrariness in choice of the embedding (all of the embeddings are isotopic for $N \gg n$) and of the maps $\tilde{f}_\alpha, \tilde{\tilde{f}}_\alpha$ in their homotopy classes.

The lemma is proved. □

Thus, for a fixed manifold $M^n_\alpha$ the homotopy class of an admissible map $f_\alpha$,

$$f_\alpha\colon S^{N+n} \to T_N(M^n),$$

is completely defined by the homotopy class of a map $\tilde{f}_\alpha\colon M^n_\alpha \to M^n$ of degree $+1$ such that

$$\nu^N(M^n_\alpha) = \tilde{f}^*_\alpha \nu^N(M^n),$$

and by the homotopy class of a map of $SO_N$-bundles

$$\tilde{\tilde{f}}_\alpha\colon \nu^N(M^n_\alpha) \to \nu^N(M^n)$$

that covers $\tilde{f}_\alpha$ (in the sequel it will be assumed without further comment that the embedding of $M^n_\alpha \subset S^{N+n}$ is fixed).

**Lemma 6.2.** *If two manifolds $M^n_{\alpha,i} \geq M^n$, $i = 1, 2$, homotopically equivalent to $M^n$, have at one time been shown to be in one and the same class $\alpha \in \bar{A}(M^n) \subset A(M^n)$, then, for any class $\alpha_1$ for which there exists an admissible map*

$$f_{\alpha_1,1}\colon S^{N+n} \to T_N(M^n)$$

*such that $f^{-1}_{\alpha_1,1}(M^n) = M^n_{\alpha,1}$, there exists another admissible map*

$$f_{\alpha_1,2}\colon S^{N+n} \to T_N(M^n)$$

*such that $f^{-1}_{\alpha_1,2}(M^n) = M^n_{\alpha,2}$.*

*Proof.* We consider the $t$-regular homotopy

$$F\colon S^{N+n} \times I(0,1) \to T_N(M^n),$$

where $F|S^{N+n} \times 0 = f_{\alpha,1}$ and $F|S^{N+n} \times 1 = f_{\alpha,2}$. We put

$$N^{n+1} = F^{-1}(M^n) \subset S^{N+n} \times I,$$

where

$$\nu^N(N^{n+1}) = F^* \nu^N(M^n).$$

Since the map $F$ becomes on the boundaries the homotopy equivalences $\tilde{f}_{\alpha,1}$ and $\tilde{f}_{\alpha,2}$, the membrane $N^{n+1}$ naturally retracts onto each of the boundaries. We denote these retractions by

$$F_i = (f_{\alpha,i})^{-1} \cdot F, \qquad i = 1, 2.$$

According to Lemma 6.1 one can obtain the element $\alpha_1$ in the following manner: on the boundary $M^n_{\alpha,1} \subset \partial N^{n+1}$ we change the map $\tilde{f}_{\alpha,1}$ into $\tilde{f}_{\alpha_1,1}$ and, analogously, we change the bundle map $\tilde{\tilde{f}}_{\alpha,1}$ into $\tilde{\tilde{f}}_{\alpha_1,1}$. Since the membrane $N^{n+1}$ retracts onto a boundary and

$$\nu^N(N^{n+1}) = F_1^* \nu^N(M^n_{\alpha,1}),$$

we can extend the maps $\tilde{f}_{\alpha_1,1}, \tilde{\tilde{f}}_{\alpha_1,1}$ to the maps

$$\tilde{F}\colon N^{n+1} \to M^n$$

and

$$\tilde{\tilde{F}}\colon \nu^N(N^{n+1}) \to T_N(M^n).$$

Then we extend this map $\tilde{\tilde{F}}$ with tubular neighborhood $T(N^{n+1}) \subset S^{N+n} \times I$ onto the entire direct product $S^{N+n} \times I$ in the manner of Thom and we denote this extension by

$$\bar{F}\colon S^{N+n} \times I \to T_N(M^n).$$

Clearly,

$$\bar{F}|S^{N+n} \times 0 = f_{\alpha_1,1}.$$

Putting

$$f_{\alpha_1,2} = \bar{F}|S^{N+n} \times 1,$$

if the extension $\bar{F}$ is smooth on $T(N^{n+1})$, and this property can always be attained, we get the statement of the lemma. The lemma is proved. □

In addition we are now able to consider only one fixed manifold $M^n_\alpha \geq M^n$, $M^n \geq M^n_\alpha$, and study the problem of determining the set of classes $\alpha_i \in \bar{A}(M^n) \subset A(M^n)$ in which it may lie. We denote by $B(M^n_\alpha)$ the set of classes $\alpha_i \in \bar{A}(M^n)$ for which there exist admissible maps

$$f_{\alpha_i}\colon S^{N+n} \to T_N(M^n)$$

such that

$$f^{-1}_{\alpha_i}(M^n) = M^n_\alpha.$$

We denote by $\pi^+(M^n_\alpha, M^n)$ the set of homotopy classes of maps $f\colon M^n_\alpha \to M^n$ of degree $+1$ such that

$$f^*\nu^N(M^n) = \nu^N(M^n_\alpha).$$

We denote by $\pi(X, Y)$ the set of homotopy classes of maps $X \to Y$ for any complexes $X, Y$. In particular, the sets $\pi^+(M^n, M^n)$ and $\pi(M^n, SO_N)$ are groups, where $\pi(M^n, SO_N)$ is an abelian group and the group $\pi^+(M^n, M^n)$ acts without fixed points and is transitive on $\pi^+(M^n_\alpha, M^n)$.

**Lemma 6.3.** *The set $B(M^n_\alpha) \subset \bar{A}(M^n)$ splits into a union of disjoint sets*

$$B(M^n_\alpha) = \bigcup_f B_f(M^n_\alpha),$$

*where $f \in \pi^+(M^n_\alpha, M^n)$ and $B_f(M^n_\alpha)$ is the subset of the set $B(M^n_\alpha)$ that consists of those classes $\alpha \in \bar{A}(M^n)$ in which is found an admissible map*

$$f_\alpha\colon S^{N+n} \to T_N(M^n)$$

*such that $f^{-1}_\alpha(M^n) = M^n_\alpha$ and, when considered on $M^n_\alpha$, having the homotopy class of $f \in \pi^+(M^n_\alpha, M^n)$.*

*Proof.* It has already been established that the set $B_f(M^n_\alpha)$ is defined correctly, i.e., to the homotopic maps $M^n_\alpha \to M^n$ there correspond identical sets of homotopy classes. Let us prove that if two sets $B_{f_1}(M^n_\alpha)$ and $B_{f_2}(M^n_\alpha)$ intersect, then they coincide. By analogy with the proof of Lemma 6.2 we consider an element

$$\alpha_0 \in B_{f_1}(M^n_\alpha) \cap B_{f_2}(M^n_\alpha)$$

and, corresponding to it, two admissible maps

$$f_{\alpha_0,i} \colon S^{N+n} \to T_N(M^n)$$

such that $f_{\alpha_0,1}|M^n_\alpha \to M^n$ and $f_{\alpha_0,2}|M^n_\alpha \to M^n$ have the homotopy classes of $f_1, f_2$.

We consider their $t$-regular homotopy

$$F \colon S^{N+n} \times I(0,1) \to T_N(M^n)$$

and the membrane

$$N^{n+1} = F^{-1}(M^n) \subset S^{N+n} \times I(0,1),$$

retracting onto each of two of its boundaries. By analogy with Lemma 6.2, on the lower boundary we change the bundle map

$$\nu^N(M^n_\alpha) \to \nu^N(M^n),$$

keeping the map $f_{\alpha_0,1}|M^n_\alpha \to M^n$ fixed. We can extend this variation of a bundle map to a variation of the bundle map

$$\nu^N(N^{n+1}) \to \nu^N(M^n),$$

keeping it fixed on $N^{n+1}$, which can be done, starting from a retraction of the membrane onto the boundary $M^n_\alpha \subset S^{N+n} \times 0$. Then, by means of a well-known method, we extend the map varied in a tubular neighborhood onto all of the product $S^{N+n} \times I(0,1)$. According to Lemma 6.1, by such a change we can obtain from $\alpha_0$ any other element $\alpha_1 \in B_{f_1}(M^n_\alpha)$. Thus

$$B_{f_1}(M^n_\alpha) \supset B_{f_2}(M^n_\alpha).$$

By symmetry

$$B_{f_1}(M^n_\alpha) = B_{f_2}(M^n_\alpha).$$

The lemma is proved. □

**Lemma 6.4.** *The group $\pi(M^n_\alpha, SO_N)$ acts transitively on each set $B_f(M^n_\alpha)$.*

*Proof.* Suppose there exist two classes $\alpha_i \in B_f(M^n_\alpha)$, $i = 1, 2$, and, representing them, admissible maps

$$f_{\alpha_i} \colon S^{N+n} \to T_N(M^n)$$

such that

$$f^{-1}_{\alpha_i}(M^n) = M^n_\alpha, \qquad i = 1, 2,$$

and the maps $f_{\alpha_i}|M^n_\alpha \to M^n$ are homotopic. By means of the homotopy constructed in Lemma 6.1 we change the map $f_{\alpha_2}$ to an admissible map $f^{(1)}_{\alpha_2}$ that is homotopic to it and such that

$$f^{(1)}_{\alpha_2} = f_{\alpha_1}|M^n_\alpha.$$

Then the bundle maps $f^{(1)}_{\alpha_2}$ and $f_{\alpha_1} \colon \nu^N(M^n_\alpha) \to \nu^N(M^n)$ differ on each fiber $D^N_x$ over a point $x \in M^n_\alpha$ by a discriminating orthogonal transformation $h_x \in SO_N$ that is smoothly dependent on the point $x \in M^n_\alpha$. Consequently, there arises a smooth map

$$h\colon M^n_\alpha \to SO_N,$$

discriminating the maps $f^{(1)}_{\alpha_2}$ and $f_{\alpha_1}$ in a neighborhood $T(M^n_\alpha) \subset S^{N+n}$ of the manifold $M^n_\alpha$. According to Lemma 6.1, if the map $h\colon M^n_\alpha \to SO_N$ is homotopic to zero, then the elements $\alpha_1$ and $\alpha_2$ are equal to each other. Thus, the discriminator $h$ is defined to within homotopy and the map $f_{\alpha_1}$, "twisted" in each fiber $D^N_x$ over a point $x \in M^n_\alpha$ by a transformation $h_x \in SO_N$, coincides with $f^{(1)}_{\alpha_2}$. On the set of classes $B_f(M^n_\alpha)$ there acts the group $\pi(M^n_\alpha, SO_N)$, and it is transitive. The lemma is proved. □

The lemmas combine into the following

**Theorem 6.5.** *On the set*

$$\bar{A}(M^n) \subset A(M^n) = H^{-1}\phi[M^n] \subset \pi_{N+n}(T_N(M^n))$$

*there acts the group* $\pi^+(M^n, SO_N)$. *On the orbit set*

$$\bar{A}(M^n)/\pi(M^n, SO_N)$$

*there acts the group* $\pi^+(M^n, M^n)$. *The elements of the orbit set*

$$B = [\bar{A}(M^n)/\pi(M^n, SO^n)]/\pi^+(M^n, M^n)$$

*are found to be in a natural one-to-one correspondence with the classes of manifolds* $M^n_\alpha \geq M^n$, $M^n \geq M^n$, *with respect to a diffeomorphism of degree* $+1$ *modulo* $\theta^n(\partial\pi)$ *for $n$ odd and a diffeomorphism of degree* $+1$ *for $n$ even.*

*Proof.* According to Lemmas 6.3 and 6.4, to the manifold $M^n$ corresponds a set

$$B(M^n_\alpha) = \bigcup_{f \in \pi^+(M^n_\alpha, M^n)} B_f(M^n_\alpha),$$

and the group $\pi(M^n_\alpha, SO_N)$ acts transitively on each set $B_f(M^n_\alpha)$. But the groups $\pi(M^n_\alpha, SO_N)$ and $\pi(M^n, SO_N)$ are isomorphic, and if a homotopy class $f \in \pi^+(M^n_\alpha, M^n)$ is given, then there corresponds to it an isomorphism

$$f^*\colon \pi(M^n, SO_N) \to \pi(M^n_\alpha, SO_N).$$

Therefore on each set $B_f(M^n_\alpha)$ there naturally acts the group $\pi(M^n, SO_N)$; here

$$h(a) = f^*h(x), \qquad \alpha \in B_f(M^n_\alpha), \quad h \in \pi(M^n, SO_N).$$

On the other hand, on the set of classes $f \in \pi^+(M^n_\alpha, M^n)$ without fixed points there acts the group $\pi^+(M^n, M^n)$ (and transitively). Therefore on the factor set $B(M^n_\alpha)/\pi(M^n, SO_N)$ there acts the group $\pi^+(M^n, M^n)$, and transitively, i.e., the factor set

$$[B(M^n_\alpha)/\pi(M^n, SO_N)]/\pi^+(M^n, M^n)$$

consists of one element. Using the action of the groups $\pi(M^n, SO_N)$ and $\pi^+(M^n, M^n)$ on each of the sets of $B(M^n_\alpha)$ for all manifolds $M^n_\alpha$, where

$$M^n_\alpha \geq M^n, \qquad M^n \geq M^n_\alpha,$$

we obtain the action of these groups on all of the set $\bar{A}(M^n)$, and the factor set with respect to both of these groups is found to be in a natural one-to-one correspondence with the set of manifolds, that are identified with each other if at least once (and consequently always according to Lemma 6.2) they lie in one and the same class $\alpha \in \bar{A}(M^n)$. Applying Theorem 5.2 we obtain the desired statement.

The theorem is proved. □

For subsequent applications it is convenient to note the following

**Lemma 6.6.** *To the automorphism of the $SO_N$-bundle*

$$h\colon \nu^N(M^n) \to \nu^N(M^n),$$

*fixed on the base $M^n$, or, what is the same thing, to the map*

$$h\colon M^n \to SO_N,$$

*there corresponds the map*

$$Th\colon T_N(M^n) \to T_N(M^n);$$

*to the homotopic maps $h_i\colon M^n \to SO_N$, $i = 0, 1$, there correspond the homotopic maps $Th_i$, and in the process there are the homotopies $Th_t$, $0 \le t \le 1$, the manifold $M^n \subset T_N(M^n)$ is fixed, and the normal ball $D_x^N$, $x \in M^n \subset T_N(M^n)$, is deformed with the use of the maps $h_t(x) \in SO_N$, $0 \le t \le 1$. If $h \in \pi(M^n, SO_N)$ and $\alpha \in \pi_{N+n}(T_N(M^n))$, where $\alpha \in \bar{A}(M^n)$, then*

$$h(\alpha) = Th_*(\alpha),$$

*where $\pi(M^n, SO_N)$ acts on $\bar{A}(M^n)$ according to Theorem* 6.5.

*Proof.* The definition of the map

$$T\colon \pi(M^n, SO_N) \to \pi(T_N(M^n), T_N(M^n))$$

follows at once from the definition of the Thom space of the bundle $\nu^N(M^n)$.

Let us prove the formula

$$h(\alpha) = Th_*(\alpha).$$

We recall how we defined the action of the group $\pi(M^n, SO_N)$ on the set $\bar{A}^N(M^n)$: suppose $f_\alpha$ is an admissible map $S^{N+n} \to T_N(M^n)$, $f_\alpha^{-1}(M^n) = M_\alpha^n$ and $f_\alpha|M_\alpha^n$ has the homotopy class of $f \in \pi^+(M_\alpha^n, M^n)$. The action of the group $\pi(M_\alpha^n, SO_N)$ and the isomorphism

$$\tilde{f}\colon \pi(M_\alpha^n, SO_N) \to \pi(M^n, SO_N)$$

are defined in the natural way. Let $h \in \pi(M^n, SO_N)$ and $\tilde{f}^{-1}h \in \pi(M_\alpha^n, SO_N)$. Then to an element $h$ corresponds a "twisting" of the bundle $\nu^N(M^n)$ in each fiber $D_x^N$ on an element $h_x \in SO_N$, $x \in M^n$. To this twisting corresponds a twisting $f_x^*$ in the fiber $D_{f_\alpha^{-1}(x)}$ on the same element $h_x \in SO_N$ at each point $f_\alpha^{-1}(x)$. This defines the map

$$f_\alpha^* = \tilde{f}^{-1}\colon \pi(M^n, SO_N) \to \pi(M_\alpha^n, SO^N).$$

One can only define the action of the group $\pi(M_\alpha^n, SO_N)$ on the set $B_f(M_\alpha^n)$ in such a way that it has the form $f_\alpha^*(h)$, since the distinction between definitions is removed in going over to homotopy classes, due to the fact that $f_\alpha|M_\alpha^n$ is a homotopy equivalence and $\tilde{f}_\alpha = f_\alpha^{*-1}$ is an isomorphism.

The lemma is proved. □

**Lemma 6.7.** *To every map* $f\colon M^n \to M^n$ *of degree* $+1$ *and such that* $f^*\nu^N(M^n) = \nu^N(M^n)$ *corresponds a nonempty set of maps*

$$(\bar{T}f)\colon T_N(M^n) \to T_N(M^n).$$

*Two maps* $\bar{T}_1, \bar{T}_2 \in (\bar{T}f)$ *differ by an automorphism* $Th$ *for some* $h\colon M^n \to SO_N$.

*To the homotopic maps* $f_1, f_2\colon M^n \to M^n$ *correspond the homotopic* $\operatorname{mod} T(\pi(M^n, SO_N))$ *maps* $\bar{T}f_1$ *and* $\bar{T}f_2\colon T_N(M^n) \to T_N(M^n)$.

*To the product* $f_1 \circ f_2$ *corresponds the product*

$$\bar{T}f_1 \circ \bar{T}f_2 = \bar{T}f_1 \circ f_2 \bmod \operatorname{Im} T.$$

*Suppose* $f \in \pi^+(M^n, M^n)$ *and* $\alpha \in \bar{A}(M^n)/\pi(M^n, SO_N)$. *Then*

$$f(\alpha) = \bar{T}f_*(\alpha),$$

*where*

$$\bar{T}f_*\colon \pi_{N+n}(T_N(M^n)) \to \pi_{N+n}(T_N(M^n)).$$

The proof of this lemma is analogous to the proof of Lemma 6.6 and follows at once from the known definition of the action of the group $\pi^+(M^n, M^n)$ on $\pi^+(M^n_\alpha, M^n)$ and the dependence of the element $\alpha \in \bar{A}(M^n)/\pi(M^n, SO_N)$ on a map $M^n_\alpha \to M^n$ of degree $+1$ (an element of the set $\pi^+(M^n_\alpha, M^n)$) (cf. Lemmas 6.1, 6.3, Theorem 6.5 and their proofs).

We now consider the particular case when $M^n$ is a $\pi$-manifold. In this case the bundle $\nu^N(M^n)$ is trivial. We define a frame field $\tau^N_x$ that is smoothly dependent on the point $x \in M^n$ and normal to $M^n$ in $T_N(M^n)$. According to [15], we call the pair $(\tau^N, M^n)$ an "equipped manifold." Then, as is easily seen, for any element $\alpha \in \bar{A}(M^n)$ and any admissible map

$$f_\alpha\colon S^{N+n} \to T_N(M^n)$$

the manifold

$$M^n_\alpha = f^{-1}_\alpha(M^n)$$

receives the natural "equipment" $f^*_\alpha \tau^N$ and becomes an equipped manifold.

In this case we have the following

**Lemma 6.8.** *There exists a single-valued homomorphism*

$$\bar{T}_0\colon \pi^+(M^n, M^n) \to \pi(T_N(M^n), T_N(M^n))$$

*such that for any* $h \in \pi(M^n, SO_n)$, $f \in \pi^+(M^n, M^n)$ *one has the following*:

a) $Th \cdot \bar{T}_0 f = \bar{T}_0 \cdot Tf^*h$, *where* $f^*\colon \pi(M^n, SO_N) \to \pi(M^n, SO_N)$;

b) $\bar{T}_0 = \bar{T} \bmod \operatorname{Im} T$.

*Proof.* Let us construct the single-valued homomorphism $\bar{T}_0$; for this purpose we consider the automorphism

$$f\colon M^n \to M^n,$$

$f \in \pi^+(M^n, M^n)$, and we cover it in a single-valued manner with respect to a map

$$\nu^N(M^n) \to \nu^N(M^n),$$

assuming that the vector with coordinates

$$(\lambda_1, \dots, \lambda_N) \in D^N_x, \qquad x \in M^n,$$

defined by a frame of $\tau_x^N$ in the fiber normal to the point $x$, goes over into the vector with coordinates $(\lambda_1, \dots, \lambda_N)$ at the point $f(x)$. Since the field $\tau^N$ is smooth, we obtain a (smooth if $f$ is smooth) map

$$\nu^N(M^n) \to \nu^N(M^n),$$

which gives the desired map

$$\bar{T}_0 f \colon T_N(M^n) \to T_N(M^n).$$

We have proved point a) of the lemma.

We consider a map $h \colon M^n \to SO_N$ and the composition

$$h \cdot \bar{T}_0 f \colon \nu^N(M^n) \to \nu^N(M^n),$$

covering the map $f \colon M^n \to M^n$. The maps $h \cdot \bar{T}_0 f$ and $\bar{T}_0 f$ differ at each point $x \in M^n$ by $h_x \in SO_N$ and at each point $f^{-1}(x) \in M^n$ by

$$f^* h_{f^{-1}(x)} \in SO_N, \qquad h_x \in f^* h_{f^{-1}(x)}.$$

Thus

$$h \cdot \bar{T}_0 f = \bar{T}_0 f \cdot f^* h$$

($f^*$ is the automorphism $f^* \colon \pi(M^n, SO_N) \to \pi(M^n, SO_N)$ induced by $f$).

Further, we have

$$Th \cdot \bar{T}_0 f = \bar{T}_0 f \cdot T f^* h.$$

Formula b) is evident from the construction of the homomorphism $\bar{T}_0$.

The lemma is proved. □

We consider the set $\pi^+(M_\alpha^n, M^n)$ defined above. On it (from the left) acts the group $\pi^+(M^n, M^n)$ and (from the right) acts the group $\pi^+(M_\alpha^n, M_\alpha^n)$, where

$$M_\alpha^n \geq M^n, \qquad M^n \geq M_\alpha^n.$$

In other words, for every

$$f \in \pi^+(M^n, M^n), \qquad g \in \pi^+(M_\alpha^n, M^n), \qquad f_1 \in \pi^+(M_\alpha^n, M_\alpha^n)$$

there is defined the composition

$$f \cdot g \cdot f_1 \in \pi^+(M_\alpha^n, M^n).$$

And what is more, for every $f \in \pi^+(M^n, M^n)$, $g \in \pi^+(M_\alpha^n, M^n)$ we have the formula

$$f \cdot g = g \cdot (g^* f), \tag{28}$$

where $g^* \colon \pi^+(M^n, M^n) \to \pi^+(M_\alpha^n, M_\alpha^n)$ is an isomorphism defined by the element $g \in \pi^+(M_\alpha^n, M^n)$.

We introduce the following notation: by means of

$$D^+(M_\alpha^n) \subset \pi^+(M_\alpha^n, M_\alpha^n)$$

we denote the subgroup consisting of those homotopy classes of maps in which there is a representative

$$h \colon M_\alpha^n \to M_\alpha^n,$$

that is a diffeomorphism; by means of

$$\tilde{D}^+ \subset \pi^+(M_\alpha^n, M_\alpha^n)$$

we denote the analogous subgroup in which a certain representative

$$\tilde{h} \colon M_\alpha^n \to M_\alpha^n$$

is a diffeomorphism everywhere except a spherical neighborhood of one point, and the obstruction to an extension of the diffeomorphism at this point belongs to the group $\theta^n(\partial\pi)$. In view of the canonical isomorphism of Smale $\theta^n = \Gamma^n$, one can assume that $\theta^n(\partial\pi) \subset \Gamma^n$ for $n \geq 5$. By means of

$$\Lambda^n(M^n_\alpha) \subset \theta^n(\partial\pi)$$

we denote the subgroup such that for each element $\gamma \in \Lambda^n(M^n_\alpha)$ there exists a map

$$\tilde{h}_\gamma \colon M^n_\alpha \to M^n_\alpha,$$

homotopic to the identity map, that is a diffeomorphism everywhere except a spherical neighborhood of one point, and the obstruction to an extension of the diffeomorphism at this point is equal to $\gamma$.

**Theorem 6.9.** *The group $D^+(M^n_\alpha)$ is a normal divisor in the group $\tilde{D}^+(M^n_\alpha)$. The factor group $\tilde{D}^+(M^n_\alpha)/D^+(M^n_\alpha)$ is isomorphically embedded in the group $\theta^n(\partial\pi)/\Lambda^n(M^n_\alpha)$. If $n$ is even, then $D^+(M^n_\alpha) = \tilde{D}^+(M^n_\alpha)$; if $n$ is odd, then the factor group $\tilde{D}^+(M^n_\alpha)/D^+(M^n_\alpha)$ is a finite cyclic group.*

*Proof.* To a representative $\tilde{h} \colon M^n_\alpha \to M^n_\alpha$ of an element of $\tilde{D}^+$ we put in correspondence the obstruction defined by it to an extension of a diffeomorphism at a point. It is easy to see that the lack of uniqueness in the definition pertains to the group $\Lambda^n(M^n_\alpha)$, and the group $D^+(M^n_\alpha)$ goes into zero. In this way the embedding

$$\tilde{D}^+(M^n_\alpha)/D^+(M^n_\alpha) \subset \theta^n(\partial\pi)/\Lambda^n(M^n_\alpha)$$

is constructed. The rest of the assertion follows from the results in [6, 8] concerning the groups $\theta^n(\partial\pi)$. The theorem is proved. ☐

**Theorem 6.10.** *The element $g^*f \in \pi^+(M^n_\alpha, M^n_\alpha)$ belongs to the subgroup $\tilde{D}^+(M^n_\alpha)$ if and only if $\bar{T}f^*(\alpha) = \alpha$, where $\alpha \in \bar{A}(M^n)/\pi(M^n, SO_N)$.*

We note a certain consequence of Theorems 6.9 and 6.10. If $M^n_\alpha = M^n$, then $g^*f = gfg^{-1}$; therefore from Theorem 6.10 follows

**Corollary 6.11.** *The subgroup $\tilde{D}^+(M^n)$ is a normal divisor in the group $\pi^+(M^n, M^n)$; the factor group $\pi^+(M^n, M^n)/\tilde{D}^+(M^n)$ is finite* (though it is not known whether or not it is abelian).

**Corollary 6.12.** *The group $D^+(M^n)$ has a finite index in $\pi^+(M^n, M^n)$.*

*Proof of Theorem* 6.10. By definition the manifold $M^n_\alpha$ is obtained as follows: a map

$$f_\alpha \colon S^{N+n} \to T_N(M^n)$$

is selected which represents one of the elements $\tilde{\alpha}$ of the class $\alpha$; it is assumed to be admissible if $f_\alpha^{-1}(M^n) \geq M^n$ and $M^n \geq f_\alpha^{-1}(M^n)$, where $f_\alpha^{-1}$ is the inverse image of $M^n$ under a map satisfying Lemma 3.2. Then we set

$$M^n_\alpha = f_\alpha^{-1}(M^n).$$

Suppose $f_\alpha | M^n_\alpha \to M^n$ has the homotopy class $g \in \pi^+(M^n_\alpha, M^n)$, and let $f$ be an element of the group $\pi^+(M^n, M^n)$ such that

$$\bar{T}f_*(\tilde{\alpha}) \equiv \tilde{\alpha} \bmod \operatorname{Im} T.$$

Since all our objects are defined to within a diffeomorphism of degree +1, from the fact that $g^*f$ is homotopic to a diffeomorphism of degree +1 it clearly follows that the sets

$$B_g(M^n_\alpha) \subset \bar{A}(M^n)$$

and

$$B_{g\cdot g^* f}(M^n_\alpha) = B_{f\cdot g}(M^n_\alpha)$$

are identical, from which follows one of the assertions of the theorem. We now show that if

$$\bar{T} f_*(\alpha) = \alpha, \qquad \alpha \in \bar{A}(M^n)/\pi(M^n, SO_N),$$

then the map $g^*f$ is homotopic to a diffeomorphism (of degree +1). We divide the proof into a number of steps.

**Step 1.** We consider homotopic admissible maps $f^{(\prime)}_\alpha$ and $f^{(\prime\prime)}_\alpha \colon S^{N+n} \to T_N(M^n)$ such that

a) $f^{(\prime)\,-1}_\alpha(M^n) = f^{(\prime\prime)\,-1}_\alpha(M^n) = M^n_\alpha$,

b) $f^{(\prime)}_\alpha | M^n_\alpha = g$, $f^{(\prime\prime)}_\alpha | M^n_\alpha = g \cdot g^* f = f \cdot g$.

We construct a homotopy $F \colon S^{N+n} \times I(0,1) \to T_N(M^n)$ that is $t$-regular and such that $F | S^{N+n} \times 0 = f^{(\prime)}_\alpha$.

**Step 2.** We define the membrane $N^{n+1} = F^{-1}(M^n) \subset S^{N+n} \times I$; clearly,

$$F^* \nu^N(M^n) = \nu^N(N^{n+1})$$

and

$$\partial N^{n+1} = M^n_\alpha \cup (-M^n_\alpha).$$

By means of Morse's reconstructions we kill the groups

$$\pi_1(N^{n+1}), \operatorname{Ker} F^{(H_2)}_*, \dots, \operatorname{Ker} F^{(H_i)}_*, \qquad i < \left[\frac{n}{2}\right],$$

at the same time carrying over onto the "new membrane" $N^{n+1}$ the map $F$ and the "equipment" (in analogy with §§4, 5). Thus one can assume that

$$\pi_1(N^{n+1}) = 0$$

and

$$\operatorname{Ker} F^{(H_i)}_* = 0, \qquad i < \left[\frac{n}{2}\right]$$

**Step 3. Case 1.** If $n$ is odd, then, following §4, we reconstruct the group $\operatorname{Ker} F^{(H_{[n/2]})}_*$. We will thereupon (see §5, Case 1) have a membrane that is diffeomorphic to $M^n_\alpha \times I(0,1)$, according to Smale [19]. The theorem is proved.

**Case 2.** If $n+1$ is even ($n+1 = 4k+2$ or $n+1 = 4k$), then it is necessary to make use of the fact that the boundaries of the manifold $N^{n+1}$ are in this case diffeomorphic. Next, by analogy with Cases 2 and 3 of y5 it is necessary to construct the membranes $\bar{M}^{n+1}(B)$ and $\bar{M}^{n+1}(\phi)$ in order to kill the obstructions encountered to Morse's reconstructions, and then to consider the unions

$$\bar{N}^{n+1}(B) = N^{n+1} \cup_{f_0} D^n \times I(0,1) \cup_{f_1} \bar{M}^{n+1}(B),$$
$$\bar{N}^{n+1}(\phi) = N^{n+1} \cup_{f_0} D^n \times I(0,1) \cup_{f_1} \bar{M}^{n+1}(\phi),$$

as in §5, Cases 2 and 3 ($B$ is the intersection matrix of the membrane $N^{n+1}$ and $\phi$ is an invariant of Kervaire). The maps

$$F \colon N^{n+1} \to M^n$$

define, ia a natural way, the maps

$$F(B)\colon \bar{N}^{n+1}(B) \to M^n$$

and

$$F(\phi)\colon \bar{N}^{n+1}(\phi) \to M^n$$

such that

$$F(B)^*\nu^N(M^n) = \nu^N(N^{n+1}(B))$$

and

$$F(\phi)^*\nu^N(M^n) = \nu^N(\bar{N}^{n+1}(\phi)).$$

It is easy to see that

$$\partial\bar{N}^{n+1}(B) = [M^n_\alpha \,\#\, \tilde{S}^n(B)] \cup (-M^n_\alpha)$$

and

$$\partial\bar{N}^{n+1}(\phi) = [M^n_\alpha \,\#\, \tilde{S}^n(\phi)] \cup (-M^n_\alpha)$$

We reconstruct by means of Morse's reconstructions the manifolds $\bar{N}^{n+1}(B)$ and $\bar{N}^{n+1}(\phi)$; the resultant manifolds $\bar{\bar{N}}^{n+1}(B)$ and $\bar{\bar{N}}^{n+1}(\phi)$ will determine a $J$-equivalence (diffeomorphism) of degree $+1$ between the manifolds $M^n_\alpha$, and $M^n_\alpha \,\#\, \tilde{S}^n(B)$, $M^n_\alpha$ and $M^n_\alpha \,\#\, \tilde{S}^n(\phi)$, where $\tilde{S}^n(B), \tilde{S}^n(\phi) \in \theta^n(\partial\pi)$. The maps $F(B), F(\phi)$ reconstructed on the membranes $\bar{\bar{N}}^{n+1}(B)$ and $\bar{\bar{N}}^{n+1}(\phi)$ are denoted by $\bar{F}(B), \bar{F}(\phi)$. Also, $\bar{\bar{N}}(B)$ is diffeomorphic to $M^n_\alpha \times I$ ($n = 4k-1$), $\bar{\bar{N}}(\phi)$ is diffeomorphic to $M^n_\alpha \times I$ ($n = 4k+1$) and $\bar{F}(B) = F|M^n_\alpha \times 1$ ($n = 4k-1$), $\bar{F}(\phi) = F|M^n_\alpha \times I$ ($n = 4k+1$).

The map

$$\bar{F}(B)\colon M^n_\alpha \to M^n$$

is homotopic to the composition

$$F_1(B)g(B)\colon M^n_\alpha \xrightarrow{g(B)} M^n_\alpha \,\#\, \tilde{S}^n(B) \xrightarrow{F_1(B)} M^n, \qquad n = 4k-1,$$

and the map

$$F(\phi)\colon M^n_\alpha \times 0 \to M^n$$

is homotopic to the composition

$$M^n_\alpha \xrightarrow{g(\phi)} M^n_\alpha \,\#\, \tilde{S}^n(\phi) \xrightarrow{F_1(\phi)} M^n, \qquad n = 4k+1,$$

where $g(B)$ and $g(\phi)$ are diffeomorphisms of degree $+1$, induced by a decomposition into a direct product

$$\bar{\bar{N}}(B) = M^n_\alpha \times I, \qquad \bar{\bar{N}}(\phi) = M^n_\alpha \times I.$$

The maps $F_1(B)$ and $F_1(\phi)$ are respectively homotopic to the maps $F|M^n_\alpha \times 1$ ($n = 4k-1$ and $n = 4k+1$), from which follows the desired statement.[5] The theorem is proved. □

[5]It remains to add that the diffeomorphism $g(B)\colon M^n_\alpha \to M^n_\alpha \,\#\, \tilde{S}^n(B)$ must be thought of as a diffeomorphism modulo a point: $M^n_\alpha \to M^n_\alpha$. An analogous statement holds for $g(\phi)$.

# Chapter II
# A processing of results

## § 7. The Thom space of a normal bundle. Its homotopy structure

In order to understand and apply the results of §§1–6 we study the homotopy structure of the Thom complex $T_N(M^n)$, where $M^n$ is a simply connected manifold, $n \geq 4$.

We select in the manifold $M^n$ an $(n-2)$-dimensional skeleton $K^{n-2}$ such that

$$H_i(K^{n-2}) = H_i(M^n), \qquad i < n.$$

Then the manifold $M^n \setminus x_0$, $x_0 \in M^n$, is contracted onto $K^{n-2}$. The embedding $j\colon K^{n-2} \subset M^n$ induces the bundle $j^*\nu^N(M^n)$ on $K^{n-2}$, the Thom space of which we denote by $T_N^{n-2}$. There exists the natural embedding $T_N^{n-2} \subset T_N(M^n)$. In an analogous way one can select skeletons of smaller dimension

$$K^0 = K^1 \subset K^2 \subset \cdots \subset K^{n-2}$$

and form the Thom complexes

$$T_N^0 = S^N \subset T_N^2 \subset \cdots \subset T_N^{n-2}.$$

The complex $T_N^i$ can be computed from the $(N+i)$-dimensional skeleton of the complex $T_N(M^n)$, $i = 0, 2, \ldots, n-2$.

**Lemma 7.1.** *The Thom complex $T_N(M^n)$ is homotopically equivalent to the union $S^{N+n} \vee T_N^{n-2}$.*

*Proof.* Lemma 7.1 is an immediate consequence of Lemma 3.1 on the sphericity of the cycle

$$\phi[M^n] \in H_{N+n}(T_N(M^n)). \qquad \square$$

We consider the group $\pi_n(M^n)$ and in it we select the subgroup $\tilde{\pi}_n(M^n) \subset \pi_n(M^n)$ consisting of those elements $\gamma \in \tilde{\pi}_n(M^n)$ such that $H(\gamma) = 0$. In the group $\tilde{\pi}_n(M^n)$ we select the even smaller subgroup $\pi_n^\nu(M^n)$ consisting of those elements $\gamma \in \pi_n^\nu(M^n)$ such that, for any map $g_\gamma\colon S^n \to M^n$ representing the element $\gamma$, the bundle $g_\gamma^*\nu^N(M^n)$ over the sphere $S^n$ is trivial.

Now suppose $L^i$ is an arbitrary $i$-dimensional complex, over which a vector $SO_N$-bundle $\nu^N$ is given. We denote the Thom complex of this bundle by $T_N(\nu^N)$. Suppose $\gamma \in \pi_n(L^i)$, and the bundle $\gamma^*\nu^N$ over the sphere $S^n$ is trivial. We will then say that $\gamma \in \pi_n(L^i, \nu^N)$. For $L^i = M^n$ and $\nu^N = \nu^N(M^n)$ we have already defined such a group. Clearly, there is defined an epimorphism

$$\pi_n(K^{n-2}, \nu^N(M^n)) \to \pi_n^\nu(M^n).$$

There is defined the natural embedding $\kappa\colon S^n \subset T_N(\nu^N)$, corresponding to an embedding of the point $x_0 = L^0 \subset L^i$. We have

**Lemma 7.2.** *There is defined the natural homomorphism*

$$T^N\colon \pi_n(L^i, \nu^N) \to \pi_{n+N}(T_N(\nu^N))/\operatorname{Im}\kappa_*. \tag{29}$$

*If there exist two bundles $\nu_1^N, \nu_2^N$ over the complexes $L_1^{i_1}, L_2^{i_2}$ respectively and a map $F\colon \nu_1^N \to \nu_2^N$ is given, then there is defined a map*

$$T(F)\colon T_N(\nu_1^N) \to T_N(\nu_2^N)$$

*such that the diagram*

$$
\begin{array}{ccc}
\pi_n(L_1^i,\nu_1^N) & \xrightarrow{\tilde{F}_*} & \pi_n(L_1^{i_2},\nu_2^N) \\
\Big\downarrow{\scriptstyle T^N} & & \Big\downarrow{\scriptstyle T^N} \\
\pi_{n+N}(T_N(\nu_1^N))/\operatorname{Im}\kappa_* & \xrightarrow{T(F)_*} & \pi_{n+N}(T_N(\nu_2^N))/\operatorname{Im}\kappa_*
\end{array}
\tag{30}
$$

*is commutative.*

*Proof.* It is easy to see that to the bundle map $F$ corresponds a map

$$\tilde{F}_*\colon \pi_n(L_1^{i_1},\nu_1^N) \to \pi_n(L_2^{i_2},\nu_2^N).$$

Namely, let the map $F$ on the bases $L_1^{i_1} \to L_2^{i_2}$ be denoted by $\tilde{F}$. Then, clearly,

$$\tilde{F}^*(\pi_n(L_1^{i_1},\nu_1^N)) \subset \pi_n(L_2^{i_2},\nu_2^N)$$

by the definition of an induced bundle. In this way the upper line of the diagram is constructed. We will denote the constructed natural map

$$\pi_n(L_1^{i_1},\nu_1^N) \to \pi_n(L_2^{i_2},\nu_2^N)$$

by $\tilde{F}_*$. The construction of the lower line of the diagram is obvious. Let us now construct the homomorphism $T^N$. For this purpose we consider an element $\gamma_s \in \pi_n(L_s^{i_s},\nu_s^N)$, $s = 1, 2$, and the map

$$\tilde{\gamma}_s\colon S^n \to L_s^{i_s},$$

representing the element $\gamma_s$. The bundle $\tilde{\gamma}_s^*\nu_s^N$ over the sphere $S^n$ is trivial. Thus there are defined the maps

$$\nu\colon S^{N+n} \to T_N(S^n,\tilde{\gamma}_s\nu_s^N),$$
$$T\tilde{\gamma}_s\colon T_N(S^n,\tilde{\gamma}_s\nu_s^N) \to T_N(L_s^{i_s},\nu_s^N),$$

where $T\tilde{\gamma}_s$ is a natural map of the Thom complexes, corresponding to the bundle map $\tilde{\gamma}_s\nu_s^N \to \nu_s^N$, and the map $\mu$ is such that

$$\mu_*[S^{N+n}] = \phi[S^n],$$

where $\phi\colon H_n(S^n) \to H_{n+N}(T_N(S^n,\tilde{\gamma}_s^*\nu_s^N))$ is a Thom isomorphism. The cycle $\phi[S^n]$ is spherical according to Lemma 3.1, since a sphere is a $\pi$-manifold. According to Lemma 7.1 the space $T_N(S^n,\tilde{\gamma}_s^*\nu_s^N)$ is homotopically equivalent to the union $S^{N+n} \vee S^N$, so that the homotopy class of the map $\mu$ is defined uniquely mod $\pi_{n+N}(S^N) = \operatorname{Im}\kappa_*$. The composition

$$T\tilde{\gamma}_s \cdot \mu\colon S^{N+n} \to T_N(L_s^{i_s},\nu_s^N)$$

also defines an element for us, which we denote by $T^N(\gamma_s)$ and is determined uniquely mod $\operatorname{Im}\kappa_*$. Once a definition is given the naturalness of it (the commutativity of the diagram in Lemma 7.2) is obvious.

The lemma is proved. □

**Remark.** We will call the groups $\pi_n(L^i,\nu^N)$ the homotopy groups of the bundle $\nu^N$, and the homomorphism $T^N$ will be called a suspension homomorphism. This nomenclature is justified by the following lemma.

**Lemma 7.3.** *If the bundle $\nu^N$ over the complex $L^i$ is trivial, then*:

a) $\pi_n(L^i, \nu^N) = \pi_n(L^i)$ *for all* $n$;

b) *the space* $T_N(L^i, \nu^N)$ *is homotopically equivalent to the union* $S^N \vee E^N L^i$, *where* $E^N$ *is an* $N$*-multiple suspension*;

c) *the homomorphism* $T^N$ *coincides with the* $N$*-times-iterated suspension homomorphism*

$$E^N\colon \pi_n(L^i) \to \pi_{n+N}(E^N L^i) = \pi_{n+N}(T_N(L^i, \nu^N))/\operatorname{Im}\kappa_*$$

*for* $N > n + 1$.

*Proof.* The Thom space of a trivial bundle of closed balls $D^N$, $\nu^N = L^i \times D^N$, clearly, is homo topic ally equivalent to a suspension for $N > 1$:

$$T_N(L^i, \nu^N) = L^i \times D^N / L^i \times \partial D^N = ET_{N-1}(L^i, \nu^{N-1}) = E(L^i \times D^{N-1}/L^i \times D^{N-1}).$$

Further, for $N = 1$ we have

$$T_1(L^i, \nu^1) = L^i \times I(0,1)/L^i \times \partial I(0,1) = E(L^i \cup x_0),$$

where $L^i \cup x_0$ denotes the union of $L^i$ with the point $x_0$. Since the space $E(L^i \cup x_0)$ is homotopically equivalent to the union $S^1 \vee EL^i$ it follows that the space $T_N(L^i, \nu^N)$ is homotopically equivalent to a suspension

$$E^{N-1}(S^1 \vee EL^i) = S^N \vee E^N L^i.$$

The second part of the lemma follows trivially from the definition of a suspension homomorphism and is actually a definition of it. The lemma is proved. □

Suppose $M^n$ is a smooth simply connected oriented manifold, $\nu^N(M^n)$ is its normal bundle,

$$T_N(M^n) = T_N(M^n, \nu^N(M^n)),$$

$j\colon K^{n-2} \subset M^n$ is its $(n-2)$-dimensional skeleton and

$$\pi_n^\nu(M^n) = \pi_n(K^{n-2}, j^*\nu^N(M^n))/\operatorname{Ker} j_*.$$

According to Lemma 7.1,

$$\pi_{n+N}(T_N(M^n)) = Z + \pi_{n+N}(T_N^{n-2}). \tag{31}$$

The generator of the group $Z = \pi_{n+N}(S^{N+n})$ depends on the choice of decomposition in the union

$$T_N(M^n) = S^{N+n} \vee T_N^{n-2}.$$

We will select this decomposition in such a way that the generator of the direct summand $Z = \pi_{N+n}(S^{N+n})$ is the generator constructed in Lemma 3.1 (in the proof of it). We denote this generator by

$$1_{N+n} \in \pi_{N+n}(S^{N+n}) \subset T_N(M^n).$$

We have the following

**Lemma 7.4.** *For any element* $\gamma \in \pi_n^\nu(M^n)$ *there exists a map* $g_\gamma\colon M^n \to M^n$ *of degree* $+1$ *such that*:

a) $g_\gamma^*\nu^N(M^n) = \nu^N(M^n)$,

b) $g_\gamma$ *is fixed on the skeleton* $K^{n-2}$,

c) *the discriminator of the map* $g_\gamma$ *and the identity map is different from zero on only one simplex* $\sigma^n \subset M^n$, *and there it is equal to* $\gamma \in \pi_n^\nu(M^n)$.

*Proof.* We consider an identity map and change it on a simplex $\sigma^n \subset M^n$ to an element $\gamma \in \pi_n^\nu(M^n)$. We denote the resultant map by $g_\gamma$. Since the degree of the map $\tilde{\gamma}\colon S^n \to M^n$ representing $\gamma$ is equal to zero by definition of the group $\pi_n^\nu(M^n)$, the degree of the map $g_\gamma\colon M^n \to M^n$ is equal to $+1$. We consider the bundles $g_\gamma^* \nu^N(M^n)$ and $\nu^N(M^n)$, which we identify, as usual, with the homotopy classes of the map $\nu\colon M^n \to B_{SO_N}$ (for the bundle $\nu^N(M^n)$) and the map $\nu \cdot g_\gamma\colon M^n \to M_n \to B_{SO_N}$ (for the bundle $g_\gamma^*\nu^N(M^n)$). The discriminator of the maps $\nu$ and $\nu \cdot g_\gamma$ is concentrated on the same simplex $\sigma^n \subset M^n$ that the discriminator of the map $g_\gamma$ and the identity map is concentrated on, and it is equal, as is easily seen, to the element

$$\nu_*(\gamma) \in \pi_n(B_{SO_N}), \qquad \nu_*\colon \pi_n(M^n) \to \pi_n(B_{SO_N}).$$

The bundle $\gamma^*\nu^N(M^n)$ over the sphere $S^n$, by definition of the group $\pi_n^\nu(M^n)$, is trivial and is defined by the composition

$$\nu \cdot \tilde{\gamma}\colon S^n \to M^n \to B_{SO_N};$$

its triviality is equivalent to the condition

$$\nu_*(\gamma) = 0.$$

Therefore the discriminator of the maps $\nu\colon M^n \to B_{SO_N}$ and $\nu \cdot g_\gamma\colon M^n \to B_{SO_N}$ is equal to zero, and they are homotopic. The lemma is proved. □

From Lemma 7.4 follows

**Lemma 7.5.** *There is defined a homomorphism $g_*\colon \pi_n^\nu(M^n) \to \pi^+(M^n, M^n)$, the image of which is composed of all elements of the group $\pi^+(M^n, M^n)$ that have representatives fixed on the skeleton $K^{n-2} \subset M^n$.*

*Proof.* The map $g_*$ has already been constructed in Lemma 7.4; namely, to the element $\gamma \in \pi_n^\nu(M^n)$ must be put in correspondence the homotopy class of the map $g_\gamma\colon M^n \to M^n$. The fact that it is a homomorphism is obvious. We calculate the image

$$\operatorname{Im} g_* \subset \pi^+(M^n, M^n).$$

We consider any map $f\colon M^n \to M^n$ of degree $+1$ representing some element of the group $\pi^+(M^n, M^n)$ and fixed on the skeleton $K^{n-2}$.

The discriminator of it and the identity map is the cocycle

$$\lambda(f) \in H^n(M^n, \pi_n(M^n)),$$

where one can assume that the cochain $\lambda(f)$ is different from zero on only one simplex $\sigma^n \subset M^n$. Then

$$\lambda(f)[\sigma^n] \in \pi_n(M^n).$$

Since the map $f$ has degree $+1$, the degree of the map of the sphere $S^n \to M^n$ representing the element

$$\lambda(f)[\sigma^n] \in \pi_n(M^n).$$

is equal to zero. Since

$$f^*\nu^N(M^n) = \nu^N(M^n),$$

the discriminator of the maps

$$\nu\colon M^n \to B_{SO_N}$$

and

$$\nu \cdot f\colon M^n \to B_{SO_N},$$

defining the bundles $\nu^N(M^n)$ and $f^*\nu^N(M^n)$, is equal to

$$\nu_*\lambda(f)[\sigma^n] \in \pi_n(B_{SO_N}).$$

and

$$\nu_*\lambda(f)[\sigma^n] = 0.$$

siace $f^*\nu^N(M^n) = \nu^N(M^n)$. Therefore

$$\lambda(f)[\sigma^n] \in \pi_n^\nu(M^n).$$

The lemma is proved. □

We recall that in §6 we defined a map

$$\bar{T}\colon \pi^+(M^n, M^n) \to \pi(T_N(M^n), T_N(M^n)),$$

homomorphic and single-valued modulo the action of the group $\pi(M^n, SO_N)$, i.e., modulo the image of the homomorphism

$$T\colon \pi(M^n, SO_N) \to \pi(T_N(M^n), T_N(M^n)).$$

**Lemma 7.6.** *The formula*

$$\bar{T}g_*\gamma(1_{N+n} + \alpha) \equiv 1_{N+n} + \alpha + T^N\gamma \pmod{\operatorname{Im} T \cup \operatorname{Im} \kappa_*} \tag{32}$$

*is valid for all* $\gamma \in \pi_n^\nu(M^n)$*, where* $1_{N+n}$ *is the generator selected above and* $\alpha$ *is an element of the group* $\pi_{N+n}(T_N^{n-2}) \subset \pi_{N+n}(T_N(M^n))$.

*Proof.* The map $g_*\gamma$ is fixed on $K^{n-2}$, and hence $\bar{T}g_*\gamma$ can be selected so that it is fixed on $T_N^{n-2} \subset T_N(M^n)$. Consequently, the map

$$\bar{T}g_*\gamma\colon T_N(M^n) \to T_N(M^n)$$

is completely defined by the map

$$\bar{T}g_*\gamma | S^{N+n} \to T_N(M^n)$$

and

$$[\bar{T}g_*\gamma]_*(\alpha) = \alpha$$

for all

$$\alpha \in \pi_{N+n}(T_N^{n-2}) \subset \pi_{N+n}(T_N(M^n)).$$

Let us investigate the image $[\bar{T}g_*\gamma]_*(1_{N+n})$. The discriminator of the maps $g_\gamma$ and $1\colon M^n \to M^n$ is concentrated on the simplex $\sigma^n \subset M^n$ and is equal to $\gamma$, the complex $M^n \setminus \sigma^n$ contracts onto $K^{n-2}$. Therefore the discriminator of the map

$$\bar{T}g_\gamma\colon T_N(M^n) \to T_N(M^n)$$

and the identity map

$$1\colon T_N(M^n) \to T_N(M^n)$$

can initially be regarded as maps of the Thom complex $T_N(S^n, \nu^N)$ ($\nu^N$ is a trivial bundle) into the Thom complex $T_N(M^n)$, where on $S^N \subset T_N(S^n, \nu^n)$ the maps are homotopic (equal). Therefore the discriminator of the maps $\bar{T}g_*\gamma$ and 1 is $T^N\gamma$ by definition of the homomorphisin $T^N$. The lack of uniqueness in the formula of Lemma 7.6 arises in consequence of the lack of uniqueness in the definition of the homomorphisms $T^N$ and $\bar{T}$. The lemma is proved. □

**Remark 7.7.** For $\pi$-manifolds the definition of the homomorphism $\bar{T}^N$ coincides with that of $E^N$ and is therefore unique; the homomorphism $\bar{T}$ in this case also admits a unique definition, according to Lemma 6.8, and the formula of Lemma 7.6 has the meaning of an exact equality instead of a congruence.

We will not prove the assertion made in the remark since we will not make use of it.

## § 8. Obstructions to a diffeomorphism of manifolds having the same homotopy type and a stable normal bundle

Let us consider a filtration

$$T_N(M^n) \supset T_N^{n-2} \supset \cdots \supset T_N^2 \supset S^N,$$

where $T_N^i$ is the Thom space of the $i$-dimensional skeleton $K^i$ of a manifold $M^n$ in minimal cell decomposition (the number of cells of dimension $i$ is equal to the number $\max \operatorname{rk} H^i(M^n, K)$ with respect to all fields $K$). We denote the numbers $\max \operatorname{rk} H^i(M^n, K)$ by $b^i_{\max}$. By $T_N^{(i,j)}$ we mean

$$T_N^{(i,j)} = T_N^i / T_N^j, \qquad j < i.$$

In particular,

$$T_N^{(i,i-1)} = \bigvee_{k=1}^{b^i_{\max}} S_k^{N+i}.$$

Clearly,

$$H_{N+i}(T_N^i, T_N^{i-1}) = H_{N+i}\left(\bigvee_{k=1}^{b^i_{\max}} S_k^{N+i}\right) = \underbrace{Z + \cdots + Z}_{b^i_{\max}}$$

The homomorphisms

$$\partial \colon H_{N+i}(T_N^i, T_N^{i-1}) \to H_{N+i-1}(T_N^{i-1}) \to H_{N+i-1}(T_N^{i-1}, T_N^{i-2})$$

define a boundary operator in the complex $T_N(M^n)$ and its homologies and cohomologies. We will have in mind exactly this interpretation of boundary homomorphisms.

**Definition of the obstruction to a diffeomorphism.** We will identify modulo $\theta^n(\partial\pi)$ the manifolds $M^n_\alpha \geq M^n$, $M^n \geq M^n_\alpha$ with the orbits of the groups $\pi(M^n, SO_N)$ and $\pi^+(M^n, M^n)$ in the set $\bar{A}(M^n)$, according to the results of §§1–6. To the manifold $M^n_\alpha$ corresponds an orbit $B(M^n_\alpha) \subset \bar{A}(M^n)$. Suppose we are given two manifolds $M^n_\alpha$ and $M^n_\beta$, $\alpha \in B(M^n_\alpha)$, $\beta \in B(M^n_\beta)$.

According to Lemma 7.1 the elements $\alpha, \beta$ have the form

$$\alpha = 1_{N+n} + \bar{\alpha}, \qquad \bar{\alpha} \in \pi_{N+n}(T_N^{n-2}),$$
$$\beta = 1_{N+n} + \bar{\beta}, \qquad \bar{\beta} \in \pi_{N+n}(T_N^{n-2}).$$

Exact sequences (for the pairs $T_N^i, T_N^j$, $j < i$)

$$(33) \qquad \cdots \to \pi_{N+n}(T_N^j) \to \pi_{N+n}(T_N^i) \to \pi_{N+n}(T_N^{i,j}) \xrightarrow{\partial} \pi_{N+n-1}(T_N^j) \to \cdots$$

are defined.

In particular, we have

$$(34)\qquad \cdots \to \pi_{N+n}(S^N) \xrightarrow{j_{0,2}} \pi_{N+n}(T_N^2) \xrightarrow{\Lambda_2} \pi_{N+n}\left(\bigvee_{k_2=1}^{b^2_{\max}} S_{k_2}^{N+2}\right) \to \cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots \to \pi_{N+n}(T_N^i) \xrightarrow{j_{i,i+1}} \pi_{N+n}(T_N^{i+1}) \xrightarrow{\Lambda_{i+1}} \pi_{N+n}\left(\bigvee_{k_{i+1}=1}^{b^{i+1}_{\max}} S_{k_{i+1}}^{N+i+1}\right) \to \cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots \to \pi_{N+n}(T_N^{n-3}) \xrightarrow{j_{n-3,n-2}} \pi_{N+n}(T_N^{n-2}) \xrightarrow{\Lambda_{n-2}} \pi_{N+n}\left(\bigvee_{k_{n-2}=1}^{b^{n-2}_{\max}} S_{k_{n-2}}^{N+n-2}\right) \to \cdots$$

We consider the difference $\bar{\alpha} - \bar{\beta} \in \pi_{N+n}(T_N^{n-2})$. We have

$$\Lambda_{n-2}(\bar{\alpha} - \bar{\beta}) \in \sum_{k_{n-2}}^{b^{n-2}_{\max}} \pi_{N+n}(S_{k_{n-2}}^{N+n-2}).$$

Thus to every sphere $S_{k_{n-2}}^{N+n-2}$ corresponds an element $d_{n-2}(\bar{\alpha}, \bar{\beta}, k_{n-2}) \in \pi_{N+n}(S_{k_{n-2}}^{N+n-2})$ (corresponding to the number $k_{n-2}$ of a direct summand of the element $\Lambda_{n-2}(\bar{\alpha} - \bar{\beta})$). The spheres $S_{k_{n-2}}^{N+n-2}$ are found in a natural one-to-one correspondence with the cells of dimension $N + n - 2$ of the complex $T_N(M^n)$ and, consequently, with the cells of dimension $n-2$ of the complex $M^n$. Therefore (under variation of $k_{n-2}$) $d_{n-2}(\bar{a}, \bar{\beta}, k_{n-2})$ runs along the chain $d_{n-2}(\bar{\alpha}, \bar{\beta})$ of the complex $T_N(M^n)$ with value in $\pi_{N+n}(S^{N+n-2})$. If the chain $d_{n-2}(\bar{\alpha}, \bar{\beta}) = 0$, then we put

$$d_{n-3}(\bar{\alpha}, \bar{\beta}, k_{n-3}) = \Lambda_{n-3} \cdot j^{-1}_{n-3,n-2}(\bar{\alpha} - \bar{\beta})$$

(on the the sphere $S_{k_{n-3}}^{N_n-3}$); if $d_{n-(i-1)}(\bar{\alpha}, \bar{\beta}) = 0$, then we define

$$d_{n-i}(\bar{\alpha}, \bar{\beta}) = \Lambda_{n-i} \cdot j^{-1}_{n-i,n-(i-1)} \cdot \cdots \cdot j^{-1}_{n-3,n-2}(\bar{\alpha} - \bar{\beta})$$

(on the sphere $S_{k_{n-i}}^{N+n-i}$ the chain $d_{n-i}(\bar{\alpha}, \bar{\beta})$ has a value equal to the corresponding direct summand of the element $\Lambda_{n-i} \cdot j^{-1}_{n-i,n-(i-1)} \cdot \cdots \cdot j^{-1}_{n-3,n-2}(\bar{\alpha} - \bar{\beta})$).

Clearly, the chain $d_{n-1}(\bar{\alpha}, \bar{\beta})$ is ambiguously defined with exactness up to

$$\Lambda_{n-i} \cdot \operatorname{Ker}(j_{n-3,n-2} \cdot \cdots \cdot j_{n-i,n-(i-1)}) = Q_{n-i}.$$

**Lemma 8.1.** *The chain $d_{n-i}(\bar{\alpha}, \bar{\beta})$ is defined if $d_{n-j}(\bar{\alpha}, \bar{\beta}) = 0$, $j < i$, and is a cycle with coefficients in the group $\pi_{N+n}(S^{N+n-i})$.*

*Proof.* Let us prove that $d_{n-i}(\bar{\alpha}, \bar{\beta})$ is a cycle. According to the definition of a boundary operator in our complex $T_N(M^n)$ in the selected cell decomposition (cf. above) it suffices to consider some element

$$j^{-1}_{n-i,n-(i-1)} \cdot \cdots \cdot j^{-1}_{n-3,n-2}(\bar{\alpha} - \bar{\beta}) \in \pi_{N+n}(T_N^{n-i})$$

and the boundary homomorphism

$$\partial\colon H_{N+n-i}(T_N^{n-i,n-i-1}) \to H_{N+n-i-1}(T_N^{n-i-1,n-i-2}).$$

We consider the homomorphisms

$$\begin{array}{ccccc} \bar\partial\colon \pi_{N+n}(T_N^{n-i,n-i-1}) & \xrightarrow{\partial} & \pi_{N+n-1}(T_N^{n-i-1}) & \longrightarrow & \pi_{N+n-1}(T_N^{n-i-1,n-i-2}) \\ \big\downarrow \approx & & & & \big\downarrow \approx \\ \sum\limits_{k_{n-i}} \pi_{N+n}(S^{N+n-i}_{k_{n-i}}) & & \xrightarrow{\qquad\bar\partial\qquad} & & \sum\limits_{k_{n-i-1}} \pi_{N+n-1}(S^{N+n-i-1}_{k_{n-i-1}}). \end{array}$$

Then we consider the chain $d_{n-i}(\bar\alpha,\bar\beta)$. Since

$$d_{n-i}(\bar\alpha,\bar\beta)=\Lambda_{n-i}\cdot j^{-1}_{n-i,n-(i-1)}\cdot\dots\cdot j^{-1}_{n-3,n-2}(\bar\alpha-\bar\beta)$$

and from the exact sequences on page 45 it follows that $\operatorname{Im}\Lambda_{n-i}\subset\operatorname{Ker}\bar\partial$, and hence

$$\bar\partial d_{n-i}(\bar\alpha,\bar\beta)=0.$$

The lemma is proved. □

In this way,

$$d_{n-i}(\bar\alpha,\bar\beta)\in H_{N+n-i}(T_N(M^n),\pi_{N+n}(S^{N+n-i})),$$

or, by the Thom isomorphism $\phi$, we obtain the element

$$\tilde d_{n-i}(\bar\alpha,\bar\beta)=\phi^{-1}d_{n-i}(\bar\alpha,\bar\beta)\in H_{N-i}(M^n,\pi_{N+n}(S^{N+n-i})),$$

defined with a large degree of ambiguity.

**Definition of the minimal difference.** We commence to arbitrarily vary the elements $\alpha\in B(M^n_\alpha)$ and $\beta\in B(M^n_\beta)$ in the sets $B(M^n_\alpha)$ and $B(M^n_\beta)$ corresponding to the manifolds $M^n_\alpha$ and $M^n_\beta$ in such a way that the difference

$$\bar\alpha-\bar\beta\in\pi_{N+n}(T_N^{n-2})$$

belongs to

$$\operatorname{Im} j_{n-3,n-2}\cdot\dots\cdot j_{n-i,n-(i-1)}(\pi_{N+n}(T_N^{n-i}))$$

for

$$i=\max_{\alpha,\beta} i[\alpha\in B(M^n_\alpha),\beta\in B(M^n_\beta)]$$

and only then we define the ("minimal") discriminator

$$d_{n-i}(M^n_\alpha,M^n_\beta)=d_{n-i}(\bar\alpha_0,\bar\beta_0),$$

where $\alpha_0\in B(M^n_\alpha)$ and $\beta_0\in B(M^n_\beta)$ are elements such that the difference $\bar\alpha_0-\bar\beta_0$ belongs to

$$\operatorname{Im} j_{n-3,n-2}\cdot\dots\cdot j_{n-i,n-i+1}$$

for $i$ largest possible. It is evident that:

1) the homology class of $d_{n-i}(M^n_\alpha,M^n_\beta)$ is defined ambiguously;

2) its degree of ambiguity has two causes:

a) generally speaking, the nontriviality of the group

$$\operatorname{Ker}(j_{n-3,n-2}\cdot\dots\cdot j_{n-i,n-i+1})$$

and

b) the ambiguity in choice of the elements $\alpha_0,\beta_0$ in the orbits $B(M^n_\alpha)$ and $B(M^n_\beta)$.

We will explain the situation more precisely in the appendices at the end of the paper by analyzing examples.

## § 9. Variation of a smooth structure under preservation of the triangulation

We begin by recalling the results of Milnor, Smale, Kervaire (cf. [4, 6, 8, 9, 10, 17, 18]). Milnor [8] defined a group of smooth structures on a sphere of dimension $n$, denoted by $\theta^n$, and introduced in it the filtration

$$\theta^n \supset \theta^n(\pi) \supset (\partial\pi).$$

An element of the group $\theta^n$ is a smooth oriented manifold having the homotopy type of a sphere. It has been shown that:

1) $\theta^n/\theta^n(\pi) = 0$, $n \neq 8k+1, 8k+2$, $k \geq 2$,[6] $\theta^n/\theta^n(\pi) = Z_2$ or 0 for $n = 8k+1, 8k+2$, $k \geq 2$;

2) there is defined an inclusion homomorphism

$$\theta^n(\pi)/\theta^n(\partial\pi) \subset \pi_{N+n}(S^N)/J\pi_n(SO_N),$$

which is an epimorphism for $n \neq 4k+2$ and for $n = 10$;

3) for $n = 4k+2$ the subgroup $\theta^n(\pi)/\theta^n(\partial\pi)$ has index 2 or 1 in the group $\pi_{N+n}(S^N)/J\pi_n(SO_N)$, and for $n = 2, 6, 14$ it has index 2;

4) the group $\theta^n(\partial\pi)$ is trivial for $n$ even and for $n \leq 6$ $(n \neq 3)$, $n = 13$; the group $\theta^{2k+1}(\partial\pi)$ is always cyclic; for $k$ even it contains not more than two elements and $\theta^9(\partial\pi) = Z_2$, while for $k$ odd its order rapidly increases, and it is nontrivial for $k = 2s-1$, $s \geq 2$ $(\theta^7(\partial\pi) = Z_{28}, \theta^{11}(\partial\pi) = Z_{992}, \dots)$.

As already stated above, an element of the group $\theta^n$, $n \geq 5$, is a smooth orientated manifold having the homotopy type of a sphere $S^n$; the inverse element is the same manifold with opposite orientation, and the group operation is the "connected sum" of oriented manifolds (cf. [10]), which has a meaning, generally speaking, for arbitrary manifolds (but the connected sum of topological spheres is a topological sphere). We will denote the elements of the group $\theta^n$ by $\tilde{S}^n_i$, emphasizing in this way their topological structure. Our first goal is a study of the connected sum $M^n \# \tilde{S}^n$, where $M^n$ is an arbitrary simply connected manifold, $n \geq 5$. Clearly, for $n \geq 5$ the manifolds $M^n$ and $M^n \# \tilde{S}^n$ are homeomorphic and even combinatorially equivalent (cf. [17]), though possibly nondiffeomorphic if the smoothness on the sphere $\tilde{S}^n$ is nonstandard (if $\tilde{S}^n \neq 0$ in the group $\theta^n$).

Below we will denote the stable group $\pi_{N+n}(S^N)$ by $G(n)$ for $N > n+1$. By an embedding of Milnor,

$$\theta^n(\pi)/\theta^n(\partial\pi) \subset \pi_{N+n}(S^N)/\operatorname{Im} J,$$

to every element $\tilde{S}^n \in \theta^n(\pi)$ corresponds a set $\tilde{B}(\tilde{S}^n) \subset G(n)$, where

$$\tilde{B}(\tilde{S}^n_1 \# \tilde{S}^n_2) = \tilde{B}(\tilde{S}^n_1) + \tilde{B}(\tilde{S}^n_2)$$

and

$$\tilde{B}(\tilde{S}^n) = \operatorname{Im} J$$

if $\tilde{S}^n \in \theta^n(\partial\pi)$. We recall that, in the preceding sections, to every manifold $M^n_1 \geq M^n$, $M^n \geq M^n_1$, there is put in correspondence in a canonical manner the sets

$$B(M^n_1) \subset \bar{A}(M^n) \subset A(M^n) \subset \pi_{N+n}(T_N(M^n)).$$

In addition there is defined the natural embedding

$$\kappa \colon S^N \subset T_N(M^n),$$

---

[6] Adams [36] showed that $\theta^n/\theta^n(\pi) = 0$ for all $n$.

where $S^N = T_N^0$ (cf. §6).

Thus there arises the homomorphism

$$\kappa^*\colon G(n) \to \pi_{N+n}(T_N(M^n)).$$

We have the following

**Lemma 9.1.** $B(M_1^n \# \tilde{S}^n) = B(M_1^n) + \kappa_* \tilde{B}(\tilde{S}^n)$.

*Proof.* Let us show that

$$B(M_1^n \# \tilde{S}^n) \supset B(M_1^n) + \kappa_* \tilde{B}(\tilde{S}^n).$$

Suppose $\alpha \in B(M_1^n)$, $\gamma \in \tilde{B}(\tilde{S}^n)$ and

$$f_\alpha\colon S^{N+n} \to T_N(M^n), \qquad f_\gamma\colon S^{N+n} \to S^N$$

are some maps representing the elements $\alpha$ and $\gamma$, that are $t$-regular in the Thom–Pontrjagin sense, where

$$f_\alpha^{-1}(M^n) = M_1^n$$

and

$$f_\gamma^{-1}(x_0) = \tilde{S}^n, \qquad x_0 \in S^N.$$

We assume that the sphere $S^N$ lies in the Thom complex $T_N(M^n)$ in the standard manner and that

$$f_\gamma\colon S^{N+n} \to T_N(M^n), \quad f_\gamma(S^{N+n}) \in \kappa S^N, \quad f_\gamma^{-1}(M^n) = f_\gamma^{-1}(x_0).$$

Then there is defined a "connected sum of maps" (cf. [15, 8, 10])

$$f_{\alpha+\gamma}\colon S^{N+n} \to T_N(M^n)$$

such that

$$f_{\alpha+\gamma}^{-1}(M^n) = M_1^n \# \tilde{S}^n$$

and the map $f_{\alpha+\gamma}$ by definition represents the element $\alpha + \kappa_*\gamma$. Let us show that

$$B(M_1^n \# \tilde{S}^n) \subset B(M_1^n) + \kappa_* \tilde{B}(\tilde{S}^n).$$

Suppose $\beta \in B(M_1^n \# \tilde{S}^n)$ and the map

$$f_\beta\colon S^{N+n} \to T_N(M^n)$$

represents the element $\beta$, satisfies Lemma 3.2 and is such that

$$f_\beta^{-1}(M^n) = M_1^n \# \tilde{S}^n \subset S^{N+n}.$$

By definition of the connected sum #, there exists in the manifold $M_1^n$ a sphere $S_0^{n-1} \subset M_1^n \# \tilde{S}^n$ such that

$$(M_1^n \# \tilde{S}^n) \setminus S_0^{n-1} = (M_1^n \setminus D_\epsilon^n) \cup (S^n \setminus D_\epsilon^n)$$

where $D_\epsilon^n \subset M^n$ and $D_\epsilon^n \subset \tilde{S}^n$ are balls of radius $\epsilon$, given in some local coordinate system by a canonical equation, and $\epsilon > 0$ is a small number. Since $\tilde{S}^n$ is a $\pi$-manifold ($\tilde{S}^n \subset \theta^n(\pi)$) it follows that every frame field $\tau^N$, that is normal to $\tilde{S}^n \subset S^{N+n}$ and defined everywhere except $D_\epsilon^n \subset S^{N+n}$, can be extended onto the ball $D_\epsilon^n$. We deform the smooth map $f_\beta$ to a map

$$\tilde{f}_\beta\colon S^{N+n} \to T_N(M^n)$$

such that

$$\tilde{f}_\beta^{-1}(x_0) \supset \tilde{S}^n \setminus D_\epsilon^n \supset M_1^n \# \tilde{S}^n, \qquad x_0 \in M^n \supset T_N(M^n)$$

(the map $\tilde{f}_\beta$ is assumed to be $t$-regular). We consider a frame $\tau^N_{x_0}$ that is normal to the manifold $M^n \supset T_N(M^n)$ at the point $x_0$. The inverse image of the frame under a $t$-regular map $\tilde{f}_\beta$ (cf. [15, 22]) provides a frame field

$$\tau^N = \tilde{f}^{-1}_\beta(\tau^N_{x_0})$$

that is normal to $\tilde{S}^N \setminus D^n_\epsilon$ in $S^{N+n}$. We now “cut” the manifold $M^n_1 \# \tilde{S}^n$ with respect to a sphere $S^{n-1}_0$ into two parts and extend the frame field $\tau^N$ of the sphere

$$S^{n-1}_0 = (\tilde{S}^n \setminus D^n_\epsilon) \cap (M^n_1 \setminus D^n_\epsilon)$$

onto the ball $D^n_\epsilon$. More rigorously, we consider the membrane

$$B^{n+1}(h) = (M^n_1 \# \tilde{S}^n) \times I\left(0, \frac{1}{2}\right) \cup_h D^N_\epsilon \times D^1,$$

where

$$h\colon \partial D^n_\epsilon \times D^1 \to S^{n-1}_0 \times D^1 \subset M^n_1 \# \tilde{S}^n,$$
$$h(x, y) = (x, y).$$

Clearly,

$$\partial B^{n+1}(h) = (M^n_1 \# \tilde{S}^n) \cup (-M^n_1 \cup -\tilde{S}^n).$$

Further, as in §1, we embed in the usual way the membrane $B^{n+1}(h)$ in the direct product $S^{N+n} \times I(0,1)$, where

$$B^{n+1}(h) \cap S^{N+n} \times 0 = M^n_1 \# \tilde{S}^n,$$

and extend the map $f_\beta | S^{N+n} \times 0$ up to the map

$$F\colon S^{N+n} \times I \to T_N(M^n),$$

where

$$F^{-1}(M^n) = B^{n+1}(h),$$

making use of the possibility of extending the field $\tau^N$ of the sphere $S^{n-1}_0 \subset S^{N+n} \times 0$ onto the ball $D^n_\epsilon \subset S^{N+n} \times I(0,1)$. This extension can obviously be selected so that

$$F^{-1}(M^n) \cap S^{N+n} \times 1 = \tilde{S}^n \cup M^n_1, \qquad \tilde{S}^n \subset F^{-1}(x_0).$$

Since

$$F^{-1}(M^n) \cap S^{N+n} \times 1 = \tilde{S}^n \cup M^n_1,$$

it follows that the map $F|S^{N+n} \times 1$ is decomposed into a sum of maps $f^{(\prime)}_\beta$ and $f^{(\prime\prime)}_\beta$, representing respectively elements of type $\beta_1 \in B(M^n_1)$ and $\beta_2 \in \kappa_* \tilde{B}(\tilde{S}^n)$.

Thus it is established that

$$B(M^n_1 \# \tilde{S}^n) \supset B(M^n_1) + \kappa_* \tilde{B}(\tilde{S}^n),$$
$$B(M^n_1 \# \tilde{S}^n) \subset B(M^n_1) + \kappa_* \tilde{B}(\tilde{S}^n),$$

The lemma is proved. □

We now investigate a more complicated operation for the variation of a smooth structure. Suppose the manifold $M^n$ is $(k-1)$-connected, where $k \leq [n/2]$. Clearly,

$$H_k(M^n) = \pi_k(M^n).$$

We consider an element $z \in H_k(M^n)$ and a smooth sphere $S^k \subset M^n$ realizing it. The tubular neighborhood $T(S^k) \subset M^n$ of the sphere represents the $SO_{n-k}$-bundle of the balls $D^{n-k}$ over the sphere $S^k$. We assume that this bundle is trivial. We consider a map

$$g\colon S^k \to \operatorname{diff} S^{n-k-1},$$

taking the entire sphere $S^k$ into a point $g(S^k) \in \operatorname{diff} S^{n-k-1}$ (we note that according to [23, 17, 8] there exists a natural isomorphism $\operatorname{diff} S^{n-k-1}/j \operatorname{diff} D^{n-k} \approx \theta^{n-k}$, $n-k \neq 3, 4$). Therefore to the map $g$ corresponds a smooth sphere $\tilde{S}^{n-k}(g) \in \theta^{n-k}$. We will only consider those maps

$$g\colon S^k \to \operatorname{diff} S^{n-k-1}$$

for which $\tilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$.

We consider the automorphism [7]

$$\tilde{g}\colon \partial T(S^k) \to \partial T(S^k)$$

induced by the map

$$g(S^k)\colon S^{n-k-1} \to S^{n-k-1}.$$

Namely, in each fiber of the bundle of $(n-k-1)$-dimensional spheres $\partial T(S^k)$ over $S^k$ we give the automorphism $g(S^k)$. We put

$$M^n(S^k, g) = (M^n \setminus T(S^k)) \cup_{\tilde{g}} T(S^k).$$

From the paper [17] and from the fact that $\tilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$ follows

**Lemma 9.2.** *The manifolds $M^n$ and $M^n(S^k, g)$ are combinatorially equivalent. The combinatorial equivalence*

$$f(g)\colon M^n(S^k, g) \to M^n$$

*can he selected so that*:

a) $f(g)^*(M^n) = \nu(M^n(S^k, g))$,

b) $f(g)|M^n(S^k, g) \setminus T(S^k)$ *is the identity map,*

c) $f(g)|S^k$ *is the identity map,*

d) $f(g)|T(S^k) \subset M^n(S^k, g)$ *fiberwise.*

*Proof.* The diffeomorphism $g(S^k)\colon \partial D^{n-k} \to \partial D^{n-k}$ is extended up to a combinatorial equivalence $G\colon D^{n-k} \to D^{n-k}$, which is a diffeomorphism everywhere except the origin $O \in D^{n-k}$. We define a map

$$f(g)\colon M^n(S^k, g) \to M^n$$

in the following way:

$$f(g) = 1 \text{ on } M^n(S^k, g) \setminus T(S^k) = M^n \setminus T(S^k),$$
$$f(g) = 1 \text{ on } S^k \subset M^n(S^k, g),$$

$f(g) = G$ on the fiber $D_x^{n-k}$ over an arbitrary point $x \in S^k$, where the identity map is denoted by 1.

For such a constructed map $f(g)$ the properties b)–d) are obvious. For the proof of property a) it is necessary to make use of the fact that $\tilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$.

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[7] We assume here that on the tube $T(S^k)$ are given coordinates, viz., a normal field of $n-k$ frames on the sphere $S^k$.

Namely, it is found that the discriminator, of the "classifying" maps $\nu_1 \cdot f(g)$ and $\nu_2$ in

$$M^n(S^k, g) \xrightarrow{f(g)} M^n \xrightarrow{\nu_1} B_{SO_N},$$
$$M^n(S^k, g) \xrightarrow{\nu_2} B_{SO_N}$$

of the bundles $f(g)^*\nu^N(M^n)$ and $\nu^N(M^n(S^k, g))$ respectively assumes a value in the group

$$H^{n-k}(M^n(S^k, g), \theta^{n-k}/\theta^{n-k}(\pi)),$$

where

$$\theta^{n-k}/\theta^{n-k}(\pi) \subset \pi_{n-k-1}(SO_N) = \pi_{n-k}(B_{SO_N})$$

(cf. [8]), and if this discriminator is equal to zero, then the maps $\nu_1 \cdot f(g)$ and $\nu_2$ are homotopic. And what is more, if $\tilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$, then this discriminator is equal to zero. From the definition of the map $f(g)$ it follows at once that this discriminator is an element

$$z(g) \in H^{n-k}(M^n(S^k, g), \pi_{n-k}(B_{SO_N}))$$

and that the equality of it to zero is sufficient for the homotopicity of the maps $\nu_1 \cdot f(g)$ and $\nu_2$. The element $z(g)$ is represented by a cocycle $\bar{z}(g)$ having the same value on each fiber $D_x^{n-k}$, $x \in S^k \subset M^n(S^k, g)$. This value (on a given fiber $D_x^{n-k}$) is by definition (cf. [8]) an element of the group $\pi_{n-k}(B_{SO_N})$ defining the normal bundle of the smooth sphere $S^{n-k}(g)$, i.e., an element of the group $\theta^n/\theta^n(\pi)$ that is equal to zero if $\tilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$.

Thus all assertions of the lemma are proved. □

Now let $M^n = S^k \times S^{n-k}$. In this case there exists the following

**Lemma 9.3.** *The manifold $M^n(S^k, g)$ is diffeomorphic with degree $+1$ to the manifold $S^k \times \tilde{S}^{n-k}(g)$.*

*Proof.* Clearly,

$$M^n(S^k, g) = (S^k \times D^{n-k}) \cup_{\tilde{g}} (S^k \times D^{n-k}).$$

The diffeomorphism

$$\tilde{g}\colon S^k \times S^{n-k-1} \to S^k \times S^{n-k-1},$$

constructed above, is such that

$$\tilde{g}(x, y) = (x, g(S^k)y).$$

At the same time the diffeomorphism of

$$S^{n-k}(g) = D^{n-k} \cup_{g(S^k)} D^{n-k}, \qquad g(S^k)\colon S^{n-k-1} \to S^{n-k-1},$$

holds by definition. Thus the diffeomorphism $\tilde{g}$ is a fiber diffeomorphism that introduces a new structure of a direct product on a manifold $S^k \times S^{n-k-1}$. As a result of the gluing

$$M^n(S^k, g) = S^k \times D^{n-k} \cup_{\tilde{g}} S^k \times D^{n-k}$$

we obtain the direct product

$$S^k \times (D^{n-k} \cup_{g(S^k)} D^{n-k}) = S^k \times \tilde{S}^{n-k}(g).$$

The lemma is proved. □

We now define the operation "sums of manifolds along a cycle." Suppose $M_1^n$ and $M_2^n$ are manifolds and the $S_i^k \subset M_i^n$, $i = 1, 2$, are smoothly situated $k$-dimensional spheres, the normal bundles $\nu^{n-k}(S_i^k, M_i^n)$, $i = 1, 2$, of which are trivial. We introduce in the tubular neighborhoods

$$T(S_i^k) \subset M_i^n, \qquad i = 1, 2,$$

the coordinates of a direct product

$$T(S_i^k) = S_i^k \times D_\epsilon^{n-k}$$

using the geodesies of an $\epsilon$-ball $D_\epsilon^{n-k}$ that are normal to the spheres $S_i^k \subset M_i^n$ in some Riemannian metric. Then we put

$$[M_1^n \setminus T(S_1^k)] \cup_h [M_2^n \setminus T(S_2^k)] = M^n(S_1^k, S_2^k, h),$$

where

$$h\colon S_1^k \times D_\epsilon^{n-k} \to S_2^k \times D_\epsilon^{n-k},$$
$$h(x, y) = (x, h_x(y)), \qquad h_x \in SO_{n-k},$$
$$d(h)\colon S_1^k \to SO_{n-k}.$$

**Lemma 9.4.** *If $k < [\pi/2]$ and $\pi_1(M_1^n) = \pi_1(M_2^n) = 0$, then the manifold $M^n(S_1^k, S_2^k, h)$ depends only on the homotopy classes $\alpha_i$ of the embeddings of a sphere $S_i^k \subset M_i^n$, $i = 1, 2$, and the homotopy class $\tilde{d}$ of the map $d(h)\colon S_1^k \to SO_{n-k}$.*

*Proof.* If two spheres $S_{i,1}^k, S_{i,2}^k$, $i = 1, 2$, are smoothly situated in the manifold $M_i^n$ and are homotopic, then for $k \leq [n/2]$ they are diffeotopic. From this fact and the results of the paper [16] it follows that two embeddings

$$f_{i,j}\colon S_{i,j}^k \times D_\epsilon^{n-k} \to M_i^n, \qquad i, j = 1, 2,$$

are defined to within diffeotopy of the pair $(\alpha_i, \tilde{d}_i)$, where $\alpha_i \in \pi_k(M_i^n)$ and $d_i \in \pi_k(SO_{n-k})$. From the fact that the manifold $M^n(S_1^k, S_2^k, h)$ is completely defined by the diffeotopy classes of the embeddings

$$f_{i,j}\colon S_{i,j}^k \times D_\epsilon^{n-k} \to M_i^n, \qquad i, j = 1, 2,$$

it immediately follows that it depends only on the quadruple

$$(\alpha_1, \tilde{d}_1, \alpha_2, \tilde{d}_2), \qquad \alpha_i \in \pi_k(M_i^n),\ \tilde{d}_i \in \pi_k(SO_{n-k}).$$

Clearly, the quadruples $(\alpha_1, \tilde{d}_1, \alpha_2, \tilde{d}_2)$ and $(\alpha_1, 0, \alpha_2, \tilde{d}_2 - \tilde{d}_1)$ define the same manifolds. The lemma is proved. □

Below we will denote the manifold $M^n(S_1^k, S_2^k, h)$ by $M^n(\alpha_1, \alpha_2, \tilde{d})$, where $\alpha_i \in \pi_k(M_i^n)$, $i = 1, 2$, and $\tilde{d} \in \pi_k(SO_{n-k})$.

**Remark.** According to our definitions the bundles $\nu^{n-k}(S_i^k, M_i^n)$ must be trivial; as a result, for $2k < n$ we have $\alpha_i \in \pi_k(M^n, \nu^N(M^n))$ (cf. §7).

The following lemma is a consequence of the definition of a connected sum along a cycle and Lemma 9.3.

**Lemma 9.5.** *Suppose $M_1^n = S^k \times \tilde{S}^{n-k}(g)$ and $M_2^n$ is a $(k-1)$-dimensional manifold, $\alpha \in \pi_k(M_2^n, \nu^N(M_2^n))$, $\beta \in \pi_k(M_1^n)$, $\tilde{d} \in \pi_k(SO_{n-k})$, where $\beta$ is a generating element. Then the manifold $M^n(\alpha, \beta, d)$ is diffeomorphic with degree $+1$ to the manifold $M_2^n(\alpha, g)$ (mod $\theta^n$) for any element $\tilde{d} \in \pi_k(SO_{n-k})$.*

*Proof.* The element $\tilde{d} \in \pi_k(SO_{n-k})$ defines a diffeomorphism

$$h(\tilde{d}) \colon S^k \times D^{n-k} \to S^k \times D^{b-k}$$

such that

$$h(\tilde{d})(x, y) = (x, h(\tilde{d})_x y), \qquad h(\tilde{d})_x \in SO_{n-k},$$

where $h(\tilde{d}) \colon S^k \to SO_{n-k}$, is a representative of $\tilde{d}$. The diffeomorphism $h(\tilde{d})$ is extended to a diffeomorphism

$$\bar{h}(\tilde{d}) \colon S^k \times \tilde{S}^{n-k}(g) \to S^k \times \tilde{S}^{n-k}(g)$$

(everywhere except a point), since

$$S^k \times \tilde{S}^{n-k}(g) = (S^k \times D^{n-k} \cup_{\tilde{g}} (S^k \times D^{n-k}),$$

where $\tilde{S}^{n-k}(g) \in \theta^{n-k}$. Therefore the result of the gluing

$$M^n(\alpha, \beta, \tilde{d}) = (M_1^n \setminus S^k \times D^{n-k}) \cup_{h(\tilde{d})} (M_2^k \setminus S^k \times D^{n-k})$$

does not depend (to within an element of $\theta^n$) on the diffeomorphism $h(\tilde{d})$. But if $\tilde{d} = 0$, then the equality

$$M^n(\alpha, \beta, 0) = M_2^n(\alpha, g)$$

is a tautology. The lemma is proved. □

We now examine the Thom complex $T_N(S^k \times S^{n-k})$ and the subset

$$A(S^k \times S^{n-k}) \subset \pi_{N+n}(T_N(S^k \times S^{n-k})).$$

The manifold $S^k \times \tilde{S}^{n-k}(g)$ is a $\pi$-manifold, if $\tilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$, and is combinatorially equivalent to the manifold $S^k \times S^{n-k}$. There is therefore (cf. §§1–6) defined the subset

$$B(S^k \times \tilde{S}^{n-k})) \subset A(S^k \times S^{n-k}).$$

In addition, to the smooth sphere $\tilde{S}^{n-k}(g)$ corresponds the subset

$$\tilde{B}(\tilde{S}^{n-k}(g)) \subset G(n-k), \qquad k < n-k.$$

**Lemma 9.6.** *The Thom complex $T_N(S^k \times S^{n-k})$ is homotopically equivalent to the union*

$$S^{N+n} \vee S^{N+n-k} \vee S^{N+k} \vee S^N.$$

*The group*

$$\pi_{N+n}(T_N(S^k \times S^{n-k}))$$

*is isomorphic to the direct sum*

$$Z + G(k) + G(n-k) + G(n).$$

*The set $A(S^k \times S^{n-k})$ consists of all elements of the form*

$$1_{N+n} + \gamma, \qquad 1_{N+n} \in Z,\ \gamma \in G(k) + G(n-k) + G(n),$$

*where the element $1_{N+n} + 0 \in B(S^k \times S^{n-k})$.*

*The decomposition into a direct sum*

$$\pi_{N+n}(T_N(S^k \times S^{n-k})) = Z + G(k) + G(n-k) + G(n)$$

*can be chosen in such a way that*:

a) $G(n) = \operatorname{Im} \kappa_*$;

b) *the subgroup* $G(n-k)$ *belongs to the image of the inclusion homomorphism* $j_*\colon \pi_{N+n}(T_N^k) \to \pi_{N+n}(T_N(S^k \times S^{n-k}))$, *where*

$$j_*\colon T_N^k \subset T_N(S^k \times S^{n-k})$$

*is the embedding constructed in* §7, *and* $T_N^k = S^{N+k} \vee S^N$; *the subgroup* $G(n-k)$ *is defined uniquely* $\operatorname{mod} G(n)$;

c) $B(S^k \times \tilde{S}^{n-k}(g) \# \theta^n(\pi)) \supset 1_{N+n} + j_*\tilde{B}(\tilde{S}^{n-k}(g)) \bmod \operatorname{Im} \kappa_*$, *where* $j\colon T_N^k \subset T_N(S^k \times S^{n-k})$ *is the natural embedding.*

*Proof.* The decomposition of the Thom space into a union of spheres follows from

$$E(S^i \times S^j) = S^{i+1} \vee S^{j+1} \vee S^{i+j+1}$$

and Lemma 7.3. All assertions of the lemma, except the last, are trivial and immediately follow from the natural decomposition of a Thom complex into a union of spheres. Further, from Lemma 9.1 it follows that

$$B(S^k \times \tilde{S}^{n-k}(g) \# \tilde{S}^n) = B(S^k \times \tilde{S}^{n-k}(g)) + \kappa_* \tilde{B}(\tilde{S}^n),$$

where $\tilde{S}^n \in \theta^n(\pi)$. Therefore, for the proof of the lemma, it is sufficient to show that

$$B(S^k \times \tilde{S}^{n-k}(g)) \supset 1_{N+n} + j_*\tilde{B}(\tilde{S}^{n-k}(g)) \bmod \operatorname{Im} \kappa_*.$$

We consider the "auxiliary Thom complex"

$$T_N(S^k) = S^{N+k} \vee S^N \subset T_N(S^k \times S^{n-k}), \quad T_N^k = T_N(S^k), \quad k < n-k.$$

We also consider a map

$$f\colon S^k \times \tilde{S}^{n-k}(g) \to S^k,$$

where

$$f(x,y) = x, \qquad x \in S^k,\ y \in \tilde{S}^{n-k}(g),$$

We extend the map $f$ to a map

$$\tilde{F}\colon S^k \times \tilde{S}^{n-k}(g) \times D^N \to S^k \times D^N,$$

putting $\tilde{F} = f \times 1$. We extend the map $\tilde{F}$ to a map $F\colon S^{N+n} \to T_N(S^k)$ in the usual way, so that

$$F | T(S^k \times \tilde{S}^{n-k}(g)) = \tilde{F},$$

since the tubular neighborhood $T(S^k \times \tilde{S}^{n-k}(g)) \subset S^{N+n}$ is diffeomorphic to $S^k \times \tilde{S}^{n-k}(g) \times D^N$ by virtue of the fact that $S^k \times \tilde{S}^{n-k}(g)$ is a $\pi$-manifold. The map $\tilde{F}$ factors into a composition of maps

$$\tilde{F} = 1 \circ \tilde{F}\colon S^k \times \tilde{S}^{n-k}(g) \times D^N \to S^k \times D^N \to S^k \times D^N,$$

where $\tilde{F}^{-1}(x_0) = \tilde{S}^{n-k}(g)$, $x_0 \in S^k$, and the maps are $t$-regular. Therefore the induced map

$$F\colon S^{N+n} \to T_N(S^k)$$

factors into a composition of maps

$$F = F_2 \circ F_1\colon S^{N+n} \to S^{N+k} \to T_N(S^k),$$

where $F_2^{-1}(S^k) = S^k$, $F_2 | S^k = 1$ and $F_1^{-1}(x_0) = \tilde{S}^{n-k}(g)$, $x_0 \in S^k$.

By definition (cf. Lemma 3.1) the map $F_2$ represents a generating element of the group

$$\pi_{N+k}(S^{N+k}) \subset \pi_{N+k}(T_N^k) = \pi_{N+k}(T_N(S^k)) = \pi_{N+k}(S^{N+k} \vee S^k).$$

The map $F_1$ represents an arbitrary element of the set

$$\tilde{B}(\tilde{S}^{n-k})) \subset \pi_{N+n}(S^{N+n}) = G(n-k).$$

We now consider the sum

$$1_{N+n} + j_* \tilde{B}(\tilde{S}^{n-k}(g)) \subset \pi_{N+n}(T_N(S^k \times S^{n-k})).$$

Let the map

$$g\colon S^{N+n} \to T_N(S^k \times S^{n-k})$$

represent the element

$$1_{N+n} \in \pi_{N+n}(S^{N+n}) \subset \pi_{N+n}(T_N(S^k \times S^{n-k}))$$

and the map

$$F\colon S^{N+n} \to T_N^k \subset T_N(S^k \times S^{n-k})$$

represent an element of the set $j_* \tilde{B}(\tilde{S}^{n-k}(g))$ (the map $F$ was constructed above). We consider the "sum" of maps

$$(g+F)\colon S^{N+n} \to T_N(S^k \times S^{n-k}),$$

where

$$\begin{aligned}(g+F)^{-1}(S^k \times S^{n-k}) &= g^{-1}(S^k \times S^{n-k}) \cup F^{-1}(S^k \times S^{n-k}) \\ &= S^k \times S^{n-k} \cup S^k \times \tilde{S}^{n-k}(g).\end{aligned}$$

We consider the product $S^k \times D_\epsilon^{n-k} \times I(0,1)$ and form the membrane $B^{n+1} \subset S^{n+N} \times I(0,1)$. We have

$$B^{n+1} = [S^k \times S^{n-k} \cup S^k \times \tilde{S}^{n-k}(g)] \times I\left(0, \frac{1}{2}\right) \cup_{h_1,h_2} S^k \times D_\epsilon^{n-k} \times I(0,1),$$

where

$$\begin{aligned} h_1&\colon S^k \times D_\epsilon^{n-k} \times 0 \to S^k \times D_\epsilon^{n-k} \subset S^k \times S^{n-k}, \\ h_2&\colon S^k \times D_\epsilon^{n-k} \times 1 \to S^k \times D_\epsilon^{n-k} \subset S^k \times \tilde{S}^{n-k}(g),\end{aligned}$$

and

$$h_i(x,y) = (x, h_{ix}(y)), \quad h_{ix} \in SO_{n-k}, \quad i = 1, 2.$$

Clearly,

$$\partial B^{n+1} = [S^k \times S^{n-k} \cup S^k \times \tilde{S}^{n-k}(g)] \cup S^k \times \tilde{S}^{n-k}(g).$$

In addition, on the manifold

$$[S^k \times S^{n-k} \cup S^k \times \tilde{S}^{n-k}(g)] = \partial B^{n+1} \cap S^{N+n} \times 0$$

is given an $N$-frame field, normal to this manifold in the sphere $S^{N+n}$ and induced by the map $(g+F)$ of some a priori given and fixed $N$-frame field, normal to the submanifold $S^k \times S^{n-k}$ in $T_N(S^k \times S^{n-k})$ (cf. §§1–6). We will place the membrane $B^{n+1}$ in the direct product $S^{N+n} \times I(0,1)$ in a smooth manner and assume, as in §§1–6, that on $S^{N+n} \times 0$ is defined the map $(g+F)$ and

$$B^{n+1} \cap S^{N+n} \times 0 = \partial B^{n+1} \cap S^{N+n} \times 0 = (g+F)^{-1}(S^k \times S^{n-k}),$$

$$B^{n+1} \cap S^{N+n} \times 1 = S^k \times \tilde{S}^{n-k}(g),$$

where the membrane $B^{n+1}$ orthogonally approaches the boundaries of the direct product $S^{N+n} \times I(0,1)$. Since the difference between the cycles $S^k \times x_0$, $x_0 \in S^{n-k}$, and $S^k \times x_1$, $x_1 \in \tilde{S}^{n-k}(g)$, belongs to the kernel

$$\operatorname{Ker}(g+F)_*^{(H_k)} \subset H_k(S^k \times S^{n-k} \cup \tilde{S}^{n-k}(g)),$$

it is possible to extend the map of a submanifold

$$(g+F)|B^{n+1}\cap S^{N+n}\times 0$$

to the map

$$\widetilde{(g+F)}\colon B^{n+1}\to S^k\times S^{n-k}\subset T_N(S^k\times S^{n-k}).$$

In addition, it is always possible to choose maps $h_1, h_2$ in such a way that the map $\widetilde{(g+F)}$ can be extended to the map

$$\widetilde{\widetilde{(g+F)}}\colon T(B^{n+1})\to T_N(S^k\times S^{n-k}),$$

where $T(B^{n+1})$ is a tubular neighborhood of the manifold $B^{n+1}\subset S^{N+n}\times I$, as in §§1–6 (or, what is the same thing, an $N$-frame field normal to the manifold $B^{n+1}\cap S^{N+n}\times 0$ can be extended to an $N$-frame field normal to the entire membrane $B^{n+1}$ in $S^{N+n}\times I(0,1)$). Then in the usual way we extend the map $\widetilde{\widetilde{(g+F)}}$ of the tube $T(B^{n+1})$ onto the entire direct product $S^{N+n}\times I(0,1)$. As a result we arrive at a certain map

$$\widetilde{\widetilde{(g+F)}}|S^{N+n}\times 1\to T_N(S^k\times S^{n-k}),$$

that is homotopic to the map $(g+F)$ and such that

$$\widetilde{\widetilde{(g+F)}}^{-1}(S^k\times S^{n-k})\cap S^{N+n}\times 1=S^k\times\tilde S^{n-k}(g).$$

We have thus proved that in any homotopy class of the manifold $1_{N+n}+j_*\tilde B(\tilde S^{n-k}(g))$ there exists a representative

$$\widetilde{\widetilde{(g+F)}}\colon S^{N+n}\times 1\to T_N(S^k\times S^{n-k})$$

such that

$$\widetilde{\widetilde{(g+F)}}^{-1}(S^k\times S^{n-k})=S^k\times\tilde S^{n-k}(g).$$

Consequently,

$$1_{N+n}+j_*\tilde B(\tilde S^{n-k}(g))\subset B(S^k\times\tilde S^{n-k}(g)) \bmod \operatorname{Im}\kappa_*.$$

Comparing our results with Lemma 7.3, we obtain the desired statement. The lemma is proved. □

From Lemma 9.6 immediately follows

**Lemma 9.7.**

$$B(S^k\times\tilde S^{n-k}(g)\,\#\,\theta^n(\pi))\supset B(S^k\times S^{n-k})+j_*\tilde B(\tilde S^{n-k}(g)) \bmod \operatorname{Im}\kappa_*.$$

The proof formally follows from Lemma 9.6. It is only necessary to note that, according to Lemma 9.6,

$$B(S^k\times\tilde S^{n-k}(g)\,\#\,\theta^n(\pi))\supset 1_{N+n}+\tilde B(\tilde S^{n-k}(g)) \bmod \operatorname{Im}\kappa_*,$$

where $1_{N+n}\in\pi_{N+n}(S^{N+n})\subset\pi_{N+n}(T_N(S^k\times S^{n-k}))$; although the decomposition

$$T_N(S^k\times S^{n-k})=S^{N+n}\vee S^{N+n-k}\vee S^{N+k}\vee S^N$$

is chosen ambiguously. Namely, if we take another element of the set $B(S^k\times S^{n-k})$ as a new generator

$$1'_{N+n}\in\pi_{N+n}(S^{N+n})$$

and choose, according to the choice of this new generator, a new decomposition of the Thom complex into a union, then under replacement of $1_{N+n}$ by $1'_{N+n}$ all the arguments of Lemma 9.6 remain in force and we get that

$$B(S^k \times \tilde{S}^{n-k}(g) \# \theta^n(\pi)) \supset 1'_{N+n} + j_* \tilde{B}(\tilde{S}^{n-k}(g)) \bmod \operatorname{Im} \kappa_*$$

for any element $1'_{N+n} \in B(S^k \times S^{n-k})$.

The lemma is proved.

Combining the results of the preceding lemmas, we can state that there have been introduced two elementary operations for changing the smoothness under preservation of the triangulation: the connected sum with a Milnor sphere from $\theta^n(\pi)$ and the "connected sum along a cycle" $S^k \subset M^n$, $k < [n/2]$ (if the normal bundle $\nu^{n-k}(S^k, M^n)$ is trivial), of the manifolds $M^n$ and $S^k \times \tilde{S}^{n-k}$, where $\tilde{S}^{n-k} \in \theta^{n-k}(\pi)$. The homotopy meaning of these operations for the case $M^n = S^k \times S^{n-k}$ was found in Lemmas 9.1–9.7.

We denote by $B_{\gamma,\delta}(M_1^n) \subset B(M_1^n)$ the subset consisting of those elements

$$\alpha_i \in B_{\gamma,\delta}(M_1^n) \subset B(M_1^n) \subset A(M^n) \subset \pi_{N+n}(T_N(M^n))$$

for which there are representatives $f_{\alpha_i} \colon S^{N+n} \to T_N(M^n)$ that satisfy Lemma 3.2 and possess the following properties:

a) the manifolds $f_{\alpha_i}^{-1}(M^n)$ are diffeomorphic to $M_1^n$, though the map $f_{\alpha_i}|M_1^n$ need not be a diffeomorphism;

b) $f_{\alpha_i *}(\delta) = \gamma$, where $\gamma \in \pi_k(M^n)$, $\delta \in \pi_k(M_1^n)$.

**Lemma 9.8.** *If there exists a diffeomorphism $h \colon M_1^n \to M_1^n$ of degree $+1$ such that $h_*(\delta_1) = \delta_2$, with $\delta_1, \delta_2 \in \pi_k(M_1^n)$, then the sets $B_{\gamma,\delta_1}(M_1^n)$ and $B_{\gamma,\delta_2}(M_1^n)$ coincide.*

The proof of the lemma follows immediately from the fact that we distinguish all our objects only to within an equivalence induced by diffeomorphisms of the manifold $M_1^n$ onto itself of degree $+1$. The lemma is proved.

Below we will always denote a "connected sum along a cycle" of two manifolds $M_1^n$ and $M_2^n$ in the following standard manner:

$$M^n(\gamma_1, \gamma_2, d) = M_1^n \#^d_{\gamma_1\gamma_2} M_2^n,$$

where $\gamma_i \in \pi_k(M_i^n, \nu^N(M_i^n)$, $d \in \pi_k(SO_{n-k})$. In the event that $M_2^n = S^k \times \tilde{S}^{n-k}$, $\gamma \in \pi_k(M_1^n, \nu^N(M_1^n))$ and $\beta \in \pi_k(S^k \times \tilde{S}^{n-k})$ is a generating element, we then, taking into account Lemma 9.5, use the notation

$$M_1^n \#^d_{\gamma,\beta} S^k \times \tilde{S}^{n-k} = M_1^n \#_\gamma S^k \times \tilde{S}^{n-k} \bmod \theta^n.$$

**Theorem 9.9.** *Suppose $M^n$ is a $(k-1)$-connected manifold and $\gamma, \delta \in \pi_k(M^n, \nu^N(M^n))$, $k < n-k$. Then in the Thom complex $T_N(M^n)$ the relation*

$$B_{\gamma,\delta}(M^n) + \tilde{B}(\tilde{S}^{n-k}(g)) \cdot T^N\gamma \subset B(M^n \#_\delta S^k \times \tilde{S}^{n-k}(g)) \bmod \operatorname{Im} \kappa_* \tag{33}$$

*is valid, where $\tilde{B}(\tilde{S}^{n-k}(g)) \subset G(n-k)$ and*

$$T^N \colon \pi_k(M^n, \nu^N(M^n)) \to \pi_{N+k}(T_N(M^n))/\operatorname{Im} \kappa_*,$$

*is the homomorphism constructed in* §7.

*Proof.* We realize the element $\gamma \in \pi_k(M^n, \nu^N(M^n))$ by a smoothly embedded sphere $\tilde{\gamma}\colon S^k \subset M^n$, which has a trivial normal bundle $\nu^{n-k}(S^k, M^n)$ in the manifold $M^n$, since the bundle $\tilde{\gamma}^*\nu^N(M^n)$ (by condition) and the bundle

$$\nu^{n-k}(S^k, M^n) \oplus \tilde{\gamma}^*\nu^N(M^n) = \nu^{N+n-k}(S^k)$$

are trivial and $k < n-k$. The embedding $\tilde{\gamma}\colon S^k \subset M^n$ determines in a natural way the embedding

$$T^N\tilde{\gamma}\colon T_N(S^k, \tilde{\gamma}^*\nu^N(M^n)) \subset T^N(M^n).$$

By analogy with the proof of Lemmas 9.6 and 9.7 we consider two maps

$$f\colon S^{N+n} \to T_N(M^n), \qquad \tilde{f} \in B_{\gamma,\delta}(M^n),$$
$$F\colon S^{N+n} \to T_N(S^k, \tilde{\gamma}^*\nu^N(M^n)) \subset T_N(M^n),$$

having the following properties:

$$\tilde{F} \in \tilde{B}(\tilde{S}^{n-k}(g) \circ B(S^k)), \qquad B(S^k) \subset \pi_{N+k}(T_N(S^k, \tilde{\gamma}\nu^N(M^n))),$$
$$T_N(S^k, \tilde{\gamma}\nu^N(M^n)) = T_N(S^k)$$

($\tilde{f}$ and $\tilde{F}$ respectively denote the homotopy classes of the maps $f$ and $F$).

It is easy to see that $f^{-1}(M^n) = M^n$ and $F^{-1}(S^k) = S^k \times \tilde{S}^{n-k}(g)$ .

Further, we consider the map

$$(f+F)\colon S^{N+n} \to T_N(M^n)$$

representing the element $\tilde{f} + T^N\tilde{\gamma}\tilde{F} \in \pi_{N+n}(T_N(M^n))$. Clearly,

$$(f+F)^{-1}(M^n) = M^n \cup S^k \times \tilde{S}^{n-k}(g) \subset S^{N+n},$$

the element $f_*^{-1}(\gamma) - F_*^{-1}(\gamma)$ belongs to the kernel $\mathrm{Ker}(f+F)_*$, and $\delta = f_*^{-1}(\gamma)$. By analogy with the proof of Lemma 6.9 we construct a membrane $B^{n+1} \subset S^{N+1} \times I(0,1)$ such that:

a) $B^{n+1} \cap S^{N+n} \times 0 = (f+F)^{-1}(M^n)$,
b) $B^{n+1} \cap S^{n+N} \times 1 = M^n \#_\delta S^k \times \tilde{S}^{n-k}$,
c) $B^{n+1} = (f+F)^{-1}(M^n) \times I(0,1/2) \cup_{h_1,h_2} S^k \times D_\epsilon^{n-k} \times I(0,1)$,
d) $h_1\colon S^k \times D_\epsilon^{n-k} \times 0 \to M^n \times 1/2$,
e) $h_2\colon S^k \times D_\epsilon^{n-k} \times 1 \to S^k \times \tilde{S}^{n-k} \times 1/2$,
f) $h_i(x, y, i-1) = (x, h_{ix}(y))$,

where $i = 1, 2$, $h_{ix} \in SO(n-k)$, $x \in S^k$, $y \in D_\epsilon^{n-k}$.

The membrane is chosen in such a way that the map

$$(f+F)|S^{N+x} \times 0$$

can be extended to a map

$$F_1\colon S^{N+n} \times I(0,1) \to T_N(M^n)$$

such that

$$F_1^{-1}(M^n) = B^{n+1}.$$

The choice of the membrane is effected by the choice of the map $h_{ix}$, $i = 1, 2$, as in Lemma 9.6, and can always be effected for $k < n-k$. On the upper boundary the map $F_1|S^{N+n} \times 1$ defines a map $\widetilde{(f+F)}$ such that

$$\widetilde{(f+F)}^{-1}(M^n) = M^n \#_\delta S^k \times \tilde{S}^{n-k}(g).$$

We have thus shown that the sum $\tilde{f} + T^N\tilde{\gamma}\tilde{F}$ belongs to the set

$$B(M^n \#_\delta S^k \times \tilde{S}^{n-k}(g))$$

when $\tilde{f} \in B_{\gamma,\delta}(M^n)$ and

$$\tilde{F} \in \tilde{B}(\tilde{S}^{n-k}(g)) \circ B(S^k), \qquad T^N\tilde{\gamma}\tilde{F} \in \tilde{B}(\tilde{S}^{n-k}(g)) \cdot T^N\gamma.$$

From the definition of a homomorphism,

$$T^N : \pi_k(M^n, \nu^N(M^n)) \to \pi_{N+n}(T_N(M^n))/\operatorname{Im}\kappa_*.$$

The theorem is proved. □

## § 10. Variation of smoothness under preservation of the triangulation. The reconstruction of Morse[8]

We assume that the manifold $M^n$ is $(k-2)$-connected and is a $\pi$-manifold for $k < n-k-1$, $k-2 \geq 1$. We select in the group

$$H_{k-1}(M^n) = \pi_{k-1}(M^n) = \pi_{k-1}(M^n, \nu^N(M^n))$$

some element $\gamma$, realize it by a sphere $S^{k-1} \subset M^n$ having by virtue of the $(k-1)$-parallelizibility of the manifold $M^n$ a trivial normal bundle $\nu^{n-k+1}(S^{k-1}, M^n)$, and form the manifold

$$B^{n+1}(h) = M^n \times I\left(0, \frac{1}{2}\right) \cup_h D^k \times D_\epsilon^{n-k+1},$$

where

$$h\colon \partial D^k \times D_\epsilon^{n-k+1} \to T(S^{k-1}) = S^{k-1} \times D_\epsilon^{n-k+1},$$
$$h(x,y) = (x, h_x(y)), \qquad h_x \in SO_{n-k+1}.$$

We select the diffeomorphism $h$ so that the manifold $B^{n+1}(h)$ is also a $\pi$-manifold, which is possible (cf. §§1–2 or §9). Clearly,

$$\partial B^{n+1}(h) = M^n \cup (-M^n(h))$$

and

$$H_k(B^{n+1}(h), M^n) = H_{n+1-k}(B^{n+1}(h), M^n(h)) = Z,$$
$$H_i(B^{n+1}(h), M^n) = H_{n+1-i}(B^{n+1}(h), M^n(h)) = 0, \qquad i \neq k.$$

Let us vary the smoothness on the manifold $M^n(h)$, keeping fixed the normal bundle $\nu^N(M^n(h))$ and the triangulation. We denote the resultant manifold by $M_1^n(h)$. To this variation of smoothness, according to the results of §8, there corresponds the set of elements $(\alpha_i) \in \pi_{N+n}(T_N^{n-2})$ representing the set of all differences

$$B(M^n(h)) - B(M_1^n(h)), \qquad T_N^{n-2} \subset T_N(M^n(h)).$$

We denote the standard combinatorial equivalence by $q\colon M_1^n(h) \to M^n(h)$. In the set $B(M_1^n(h))$ we select the subset $B^{(q)}(M_1^n(h))$ consisting of those elements $\alpha \in B^{(q)}(M_1^n(h))$ which have $t$-regular representatives

$$f_2\colon S^{N+n} \to T_N(M^n(h))$$

such that

$$f_2^{-1}(M^n(h)) = M_1^n(h)$$

[8]The principal theorem of this section, Theorem 10.2, is proved here incompletely. The reader can omit this section, since its results are not used later. A detailed proof of Theorem 10.2 will be given in the second part of the paper.

and

$$f_2|M_1^n(h) = q.$$

We fix the standard element $1_{N+n} \in B(M^n(h))$, constructed for the proof of Lemma 3.1, and consider the subset of the set of differences of the form

$$\{1_{N+n} - B^{(q)}(M^n(h))\} \in \pi_{N+n}(T_N^{n-2}), \qquad T_N^{n-2} \subset T_N(M^n(h)).$$

We extend the smoothness of the manifold $M_1^n(h)$ onto the entire membrane $B^{n+1}(h)$. In this connection there arise the obstructions

$$\phi^s \in H^s(B^{n+1}(h), M^n(h), \theta^{n-s}), \qquad \Gamma^{n-s} \subset \theta^{n-s}$$

with coefficients in Milnor groups (cf. [12, 23]). But since

$$H^s(B^{n+1}(h), M^n(h)) = 0, \qquad s \neq n+1-k,$$

there arises only one obstruction

$$\phi^{n+1-k} \in H^{n+1-k}(B^{n+1}(h), M^n(h), \theta^{n-k}) = \theta^{n-k}.$$

Thus, to every manifold $M^n(h)$ that is combinatorially equivalent to the manifold $M^n(h)$ there corresponds an element $\phi^{n+1-k} \in \theta^{n-k}$. According to certain results of Munkres [12], if $\phi^{n+1-k} = 0$, then the variation of smoothness can be extended onto $B^{n+1}(h)$ with boundary $M^n(h)$ without varying the triangulation on it.

We select in the group

$$H_{k-1}(M^n) = \pi_{k-1}(M^n)$$

a minimal system of generators $\gamma_1, \dots, \gamma_l$; and realize them by spheres $S_1^{k-1}, \dots, S_l^{k-1} \subset M^n$, that are smoothly embedded and mutually disjoint. For each of these spheres the bundles $\nu^{n-k+1}(S_i^{k-1}, M^n)$, $i = 1, \dots, l$, are trivial. We form the manifold

$$B_l^{n+1} = M^n \times I\left(0, \frac{1}{2}\right) \cup_{h_1,\dots,h_l} [(D_1^k \times D_\epsilon^{n-k+1}) \cup \dots \cup (D_l^k \times D_\epsilon^{n-k+1})],$$

where

$$h_i\colon \partial D_i^k \times D_\epsilon^{n-k+1} \to S_i^{k-1} \times D_\epsilon^{n-k+1} \subset M^n, \qquad i = 1, \dots, l,$$

so that

$$h_i(x, y) = (x, h_{ix}(y)), \qquad x \in S_i^{k-1}, \qquad y \in D_\epsilon^{n-k+1}, \qquad h_{ix} \in SO_{n-k+1}.$$

We select the diffeomorphism $h_i$ according to §§1–2 so that the manifolds

$$M_l^n(h) = \left(M^n \setminus \bigcup_i T(S_i^{k-1})\right) \cup_{h_1,\dots,h_l} \left[\bigcup_i D_i^k \times S_\epsilon^{n-k}\right]$$

and $B_l^{n+1}(h)$ are $\pi$-manifolds, which is possible for $k < n-k$. Clearly,

$$\partial B_l^{n+1}(h) = M^n \cup (-M_l^n(h))$$

and

$$H^s(B_l^{n+1}(h), M^n) = H^{n+1-s}(B_l^{n+1}(h), M_l^n(h)) = 0, \qquad s \neq k.$$

Since $k < n-k-1$, the manifold $M_l^n(h)$ is $(k-1)$-connected. By analogy with the above, to every variation of smoothness on $M_l^n(h)$ without variation of the triangulation there corresponds an element

$$\phi^{n+1-k} \in H^{n+1-k}(B_l^{n+1}(h), M_l^n(h), \theta^{n-k}) = \theta_{(1)}^{n-k} + \dots + \theta_{(l)}^{n-k}.$$

Let

$$H_i(M_l^n(h)) = 0, \qquad i < k+p \qquad (p \geq 0)$$

and

$$H_{k+p}(M_l^n(h)) = \pi_{k+p}(M_l^n(h)) \neq 0,$$

where $k+p < n-k-p-1$. On the manifold $M_l^n(h)$ we vary a smooth structure, using the results of §9; namely, we select in the group $\pi_{k+p}(M_l^n(h))$ a base $\delta_1, \dots, \delta_m$ and consider the sum

$$M_l^n(h) \#_{\delta_1} S^{k+p} \times \tilde{S}_1^{n-k-p} \#_{\delta_2} \cdots \#_{\delta_m} S^{k+p} \times \tilde{S}_m^{n-k-p},$$

where $\tilde{S}_m^{n-k-p} \in \theta^{n-k-p}(\pi)$. Let us attempt to "carry over" the new smoothness with respect to the membrane $B_l^{n+1}(h)$ on $M^n$. There arises an obstruction

$$\phi^{n+1-k} \in \theta^{n-k} + \cdots + \theta^{n-k} \qquad (l \text{ terms});$$

this obstruction defines a map

$$\phi^{n+1-k} \colon \sum_{i=1}^{m} \theta_i^{n-k-p} \to \sum_{j=1}^{l} \theta_j^{n-k} \tag{35}$$

(to the variation of smoothness of the manifold $M_l^n(h)$ by an element $\theta \in \sum_{i=1}^m \theta^{n-k-p}$ corresponds an obstruction $\phi^{n+1-k}(\theta) \in \sum_{i=1}^l \theta_j^{n-k}$). If $\theta \in \operatorname{Ker} \phi^{n+1-k}$, then the variation of smoothness by $\theta$ permits a carrying over. We now study the homotopy nature of the constructed map $\phi^{n+1-k}$ in terms of a Thom complex. In this connection we recall the filtration of a Thom complex

$$T_N(M^n) \supset T_N^{n-2} \supset \cdots \supset T_N^2 \supset S^N = T_N^0.$$

If the manifold $M^n$ is $(k-2)$-connected, then

$$T_N^2 = T_N^3 = \cdots = T_N^{k-2} = T_N^0 = S^N$$

and

$$T_N^{n-2} = \cdots = T_N^{n-k+1}.$$

In general, we will always select a filtration

$$T_N^i = T_N(K^i, j^* \nu^N(M^n)),$$

where $K^i$ is the $i$-dimensional skeleton in a minimal triangulation and $j \colon K^i \subset M^n$ (the number of cells $\sigma^i \subset M^n$ is equal to $\max \operatorname{rk} H^i(M^n, k)$ with respect to all fields $k$). To each cell $\sigma^i \subset M^n$ corresponds a cell

$$T_N \gamma^i \subset T_N^i \subset T_N(M^n),$$

and the boundary operators in the complexes $M^n$ and $T_N(M^n)$ are applied identically:

$$\partial T_N(\gamma^i) = T_N(\partial \sigma^i).$$

It was proved in §7 that, if $M^n$ is a $\pi$-manifold, then the space $T_N(M^n)$ is homotopically equivalent to the union

$$E^N(M^n) \vee S^N = E^N(M^n \cup x_0),$$

where $x_0$ is a point. In this case we can assume that

$$E^N(K^i \cup x_0) = E^N K^i \vee S^N = T_N^i,$$

and

$$T_N(M^n) = S^{N+n} \vee E^N K^{n-2} \vee S^N.$$

Let us consider the Thom complex $T_N(B_l^{n+1}(h))$, which is a pseudomanifold with boundary

$$\partial T_N(B_l^{n+1}(h)) = T_N(M^n) \vee T_N(M_l^n(h)).$$

As is well known (cf. §1), the space $B_l^{n+1}(h)$ contracts to its part

$$M^n \times \frac{1}{2} \cup_{h_1,\dots,h_l} (D_1^k \times 0 \cup \dots \cup D_l^k \times 0).$$

The homotopy type of a Thom complex depends only on the homotopy type of the base. Therefore the Thom complex $T_N(B_l^{n+1}(h))$ is homotopically equivalent to the Thom complex

$$T_N(M^n) \cup_{T_N h_1,\dots,T_N h_l} (D_1^{N+k} \cup \dots \cup D_l^{N+k}),$$

where

$$T_N h_i \colon \partial D_i^{N+k} \to E^N K^{k-1} \subset T_N^{k-1}, \qquad i = 1, \dots, l.$$

In this connection it is evident that

$$K^{k-1} = S_1^{k-1} \vee \dots \vee S_l^{k-1}$$

and

$$T_N h_i \colon \partial D_i^{N+k} \to E^N S_i^k = S_i^{n+k-1},$$

if the spheres $S_i^{k-1} \subset K^{k-1}$ are chosen according to the previously selected system of generators $\gamma_1, \dots, \gamma_l$ of the group

$$H_{k-1}(M^n) = \pi_{k-1}(M^n)$$

for the definition of the manifold

$$B_l^{n+1}(h), \qquad h = (h_1, \dots, h_l).$$

We now investigate the Thom complex $T_N(M_l^n(h))$. If an element $\gamma_s$ is an element of infinite order, then, under passage from $M^n$ to $M_l^n(h)$ of a cycle $\tilde{\gamma}_s \in H_{n-k+1}(M^n)$ such that $\tilde{\gamma}_s \cdot \gamma_s = 1$, a neighborhood of a point that is orthogonal to the sphere $S_s^{k-1} \subset M^n$ will be discarded. If for all generators of cycles of infinite order

$$\gamma_{i_1}, \dots, \gamma_{i_s} \in \pi_{k-1}(M^n)$$

there exists a system of dual generators

$$\tilde{\gamma}_{i_1}, \dots, \tilde{\gamma}_{i_s} \in H_{n-k-1}(M^n)$$

such that

$$\tilde{\gamma}_{i_j} \cdot \gamma_{i_t} = \delta_{jt},$$

and each generating element $\tilde{\gamma}_{i_j}$ is defined by precisely one cell $\sigma_j^{n-k+1} \subset M^n$, then, under passage from $M^n$ to $M_l^n(h)$ of the interior of each cell $\sigma_j^{n-k+1}$, a small spherical neighborhood of a point will be discarded, and the complement can be contracted to $K^{n-k}$. If an element $\gamma_t$ has a finite order $q_t$, then there exists an element

$$\tilde{\gamma}_t \in H^{n-k+1}(M^n, Z_{q_t})$$

such that

$$\gamma_t \cdot \tilde{\gamma}_t = 1 \pmod{q_t};$$

if the element $\tilde{\gamma}_t$ is also defined by a single cell

$$\sigma_t^{n-k+1} \in K^{n-k+1} \subset M^n$$

(which can always be assumed if $n-k+1\neq k-1$), then, under passage from $M^n$ to $M_l^n(h)$ of this cell, only one spherical neighborhood of a point of intersection of $\sigma_t^{n-k+1}$ and $S_1^{k-1}$ will be discarded, and after this operation the complement can be contracted onto the boundary $\partial\sigma_t^{n-k+1}\subset K^{n-k}$. Furthermore the entire group $\pi_{k-1}(M^n)$ goes into zero under passage from $M^n$ to $M_l^n(h)$ (a ball $D_i^k$ is stretched onto each sphere $S_i^{k-1}$, $i=1,\dots,l$, that is shifted onto the boundary of a tubular neighborhood $\partial T(S_i^{k-1})\subset M^n$). Thus we have obtained the following statement.

**Lemma 10.1.** *The complex $T_N(B_l^{n+1}(h))$ is homotopically equivalent to the Thom complex*

$$T_N(M^n)=S^{N+n}\vee E^NK^{n-k+1}\vee S^N$$

*with cone stretched onto the $(N+k-1)$-dimensional subcomplex*

$$E^NK^{k-1}=S_1^{N+k-1}\vee\dots\vee S_l^{N+k-1}\subset E^NK^{n+k-1}\subset T_N(M^n).$$

*If $k-1<n-(k-1)-1$, then the Thom complex $T_N(M_l^n(h))$ is a subcomplex of the complex $T_N(B_l^{n+1}(h))$ and contracts on itself to the subcomplex*

$$(S^{N+n}\vee E^NK^{n-k}\vee S^N)/E^NK^{k-1}$$

*of the complex*

$$T_N(B_l^{n+1}(h))=(S^{N+n}\vee E^NK^{n-k+1}\vee S^N)/E^NK^{k-1}.$$

The proof of Lemma 10.1 is obtained from the arguments preceding the formulation (passage to Thom complexes).

The lemma is proved.

We have already considered in §8 the exact sequences (33) and (34) of the form

$$\dots\to\pi_{N+i}(T_N^{k-1})\to\pi_{N+i}(T_N^{k+p})\to\pi_{N+i}(T_N^{k+p}/T_N^{k-1})\xrightarrow{\partial}\pi_{N+i-1}(T_N^{k-1})\to\cdots$$

for $i=n$, $p\geq 0$. In our case

$$T_N^i=E^NK^i\vee S^N,$$
$$T_N^{k-1}=S_1^{N+k-1}\vee\dots\vee S_l^{N+k-1}\vee S^N=E^NK^{k-1}\vee S^N.$$

Now suppose again that $i=n$. We consider the exact sequence

$$(36)\quad \pi_{N+n}(E^NK^{k-1})\to\pi_{N+n}(E^NK^{k+p})\to\pi_{N+n}(E^NK^{k+p}/E^NK^{k-1})$$
$$\xrightarrow{\partial}\pi_{N+n-1}(E^NK^{k-1})\to\pi_{N+n-1}(E^NK^{k+p}),\qquad p\geq 0,$$

corresponding to the exact sequences (33) and (34), since

$$T_N^m=E^NK^m\vee S^N.$$

In order to emphasize the dependence on a manifold, we will write

$$T_N^m=T_N^m(M^n)\subset T_N(M^n),\qquad T_N^m(M_l^n(h))\subset T_N(M_l^n(h)),$$
$$T_N^m(B_l^{n+1}(h))\subset T_N(B_l^{n+1}(h)).$$

It follows from Lemma 10.1 that

$$T_N^m(M_l^n(h))=T_N^m(B_l^{n+1}(h))=(E^NK^m/E^NK^{k-1})\vee S^N$$

for $m\leq n-k$ and

$$T_N^{n-k+1}(B_l^{n+1}(h))=(E^NK^{n-k+1}/E^NK^{k-1})\vee S^N.$$

We will also write

$$K^m = K^m(M^n) \subset M^n,$$

$$K^m(M_l^n(h)) \subset M_l^n(h), \qquad K^m(B_l^{n+1}(h)) \subset B_l^{n+1}(h),$$

denoting the skeletons of dimension $m$ of the corresponding manifolds $M^n$, $M_l^n(h)$ or $B_l^{n+1}(h)$ by symbols depending on the manifold. We note in addition that

$$\pi_{N+n-1}(E^N K^{k-1}) = G(n-k) + \cdots + G(n-k) \qquad (l \text{ terms}).$$

We rewrite the exact sequence (36) in the form

$$(37) \quad \sum_{i=1}^{l} G_i(n-k+1) \to \pi_{N+n}(E^N K^{k+p}(M^n)) \xrightarrow{\wedge} \pi_{n+N}(E^N K^{k+p}(B_l^{n+1}(h))) \xrightarrow{\partial} \sum_{i=1}^{l} G_i(n-k) \to \pi_{N+n-1}(E^N K^{k+p}(M^n)),$$

where, if $k+p \leq n-k$, then

$$E^N K^{k+p}(B_l^{n+1}(h)) = E^N K^{k+p}(M_l^n(h)).$$

In accordance with the notation of §8, we obtain from Lemma 10.1

$$T_N^{k+p,k-1}(M^n) = T_N^{k+p}(M_l^n(h)) = T_N^{k+p}(B_l^{n+1} + (h)),$$
$$k-1 < n-k-2, \qquad p \geq 0, \qquad k+p \leq n-k.$$

Let us now consider the "equipped" smooth spheres $\tilde{S}^i \subset S^{N+1}$ in the sense of Pontrjagin [15]. In this case a sphere $\tilde{S}^i$ with normal frame field $\tau^N$ ("equipment") in $S^{N+i}$ defines an element of the group $G(i)$. We will also always carry out the operation of a "connected sum along a cycle," defined in §9, for "equipped" $\pi$-manifolds $M_1^n, M_2^n \subset S^{N+n}$, so that the manifold

$$M_1^n \#_{\gamma_1;\gamma_2}^d M_2^n$$

receives the natural equipment for a suitably chosen element $d$. Since an "equipped" smooth sphere $\tilde{S}_i$ defines only one element $\alpha(\tilde{S}^i, \tau^N) \subset G(i)$, we obtain a new formulation for Theorem 9.9:

*Every element $\beta \in B_{\gamma,\delta}(M_1^n) \subset A(M^n)$ represents*

a) *an "equipped'" manifold $M_1^n$ plus*

b) *a fixed to within homotopy map $f\colon M_1^n \to M^n$ of degree $+1$ such that*

$$f_*\delta = \gamma, \qquad \gamma \in \pi_k(M^n, \nu^N(M^n)), \qquad \delta \in \pi_k(M_1^n, \nu^N);$$

*on the manifold $M_1^n \#_\delta S^k \times \tilde{S}^{n-k}$ appears the natural equipment and the natural map*

$$\tilde{f}\colon M_1^n \#_\delta S^k \times \tilde{S}^{n-k} \to M^n;$$

*this equipment and map $\tilde{f}$ jointly define an element*

$$\beta + \alpha(\tilde{S}^{n-k}, \tau^N) \circ T^N\gamma \in B_{\gamma,\delta}(M_1^n \#_\delta S^k \times \tilde{S}^{n-k}),$$

*where*

$$\alpha(\tilde{S}^{n-k}, \tau^N) \in \tilde{B}(\tilde{S}^{n-k}), \qquad \beta \in B_{\gamma,\delta}(M_1^n).$$

This new formulation is somewhat stronger than the old one, but it is proved in essentially the same way. We will call this (stronger) assertion Theorem 9.9′ In addition, in carrying over a smooth structure with respect to the membrane $B_l^{n+1}(h)$ we will attempt to carry over the new "equipment" obtained in varying the boundary $M_l^n(h)$ onto the "equipped" smooth sphere $\tilde{S}^{n-k-p}$, $p \geq 0$ (the manifold $M^n$ is $(k-2)$-connected and the manifold $M_l^n(h)$ is $(k+p-1)$-connected). We recall that the manifold $M^n$ was "equipped" and, according to §2, we defined the membrane $B_l^{n+1}(h)$ in such a way that the "equipment" given on the manifold $M^n$ was extended to the "equipment" of the membrane

$$B_l^{n+1}(h) \subset S^{N+n} \times I(0,1), \qquad M^n \subset S^{n+N} \times 0,$$
$$M_l^n(h) \subset S^{N+n} \times 1$$

In this case an obstruction to the carrying over of the new "equipment" (together with the smoothness) of the boundary $M_l^n(h)$ onto the membrane $B_l^{n+1}(h)$ will be the class of cohomologies

$$\begin{aligned}\tilde{\phi}^{n+1-k} \in h^{n+1-k}&(B_l^{n+1}(h), M_l^n(h); G(n-k))\\ &= G(n-k) + \cdots + G(n-k) \qquad (l \text{ terms}).\end{aligned}$$

This obstruction to extending the smoothness and equipment of a boundary onto a membrane falls into the following parts:

1) There is defined on the boundary $\partial\sigma^{n+1-k} = S^{n-k}$ of each simplex

$$\sigma^{n+l-k} \in B_l^{n+1}(h)$$

a new smoothness

$$\tilde{S}^{n-k}(\sigma^{n+1-k}) \in \theta^{n-k}$$

(cf. [12, 23]).

2) There is defined on the boundary $\partial\sigma^{n+1-k}$ a frame field $\tau^N$ that is normal to the entire membrane $B_l^{n+1}(h) \subset S^{N+n} \times I(0,1)$, which has meaning, since the new smoothness is already defined, simultaneously with the new "equipment" of the membrane $B_l^{n+1}(h)$, on a neighborhood of the $(n-k)$-dimensional skeleton.

3) There is defined on $\partial\sigma^{n+1-k}$ a frame field normal to $\partial\sigma^{n+1-k}$ in $B_l^{n+1}(h)$ (in the new smoothness). We denote this frame field by $\tau^k$; it must have meaning in the new smoothness.

4) The smoothness $\tilde{S}^{n-k}$ on $\partial\sigma^{n+1-k}$ and of the field $(\tau^N, \tau^k)$ jointly define an element

$$\alpha(\sigma^{n+1-k}) \in G(n-k);$$

if the smoothness and equipment of $(\tau^N, \tau^k)$ are extended from a neighborhood of the boundary onto a neighborhood of the simplex $\sigma^{n+1-k}$ and define a "smoothness with equipment" on a neighborhood of the $(n-k)$-dimensional skeleton plus a neighborhood of the simplex (cf. [12, 23]), then

$$\alpha(\sigma^{n+1-k}) = 0.$$

According to the preceding results, we can vary the smoothness and equipment on the $(k+p)$-dimensional skeleton of the manifold $M_l^n(h)$ onto an element

$$\alpha \in \sum_{i=1}^{m} G_i(n-k-p),$$ [9]

where $m$ is the number of generators of the group

$$H_{k+p}(M_l^n(h)) = \pi_{k+p}(M_l^n(h)).$$

[9] It is important to note that equipped smooth spheres do not take up the entire group $G(i)$ for $i = 4q+2$, so that $\tilde{\phi}^{n+1-k}$ is not defined everywhere.

To an element $\alpha \in \sum_{i=1}^{m} G_j(n-k-p)$ corresponds an element

$$\tilde{\phi}^{n+1-k}(\alpha) \in \sum_{j=1}^{l} G_j(n-k) = H^{n+1-k}(B_l^{n+1}(h), M_l^n(h); G(n-k)).$$

On the other hand, we have constructed an exact sequence (37)

$$\cdots \to \pi_{N+n}(E^N K^{k+p}(M^n)) \xrightarrow{\triangle} \pi_{N+n}(E^N K^{k+p}(M_l(h))) \xrightarrow{\partial} \sum_{i=1}^{l} G_i(n-k) \to \dots,$$

where

$$\pi_{N+1}(E^N K^{k+p}(M_l(h))) = \sum_{j=1}^{m} G_j(n-k-p),$$

so that

$$\partial\colon \sum_i G_i(n-k-p) \to \sum G_i(n-k).$$

There occurs the following

**Theorem 10.2.** *The homomorphism*

$$\partial\colon \sum_{j=1}^{m} G(n-k-p) \to \sum_{i=1}^{l} G_i(n-k).$$

*of the exact sequence* (37) *coincides in the common domain of definition with the the map* $\tilde{\phi}^{n+1-k}$.

*Sketch of the proof.* The definition of the homomorphism $\partial$ bears an algebraic character, whereas the map $\tilde{\phi}^{n+1-k}$ was defined in terms of geometric concepts. Consequently, in order to establish a connection between them it is necessary to translate the definition of the homomorphism $\tilde{\phi}^{n+1-k}$ into algebraic language. Let us consider the manifold

$$\left(M^n \setminus \bigcup_{i=1}^{l} T(S_i^{k-1})\right) = B^n,$$

where

$$\partial B^n = \bigcup_{i=1}^{l} S_i^{n-k} \times S_i^{k-1}.$$

Clearly,

$$M_l^n(h) = B^n \cup_{h_1,\dots,h_l} \left[\bigcup_{i=1}^{l} S_i^{n-k} \times D_i^k\right]$$

and

$$M^n = B^n \cup \left[\bigcup_{i=1}^{l} D_i^{n-k+1} \times S_i^{k-1}\right].$$

We vary in the manner described above the smoothness of the manifold $M_l^n(h)$ (together with the equipment if it exists) onto an element

$$\alpha \in \sum_{j=1}^{m} G_j(n-k-p), \qquad \alpha = \sum_j \alpha_j,\ \alpha_j \in G_j(n-k-p).$$

Thus the smoothness and equipment are varied only in a neighborhood of the cycles $S_j^{k+p} \subset M_l^n(h)$. The intersection

$$S_j^{k+p} \cdot S_i^{n-k} = M_{ij}^p$$

represents a smooth submanifold $M_{ij}^p \subset S_i^{n-k}$, equipped in the sphere $S_i^{n-k}$ by a frame field that is induced by the coordinate system in a neighborhood of the sphere $S_j^{k+p}$; we assume that the spheres $S_j^{k+p}$ and $S_i^{n-k}$ are orthogonal to each other at the common points of intersection. This equipped manifold defines an element $\beta_{ij} \in \pi_{n-k}(S^{n-k-p})$; under variation of the smoothness of the manifold $M_l^n(h)$ in a neighborhood of the cycle $S_j^{k+p}$ by a sphere $\tilde{S}_j^{n-k-p}(\alpha_j) \in \theta^{n-k-p}(\pi)$ the smoothness on the sphere $S_i^{n-k} \subset M_l^n(h)$ is varied in a tubular neighborhood of the manifold $M_{ij}^p \subset S_i^{n-k}$; namely,

$$T(M_{ij}^p) \subset S_i^{n-k},$$
$$T(M_{ij}^p) = M_{ij}^p \times D_\epsilon^{n-k-p},$$
$$\partial T(M_{ij}^p) = M_{ij}^p \times S_\epsilon^{n-k-p-1}.$$

Let us consider the map

$$\tilde{g}\colon M_{ij}^p \to \operatorname{diff} S_\epsilon^{n-k-p-1},$$

induced by the map

$$g\colon S_j^{k+p} \to \operatorname{diff} S_\epsilon^{n-k-p-1}$$

taking the entire sphere $S_j^{k+p}$ into the point $g(S_j^{k+p})$, where

$$\tilde{S}^{n-k-p}(g(S_j^{k+p})) = \tilde{S}^{n-k-p}(\alpha_j).$$

Further, we set

$$\tilde{S}_i^{n-k}(\alpha_j) = [S_j^{n-k} \setminus T(M_{ij}^p)] \cup_{\tilde{g}} T(M_{ij}^p), \tag{38}$$

where

$$\tilde{g}\colon \partial T(M_{ij}^p) \to \partial T(M_{ij}^p),$$
$$\tilde{g}(x,y) = (x, \tilde{g}(M_{ij}^p) \circ y), \qquad x \in M_{ij}^p,\ y \in S_\epsilon^{n-k-p-1}.$$

The following lemma clarifies the meaning of the elements $\beta_{ij} \in G(p)$.

**Lemma 10.3.** *The complex $T_N^{k+p}(M^n)$ is homotopically equivalent to the union*

$$S^N \vee \left[\left(\bigcup_{j=1}^m D_i^{k+p+N}\right) \cup_{\beta_{ij}} \bigvee_{i=1}^n S_i^{N+k-1}\right],$$

*where $\beta_{ij} \in \pi_{N+k+p-1}(S_i^{N+k-1}) = G(p)$.*

*Proof.* Clearly,

$$M^n = B^n \cup \left[\bigcup_i D_i^{n-k+1} \times S_i^{k+1}\right],$$

where

$$B^p = M_l^n(h) \setminus \left(\bigcup_i S_i^{n-k} \times D_i^k\right);$$

the manifold $M^p_{ij} \subset S^{n-k}_i$ represents the intersection

$$S^{n-k}_i \cdot S^{k+p}_j \subset M^n_l(h), \qquad i = 1, \dots, l,\ j = l, \dots, m.$$

We will assume without further ado that the spheres $S^{n-k}_i$ and $S^{k+p}_j$ intersect at right angles at each point of the manifold $M^p_{ij}$. We consider a tubular neighborhood $\bar{T}(M^p_{ij}) \subset S^{k+p}_j$ of the manifold $M^p_{ij}$ in the sphere $S^{k+p}_i$. Clearly,

$$T(M^p_{ij}) = M^p_{ij} \times D^k_{\epsilon i}$$

and

$$\partial T(M^p_{ij}) = M^p_{ij} \times S^{k-1}_{\epsilon} \subset S^{k+p}_j.$$

We note that on the manifold $M^p_{ij}$ there is equipment normal to $M^p_{ij}$ in $S^{n-k}_i$, on the entire manifold $M^p_{ij} \times S^{k-1}_{\epsilon i}$ there is equipment normal to $M^p_{ij} \times S^{k-1}_{\epsilon i}$ in the manifold

$$\partial T(S^{n-k}_i) = S^{n-k}_i \times S^{k-1}_{\epsilon i},$$

and on $M^p_{ij} \times S^{k-1}_{\epsilon i}$ there is an $N$-frame field normal to the manifold $M^n_l(h)$ in the sphere $S^{N+n}$. We consider the Thom complex $T_N(S^{k-1}_{\epsilon i})$ and note that the sphere $S^{k-i}_{\epsilon i} \subset B^n$ defines in general a nontrivial cycle in the homologies $H_k(M^n)$, where the group $H_k(M^n)$ is generated by the cycles $S^{k-1}_{\epsilon i} \subset B^n$, formed under a passage from $M^n_l(h)$ to $B^n \subset M^n$ as a result of discarding the tubular neighborhoods $T(S^{n-k}_i) \subset M^n_l(h)$. The pair of equipments on the manifold

$$M^p_{ij} \times S^{k-1}_i \subset M^n_l(h) \subset S^{N+n},$$

mentioned above, define together with the natural projection

$$p\colon M^p_{ij} \times S^{k-1}_i \to S^{k-1}_i$$

a map

$$F(\beta_{ij})\colon S^{N+k+p-1} \to T_N(S^{k-1}_i) = S^N \vee S^{N+k-1},$$

satisfying Lemma 3.2 and such that

$$F(\beta_{ij})^{-1}(S^{k-1}_i) = M^p_{ij} \times S^{k-1}_i, \qquad F(\beta_{ij}) = p/M^p_{ij} \times S^{k-1}_i,$$

and the map $F(\beta_{ij})$ is defined on a tubular neighborhood of the manifold $M^p_{ij} \times S^{k-1}$ by the pair of equipments constructed above, that are normal to $M^p_{ij} \times S^{k-1}_i \subset S^{n-k}_i \times S^{k-1}_i$ and to $M^n_l(h) \subset S^{N+n}$. It is easy to see that the map

$$F(\beta_{ij})\colon S^{N+n} \to T_N(S^{k-1})$$

has the homotopy class $\beta_{ij} \circ T_N \gamma_i$, where $\gamma_i$ is a generating element of the group $\pi_{k-1}(S^{k-1}_i)$. We recall that the equipment normal to $M^n_l(h)$ was given on the entire membrane

$$B^{n+1}_l(h) \subset S^{N+n} \times I(0,1)$$

and consequently on the manifold $M^n \subset S^{N+n} \times 0$, where

$$M^n_l(h) \subset S^{N+n} \times 1.$$

Therefore the constructed map

$$\sum_i F(\beta_{ij})\colon S^{N+n} \to T_N\left(\bigvee_i S^{k-1}_i\right)$$

is homotopic to zero in the complex $T_N(M^n)$, since the equipment of

$$\bigcup_i M^p_{ij} \times S^{k-1}_i \subset B^n \subset M^n$$

is already extended onto the membrane

$$\left(S^{k+p}_j \setminus \bigcup_i (M^p_{ij} \times D^k_{\epsilon i})\right) \subset B^n$$

by definition of this equipment, and the equipment normal to the entire manifold $M^n_l(h)$ is extended onto the membrane $B^{n+1}_l(h)$. Therefore the element

$$\sum_i \beta_{ij} \circ T^N \gamma_i \in \pi_{N+k+p-1}(T^{k+p}_N(M^n))$$

is equal to zero. It is easy to see that every element

$$\beta \in \pi_{N+k+p-1}(T^{k-1}_N(M^n))$$

belonging to the kernel of the inclusion homomorphism

$$T^{k-1}_N(M^n) \subset T^{k+p}_N(M^n),$$

is a linear combination of the elements $\sum_i \beta_{ij} \circ T^N_{\gamma_i}$, from which also follows the desired statement. The lemma is proved. □

**Remark.** If $p = 0$, then the manifold $M^p_{ij}$ represents a collection of points and there is defined the index of the intersection

$$\beta_{ij} = S^{k+p}_j \cdot S^{n-k}_i, \qquad i = 1, \dots, l, \ j = 1, \dots, m.$$

The proof of Lemma 10.3 is trivial in this case, and the boundary operator in the complex $T^{k+p}_N(M^n)$ can be expressed in terms of the indices of the intersections $S^{k+p}_j \cdot S^{n-k}_i$ (the elements $\beta_{ij} \in G(0) = Z$ represent integers).

Let us study how the smoothness on the spheres $S^{n-k}_i \supset M^p_{ij}$ is varied under the variation of the smoothness in a tubular neighborhood

$$T(M^p_{ij}) = M^p_{ij} \times D^{n-k-p},$$

described above. Namely,

$$\tilde{S}^{n-k} = (S^{n-k} \setminus T(M^p_{ij})) \cup_g T(M^p_{ij}),^{10}$$
$$g\colon M^p_{ij} \to \operatorname{diff} S^{n-k-p-1},$$

and $g(M^p_{ij})$ consists of one point (one diffeomorphism) corresponding to the sphere $\tilde{S}^{n-k-p}(g) \in \theta^{n-k-p}(\pi)$. We consider separately the manifold

$$M^p_{ij} \times \tilde{S}^{n-k-p}(g)$$

and on it we assign the equipment $\tau^N$ in the sphere $S^{N+n-k}$ in such a way that the equipped manifold

$$M^p_{ij} \times \tilde{S}^{n-k-p}(g) \subset S^{N+n-k}$$

has defined an element of the set

$$\beta_{ij} \circ \tilde{B}(\tilde{S}^{n-k-p}(g)) \in G(n-k).$$

[10]The operation indicated here for varying the smoothness depends essentially on the choice of the map $M^p_{ij} \to SO_{n-k-p}$ defining the normal coordinates.

On the sphere $S^{n-k} \subset S^{N+n}$ we initially assign the null equipment $\tau_0^N$. We consider the equipped map

$$M^{n-k} = (S^{n-k} \cup M_{ij}^p \times \tilde{S}^{n-k-p}(g))$$

in the sphere $S^{N+n-k} \times 0$ and the membrane

$$N_q^{n-k+1} = M^{n-k} \cup_q D^{n-k-p} \times I(0,1) \times M_{ij}^p,$$

where

$$q = (q_0, q_1),$$
$$q_0 \colon D^{n-k-p} \times M_{ij}^p \times 0 \to S^{n-k},$$
$$q_1 \colon D^{n-k-p} \times M_{ij}^p \times 1 \to M_{ij}^p \times \tilde{S}^{n-k-p}(g),$$

and

$$q_i(x,y,i) = (q_{iy}(x), y, i), \qquad i = 0,1,\ q_{iy} \in SO_{n-k-p}.$$

We will assume that

$$N_q^{n-k+1} \subset S^{N+n-k} \times I(0,1),$$

where it is evident that

$$N_q^{n-k+1} \cap S^{N+n-k} \times 0 = M^{n-k},$$
$$N_q^{n-k+1} \cap S^{N+n-k} \times 1 = \tilde{S}^{n-k},$$

and the membrane $N_q^{n-k+1}$ orthogonally approaches the boundaries.

**Lemma 10.4.** *The maps $q_i \colon M_{ij}^p \to SO_{n-k-p}$, $i = 0, 1$, can be chosen in such a way that the equipment $\tau^N \cup \tau_0^N$, given on the manifold $M^{n-k} \subset S^{N+n-k} \times 0$, can be extended onto the entire membrane $N_q^{n-k+1} \subset S^{N+n-k} \times I(0,1)$.*

*Proof.* Since, by condition, the number $p$ is small in comparison to the number $n-k-p$, the natural inclusion homomorphism

$$\pi(M_{ij}^p, SO_{n-k-p}) \to \pi(M_{ij}^p, SO_N)$$

is an epimorphism. Therefore for a fixed map

$$q_0 \colon M_{ij}^p \to SO_{n-k-p}$$

it is possible to select a map $q_1$,

$$q_1 \colon M_{ij}^p \to SO_{n-k-p},$$

such that the equipment $\tau^N \cup \tau_0^N$ is extended from $M^{n-k}$ onto the membrane $N_q^{n-k+1}$, $q = (q_0, q_1)$, since the membrane $N_q^{n+1-k}$ always contracts to the subcomplex

$$M^{n-k} \cup_q 0 \times M_{ij}^p \times I(0,1),$$

and it is sufficient to extend the equipment only onto this subcomplex, which is done in exact analogy with the proof of Lemma 2.1.

The lemma is proved. □

Thus Lemma 10.4 gives us information on the new smoothnesses and equipments on the spheres $S_i^{n-k}$, $i = 1, \dots, l$, under a variation of the smoothness and equipment on the original manifold $M_l^n(h)$. Namely, under a variation of the smoothness (and equipment) on the $j$th base cycle of the group

$$H_{k+p}(M_l^n(h)) = \pi_{k+p}(M_l^n(h))$$

onto the Milnor sphere $\tilde{S}^{n-k-p}(\alpha_j) \in \theta^{n-k-p}(\pi)$ (which, together with the equipment, is the element $\alpha_i$ of the group $G(n-k-p)$), the smoothness and equipment on the sphere $S_i^{n-k}$ define the element

$$\sum_j \beta_{ij} \circ \alpha_j \in G(n-k).$$

Since the homomorphism

$$\partial\colon \pi_{N+n}(T_N^{k+p}(M_l^n(h))) \to \pi_{N+n-1}(T_N^{k-1}(M^n)),$$

constructed above, is defined, as is well known in homotopy topology, so that

$$\alpha \to \sum_{i,j} \alpha_j \circ \beta_{ij},$$

where $\alpha = \sum \alpha_j$ for all

$$\alpha \in \sum_{j=1}^{m} G_j(n-k-p) = \pi_{N+n}(T_N^{k+p}(M_l^n(h))),$$

and the elements

$$\beta_{ij} \in \sum_{j=1}^{l} G_i(p) \subset \pi_{N+k+p-1}(T_N^{k-1}(M^n))$$

possess the properties indicated in Lemma 10.3, our theorem is proved. □

Summarizing the results of Chapter II, we can state that we have partially studied the homotopy structure of a Thom complex, the action of the group $\pi^+(M^n, M^n)$, the operation of the connected sum of a manifold with a Milnor sphere, and the variation of the smoothness along a cycle of minimal nonzero dimension (for the case of $\pi$-manifolds). In addition, we observed the variation in the homotopy sturcture of a Thom complex under Morse reconstructions and, finally, we studied the connection between a variation of the smoothness in a reconstructed manifold and a homomorphism in a certain exact sequence that is closely connected with the homotopy structure of a Thom complex. The study of this latter connection was conducted only for elementary operations of varying the smoothness, but in a forthcoming paper the author will conduct a more complete investigation of the operations of varying the smoothness on manifolds and their connection with the homomorphisms of type $\partial$.

In the next chapter we turn to the derivation of corollaries of the established general theorems and an analysis of examples.

# Chapter III
# Corollaries and appendices

## § 11. Smooth structures on a direct product of spheres

Let us apply the results of the preceding sections to the important example

$$M^n = S^k \times S^{n-k}, \qquad n-k > k.$$

From §7 it follows that

$$T_N(M^n) = S^{N+n} \vee S^{N+n-k} \vee S^{N+k} \vee S^N$$

and

$$\pi_{N+n}(T_N) = Z + G(k) + G(n-k) + G(n).$$

The set $A(M^n)$ consists of all possible elements of the form

$$l_{N+n} + \alpha, \qquad 1_{N+n} \in Z, \qquad \alpha \in G(k) + G(n-k) + G(n),$$

where $1_{N+n} + 0 \in B(S^k \times S^{n-k})$.

We will study the action of the group $\pi(M^n, SO_N)$ on the set $A(M^n)$. It is easy to see that the sequence

$$\pi_n(SO_N) \to \pi(M^n, SO_N) \xrightarrow{p} \pi_{n-k}(SO_N) + \pi_k(SO_N) \to 0 \tag{39}$$

is exact.

**Lemma 11.1.** *If $b \in \pi_n(SO_N) \subset \pi(M^n, SO_N)$, then for each element $1_{N+n} + \alpha \in A(M^n)$ we have*

$$b(1_{N+n} + \alpha) = 1_{N+n} + \alpha + J(b). \tag{40}$$

*Proof.* We discern two maps

$$f_i \colon S^{N+n} \to T_N(M^n), \qquad i = 1, 2,$$

representing respectively the elements $1_{N+n} + \alpha$ and $b(1_{N+n} + \alpha)$ where

$$f_1^{-1}(M^n) = f_2^{-1}(M^n) = M^n_\alpha$$

and

$$f_1|M^n_\alpha = f_2|M^n_\alpha.$$

But in a tubular neighborhood $T(M^n_\alpha)$ the maps $f_1$ and $f_2$ are discriminated by an element

$$b \in \pi_n(SO_N) \subset \pi(M^n, SO_N),$$

and this discriminator is concentrated near a point $x_0 \in M^n_\alpha$. It is also possible to say: the manifold $M^n_\alpha$ is equipped in two distinct ways $\tau_i^N$, $i = 1, 2$, and these equipments differ only near the point $x_0$ by the element $b \in \pi_n(SO_N)$. In this case there exists on the sphere $S^n$ the equipment $\tau^N$, corresponding to the element $b$, such that for the equipped manifolds $(\tau_1^N, M^n_\alpha)$, $(\tau_2^N, M^N_\alpha)$, $(\tau^N, S^n)$ we have

$$(\tau_1^N, M^n_\alpha) \mathbin{\#} (\tau^N, S^n) = (\tau_2^N, M^n_\alpha).$$

Therefore the equipments $\tau_i^N$, $i = 1, 2$, on the manifold $M^n_\alpha$ are discriminated by an equipped sphere $S^n$, and in the homotopy groups $\pi_{N+n}(T_N)$

$$b(1_{N+n} + \alpha) = 1_{N+n} + \alpha + J(b).$$

The lemma is proved. □

**Lemma 11.2.** *If $a \in \pi(M^n, SO_N)$ and $p(a) \in \pi_{n-k}(SO_N)$, then for each element $1_{N+n} + \alpha \in B(S^k \times \tilde{S}^{n-k})$*

$$a(1_{N+n} + \alpha) = 1_{N+n} + \alpha + J(p(a)) \pmod{\operatorname{Im} \kappa_* \in G(n)}. \tag{41}$$

*Proof.* Suppose, as above,

$$f_i \colon S^{N+n} \to T_N(M^n), \qquad i = 1, 2,$$

represent the elements $a(1_{N+n} + \alpha)$ and $1_{N+n} + \alpha$, where the manifold

$$M^n_\alpha = f_i^{-1}(M^n), \qquad i = 1, 2,$$

is diffeomorphic to the manifold $S^k \times \tilde{S}^{n-k}$ and is equipped in two distinct ways. These equipments $\tau_i^N$, $i = 1, 2$, are discriminated by the base cycle $\tilde{S}^{n-k} \subset M_\alpha^n$ and, moreover,

$$f_1 | M_\alpha^n = f_2 | M_\alpha^n.$$

We select the standard frame $\tau_0^k$, that is tangent to $S^k$ at the point $x_0 \in S^k$, and discern the frame fields $(\tau_i^N, \tau_0^k)$ on the sphere

$$x_0 \times \tilde{S}^{n-k} \subset M_\alpha^n,$$

that are discriminated by an element $j_* p(\alpha)$, where

$$j_* \colon \pi_{n-k}(SO_N) \to \pi_{n-k}(SO_{N+k}).$$

We discern separately the manifold

$$S^k \times S^{n-k} \subset S^{N+n}$$

and assign on the cycle $x_0 \times S^{n-k}$ the equipment $\tau^{N+k}$, defined by the element $j_* p(a)$, where the last $k$ vectors are tangent to the factor $S^k$, and the first $N$ vectors are normal to $S^k \times S^{n-k}$ (also given on $x_0 \times S^{n-k}$). We extend this frame field $\tau^N$, defined by the first $N$ vectors of the frame $\tau^{N+k}$, onto all of the manifold

$$S^k \times S^{n-k} \subset S^{N+n},$$

which is possible; then we define a map

$$F \colon S^k \times S^{n-k} \to S^k \times y_0 \subset S^k \times S^{n-k},$$

putting $F(x, y) = x$. We discern an element $\beta$ of the group $\pi_{N+n}(T_N^k(M^n))$, defined by the extended equipment and the map $F$ and representable, clearly, by a map

$$f_\beta \colon S^{N+n} \to T_N^k$$

such that

$$f_\beta^{-1}(S^k) = S^k \times S^{n-k}, \qquad f_\beta | S^k \times S^{n-k} = F.$$

It is easy to see that the sum $1_{N+n} + \alpha + \beta$ is represented by the map

$$(f_2 + f_\beta) \colon S^{N+n} \to T_N,$$

where

$$\begin{aligned}(f_\alpha + f_\beta)^{-1}(S^k \times S^{n-k}) &= (S^k \times \tilde{S}^{n-k}) \cup (S^k \times S^{n-k}) \\ &= f_1^{-1}(S^k \times S^{n-k}) \cup f_\beta^{-1}(S^k \times S^{n-k}).\end{aligned}$$

In analogy with §10 we make use of the "connected sum

$$S^k \times \tilde{S}^{n-k} \#_\gamma S^k \times S^{n-k}$$

along a cycle" $\gamma = S^k$ for the equipped manifolds $S^k \times \tilde{S}^{n-k}$ and $S^k \times S^{n-k}$ to construct the map

$$(\widetilde{f_2 + f_\beta}) \colon S^{N+n} \to T_N$$

of homotopy class $1_{N+n} + \alpha + \beta$ such that

$$\begin{aligned}&(\widetilde{f_2 + f_\beta})^{-1}(S^k \times S^{n-k}) \\ &\qquad = (S^k \times S^{n-k} \#_\gamma S^k \times \tilde{S}^{n-k}) = S^k \times \tilde{S}^{n-k} \bmod \theta^n.\end{aligned}$$

The map $(\widetilde{f_2 + f_\beta})$, considered on $S^k \times \tilde{S}^{n-k}$, coincides with both of the maps $f_1$ and $f_2$ on $S^k \times \tilde{S}^{n-k}$, and in a tubular neighborhood is different from $f_1$ only in the neighborhood of a point (the discrinainator between them has a value in dimension $n$, since we killed the discriminator $p(a)$ on the $(n-k)$-dimensional skeleton). Thus we conclude that

$$1_{N+n} + \alpha + \beta = a(1+\alpha) \bmod \operatorname{Im} \kappa_* \subset G(n)$$

according to Lemma 1. By virtue of Theorem 9.9 (or its modification, Theorem 9.9′, given in §10),

$$\beta = Jp(a) \circ E^N_\gamma \bmod \operatorname{Im} \kappa_*,$$

where $\gamma$ is the fundamental class of the sphere $S^k$.

The lemma is proved. □

We will now study the action of the group $\pi^+(M^n, M^n)$ on the set $A(M^n)$, beginning with the results of §7.

It is easy to see that

$$\pi^\nu_n(S^k \times S^{n-k}) = \pi_n(S^k \times S^{n-k}) = \pi_n(S^k) + \pi_n(S^{n-k})$$

and that the sequence

$$0 \to \pi_n(S^k) + \pi_n(S^{n-k}) \to \pi^+(S^k \times S^{n-k})$$

is exact. Since $n-k > k$, the homomorphism

$$T^N = E^N \colon \pi_n(S^k) \to G(k) \subset \pi_{N+n}(T_N(M^n))/G(n)$$

constructed in §7, is an epimorphism. Applying Lemma 7.6, we obtain the following statement.

**Lemma 11.3.** *The set $B(M^n_\alpha)$ contains all elements of the form*

$$1_{N+n} + \alpha + \beta \quad (\bmod G(n)),$$

*where $\beta \in G(k)$, $\alpha \in G(k) + G(n-k) + G(n)$.*

*Proof.* Let $\gamma \in \pi_n(S^{n-k}) \subset \pi^+(M^n, M^n)$. According to §6, the group $\pi^+(M^n, M^n)$ acts on the set

$$B(1_{N+n} + \alpha) \subset A(M^n)$$

and, according to §7 (Lemma 7.6), we have

$$\gamma(1_{N+n} + \alpha) = E^N\gamma + 1_{N+n} + \alpha \quad (\bmod G(n));$$

but the homomorphism $E^N$ is an epimorphism, from which follows the desired statement. The lemma is proved. □

A comparison of Lemmas 11.2 and 11.3 and the results of §10 leads to the following lemma.

**Lemma 11.4.** *For each smooth sphere $\tilde{S}^{n-k} \subset \theta^{n-k}(\pi)$ the set*

$$B(S^k \times \tilde{S}^{n-k}) \subset A(M^n)$$

*contains all elements of the form*

$$1_{N+n} + \tilde{B}(\tilde{S}^{n-k}) + G(k) \quad (\bmod G(n)),$$

*where to the element $1_{N+n} + 0$ corresponds the manifold*

$$M^n = S^k \times S^{n-k},$$

*and the set $\tilde{B}(\tilde{S}^{n-k})$ represents a coset* $\bmod \operatorname{Im} J$ *in the group $G(n-k)$.*

The proof of the lemma consists of a formal comparison of the preceding lemmas.

**Theorem 11.5.** 1) *If $n-k \not\equiv 2 \bmod 4$, then each element of the set $A(M^n) \bmod G(n)$ belongs to one of the sets $B(S^k \times \tilde{S}^{n-k})$, $\tilde{S}^{n-k} \in \theta^{n-k}(\pi)$, and there exists the following embedding*

$$B(S^k \times \tilde{S}^{n-k}) \supset 1_{N+n} + \tilde{B}(\tilde{S}^{n-k}) + G(k) \bmod G(n). \tag{42}$$

*For any pair $\tilde{S}^k \in \theta^k(\pi)$, $\tilde{S}^{n-k} \in \theta^{n-k}(\pi)$ there exists a smooth sphere $\tilde{S}_1^{n-k} \in \theta^{n-k}(\pi)$ such that*

$$B(\tilde{S}^k \times \tilde{S}^{n-k}) = B(S^k \times \tilde{S}_1^{n-k}) \bmod G(n). \tag{43}$$

2) *If a manifold $M_1^n$ is such that*

$$B(M_1^n) \neq B(\tilde{S}^k \times \tilde{S}^{n-k}) \bmod G(n)$$

*for $\tilde{S}^k \in \theta^k$, $\tilde{S}^{n-k} \in \theta^{n-k}$, then the manifold $M_1^n$ is not combinatorially equivalent to the manifold $M^n = S^k \times S^{n-k}$.*

3) *If $B(M_1^n) = B(M_2^n) \bmod G(n)$, then the manifolds $M_1^n$ and $M_2^n$ are diffeomorphic modulo a point.* [11]

*Proof.* If $n-k \not\equiv 2 \bmod 4$, then $\tilde{\theta}(n-k) = G(n-k)$ and, according to Lemma 11.4, every element of the set $A(M^n)$ belongs to one of the sets of the form

$$B(S^k \times \tilde{S}_1^{n-k}) \bmod G(n),$$

from which follows assertion 1).

If $n-k \equiv 2 \bmod 4$ and $G(n-k)/\tilde{\theta}(n-k) = Z_2$ (cf. [6]), then it is possible to have a situation such that

$$B(M_1^n) \neq B(\tilde{S}^k \times \tilde{S}^{n-k}) \bmod G(n)$$

for $\tilde{S}^k, \tilde{S}^{n-k}$ such that $\tilde{S}^k \times \tilde{S}^{n-k}$ is a $\pi$-manifold. We assume in the latter case, arguing by contradiction, that $M_1^n$ is combinatorially equivalent to $S^k \times S^{n-k}$ and some map

$$f\colon M_1^n \to S^k \times S^{n-k}$$

effects this combinatorial equivalence. According to [11] there arises a first obstruction

$$p^k(f) \in H^{n-k}(M_1^n, \theta^k) = \theta^k,$$

i.e., $p^k(f) \in \theta^k$ and to the element $p^k(f)$ corresponds the sphere $\tilde{S}^k \in \theta^k$.

We consider the standard combinatorial equivalence

$$f_0\colon S^k \times S^{n-k} \to \tilde{S}_1^k \times S^{n-k}, \qquad \tilde{S}_1^k = -p^k(f),$$

such that

$$p^k(f_0) = -p^k(f) = \tilde{S}_1^k \in \theta^k.$$

Clearly,

$$p^k(f_0 \circ f) = p^k(f) + p^k(f_0) = 0.$$

We consider the second obstruction

$$p^{n-k}(f_0 \cdot f) \in H^k(M_1^n, \theta^{n-k}) = \theta^{n-k},$$

the sphere

$$\tilde{S}_1^{n-k} = -p^{n-k}(f_0 \cdot f)$$

and the map

$$f_1\colon \tilde{S}_1^k \times S^{n-k} \to \tilde{S}_1^k \times \tilde{S}_1^{n-k}.$$

---

[11] $\tilde{\theta}(n-k) \subset G(n-k)$ consists of equipped Milnor spheres.

Clearly,

$$p^{n-k}(f_1 \circ f_0 \cdot f) = p^{n-k}(f_1) + p^{n-k}(f_0 \cdot f) = 0.$$

According to the results of the papers [9, 11, 17] the manifolds $M_1^n$ and $\tilde{S}_1^k \times \tilde{S}_1^{n-k}$ are diffeomorphic modulo a point, and from §9 we have

$$B(M_1^n) \equiv B(\tilde{S}_1^k \times \tilde{S}_1^{n-k}) \bmod G(n).$$

Thus we arrive at a contradiction with our assumption, and therefore assertion 2) is proved. As to assertion 3), it was essentially proved in §9 (cf. Lemma 9.1).

The theorem is proved. [12] □

**Remark.** Since it is always possible to smoothly realize a sphere $\tilde{S}^{n-k} \subset \theta^{n-k}(\partial\pi)$ in the space $R^n$ for $k \geq 2$, it follows from a paper of Smale [19] that $\tilde{S}^{n-k} \times D^{k+1}$ is diffeomorphic to $S^{n-k} \times D^{k+1}$, $k \geq 2$. Therefore $\tilde{S}^{n-k} \times S^k$ is diffeomorphic to $S^{n-k} \times S^k$.

**Corollary 11.6.** *If $n - k \not\equiv 2 \bmod 4$, then every direct product $\tilde{S}^k \times \tilde{S}_1^{n-k}$ is diffeomorphic modulo a point to a direct product $\tilde{S}^k \times \tilde{S}_2^{n-k}$ for some sphere $\tilde{S}_2^{n-k}$, where*

$$\tilde{S}^k \in \theta^k(\pi), \qquad \tilde{S}_i^{n-k} \in \theta^{n-k}(\pi), \qquad i = 1, 2,\ k \geq 2,\ n-k > k.$$

This fact immediately follows from Theorem 11.5 and Lemma 9.1.

**Example 1.** Let $M^n = S^2 \times S^6$. Then $\pi(M^n, SO_N) = Z_2$ and the sequence

$$0 \to \pi_8(S^2) + \pi_8(S^6) \to \pi^+(S^2 \times S^6, S^2 \times S^6) \xrightarrow{q} \pi_6(S^2) + Z_2 \to 0$$

is exact. Further,

$$T_N(S^2 \times S^6) = S^{N+8} \vee S^{N+6} \vee S^{N+2} \vee S^N,$$

the set $A(M^n) = \tilde{A}(M^n)$ consists of all elements of the form

$$1_{N+n} + G(2) + G(6) + G(8)$$

and

$$B(S^2 \times S^6) \supset 1_{N+n} + 0.$$

What is the action of the group $\pi^+(M^n, M^n)$? If $a \in \pi_8(S^2)$ and $b \in \pi_8(S^6)$, then, according to §7, we have

$$(b + a)(1_{N+n} + \alpha) \equiv 1_{N+n} + \alpha + E^N a + E^N b \bmod G(8). \tag{44}$$

We discern the subgroup $Z_2 \in \pi^+(M^n, M^n)$, generated by a diffeomorphism

$$f\colon S^2 \times S^6 \to S^2 \times S^6$$

such that $f(x, y) = (-x, -y)$.

According to §6 we have

$$T^N f(1_{N+n} + \alpha) = 1_{N+n} - \alpha \bmod G(8). \tag{45}$$

We know that $\pi_6(S^2) = Z_{12}$; let $\eta$ be a generating element of the group $\pi_6(S^2) = Z_{12}$ and $\tilde{\eta} \in q^{-1}(\eta)$. Let also $\alpha \in G(2) + G(6)$. We will show that

$$\tilde{\eta}(1_{N+n} + \alpha) = 1_{N+n} + \alpha \bmod G(2) + G(8).$$

[12] In part II it will be proved that if the factor group $G(n)/\tilde{\theta}(n) = Z_2$, then for all $M^n$ the set $\tilde{A}(M^n)$ contains half (and only half) of the set $A(M^n)$, $n = 4k + 2$.

According to the results of §6 the map

$$f_{\tilde\eta}\colon S^2 \times S^6 \times S^2 \times S^6,$$

representiag the element $\tilde\eta \in \pi^+(S^2 \times S^6, S^2 \times S^6)$, induces a map

$$E^N f_{\tilde\eta}\colon E^N(S^2 \times S^6) \to E^N(S^2 \times S^6)$$

and, since $T_N(S^2 \times S^6) = S^N \vee E^N(S^2 \times S^6)$, it follows from §6 that

$$E^N f_{\tilde\eta *}(1_{N+n} + \alpha) = \tilde\eta(1_{N+n} + \alpha) \bmod G(8).$$

We consider the map

$$f_{\tilde\eta}\colon E^N(S^2 \times S^6) \to E^N(S^2 \times S^6).$$

We note that the space $E(S^2 \times S^6)$ is homotopically equivalent to the complex $S^3 \vee S^7 \vee S^9$ and that

$$\pi_9(E(S^2 \times S^6)) = \pi_9(S^3) + \pi_9(S^7) + \pi_9(S^9) + \operatorname{Ker} E^{N-1},$$

where

$$\pi_9(S^3) = Z_3, \qquad \pi_9(S^7) = Z_2, \qquad \pi_9(S^9) = Z.$$

It is evident that

$$E f_{\tilde\eta}(\lambda_9) = \lambda_9 + \mu_9^{(1)} + \mu_9^{(2)} \pmod{\operatorname{Ker} E^{N-1}},$$

where

$$\mu_9^{(1)} \in \pi_9(S^3), \qquad \mu_9^{(2)} \in \pi_9(S^7), \qquad \lambda_9 \in \pi_9(E(S^2 \times S^6)).$$

Since

$$E^N f_{\tilde\eta}(1_{N+n} + \alpha) = 1_{N+n} + \alpha + E^{N-1}(\mu_9^{(1)} + \mu_9^{(2)})$$

and

$$E^{N-1}(\mu_9^{(1)}) = 0, \qquad E^{N-1}(\mu_9^{(2)}) \in G(2),$$

we get that

$$\tilde\eta(1_{N+n} + \alpha) \equiv 1_{N+n} + \alpha \pmod{G(2) + G(8)}.$$

We have thus proved that the set $A(S^2 \times S^6)$ decomposes into the following sets:

a) $\bigcup_{\tilde S^8 \in \theta^8} B(S^2 \times S^6 \# \tilde S^8) = 1_{N+n} + G(2) + G(8)$.

b) Since $G(6) = Z_2$ and $G(6) \neq \operatorname{Im} E^{N-1}\pi_8(S^2)$, the set

$$A(S^2 \times S^6) \setminus \bigcup_{\tilde S^8 \in \theta^8} B(S^2 \times S^6 \# S^8)$$

is not empty. There exists a $\pi$-manifold $M_1^n$ of the homotopy type of $S^2 \times S^6$, that is not diffeomorphic to $S^2 \times S^6 \bmod \theta^8$ .

c) Since $\theta^2 = \theta^6 = 0$, we find that the manifold $M_1^n$ is not combinatorially equivalent to $S^2 \times S^n$.

**Corollary 11.7.** *There exist simply connected manifolds, which are not combinatorially equivalent, having the homotopy type of* $S^2 \times S^6$*.*

## § 12. Manifolds of small dimension.[13] The case $n = 4, 5, 6, 7$.

Let $M^n$ be a simply connected manifold of dimension $n$. We consider the Thom complex $T_N(M^n)$ and the Thom isomorphism

$$\phi\colon H^i(M^n) \to H^{N+i}(T_N(M^n)), \qquad i \geq 0.$$

As usual, $u_N \in H^N(T_N)$ denotes the fundamental class of the Thom complex. Let $\bar{w}_i \in H^i(M^n, Z_2)$ be the Stiefel–Whitney normal classes. A well-known fact is the following

**Lemma 12.1.** *There exists the formula*

$$\phi(\bar{w}_i) = Sq^i u_N. \tag{46}$$

The proof of this lemma belongs (in the case of a tangent bundle and its Thom complex) to Thom [21] and Wu [26] and is analogous for the Thom complexes of any bundle (in our case a normal one).

If $p_1 \in H^4(M^n, Z_3)$ denotes the Pontrjagin class of a normal bundle, reduced modulo 3, then (for $n \geq 6$) there exists the analogous formula

$$\phi(p_1) = P^1 u_N \tag{46'}$$

where

$$P^1\colon H^k(x, Z_3) \to H^{4+k}(x, Z_3)$$

is a Steenrod square. For $n = 4$ the Pontrjagin class is equal to $\tau/3$, where $\tau$ is the signature of the manifold $M^n$ (cf. [16, 3]) and for $n = 5$ the class $p_1$ is equal to zero in view of the simple connectedness of the manifold $M^5$.

Let $n = 4$. Then there holds the following

**Lemma 12.2.** *The group $\pi(M^4, SO_N)$ is trivial for any simply connected manifold $M^4$.*

The proof of the lemma follows from the fact that

$$\pi_2(SO_N) = \pi_4(SO_N) = 0.$$

It is also easy to prove

**Lemma 12.3.** *The map*

$$T^N\colon \pi_4(M^4, \nu^N(M^4)) \to \pi_{N+4}(T_N^2(M^4))$$

*is an epimorphism for any simply connected manifold $M^4$; the group* $\operatorname{Im} \kappa^*(\pi_{N+4}(S^N))$ *is equal to zero.*

*Proof.* Since the group $G(4)$ is equal to zero, the image $\operatorname{Im} \kappa_*$ is trivial. Inasmuch as the suspension homomorphism

$$E^N\colon \pi_4(S^2) \to G(2)$$

is an epimorphism, the map

$$T^N\colon \pi_4(K^2(M^4), \nu^N) \to T_N^2(M^4),$$

which easily reduces to a suspension homomorphism, is also an epimorphism (we note that $\pi_4(K^2(M^4), \nu^N) = \pi_4(K^2(M^4))$. Since the natural map

$$\pi_4(K^2(M^4), \nu^N) \to \pi_4(M^4, \nu^N(M^n))$$

---

[13] A detailed proof of the theorem of this section will be given in a subsequent part of the paper.

is an epimorphism, the lemma is proved. □

Taking into account the fact that

$$T_N(M^4) = T_N^2(M^n) \vee S^{N+4},$$

we obtain the following statement.

**Theorem 12.4.** *The set* $B(M^4) \subset A(M^4) \subset \pi_{M+4}(T_N)$ *coincides with the whole set* $A(M^4)$. *Therefore*

$$\tilde{A}(M^4) = A(M^4) = B(M^4)$$

*and every two simply connected homotopically equivalent four-dimensional manifolds are* $J$*-equivalent.*

The proof of the theorem is obtained directly from Lemma 12.3 and the results of §7.

**Lemma 12.5.** 1) *If* $n = 5, 6$, *then there is defined a canonical epimorphism*

$$H^3(M^n, Z) \to \pi(M^n, SO_N).$$

2) *If* $n = 7$, *then the sequence*

$$Z = \pi_7(SO_N) \to (M^7, SO_N) \to H^3(M^7, Z) \to 0$$

*is exact.*

*Proof.* Since

$$\pi_7(SO_N) = \pi_3(SO_N) = Z$$

and

$$\pi_2(SO_N) = \pi_4(SO_N) = \pi_5(SO_N) = \pi_6(SO_N) = 0, \qquad \pi_1(M^n) = 0,$$

the lemma is trivial, as follows from the theory of obstructions to homotopy maps. □

Let us investigate the action of the group $\pi(M^n, SO_N)$ on the set

$$\tilde{A}(M^n) \subset \pi_{N+n}(T_N(M^n)).$$

We note that the filtration

$$T_N \supset T_N^{n-2} \supset \cdots \supset T_N^2 \supset S^N$$

for $n \le 7$ consists of not more than six terms. Taking into account the fact that $G(4) = G(5) = 0$, we find that there are defined the exact sequences

$$\pi_{N+n}(T_N^{n-3}) \xrightarrow{\Delta^{(2)}} \pi_{N+n}(T_N^{n-2}) \xrightarrow{\Delta} \sum_{i=1}^{l} G_i(2),$$

$$\pi_{N+n}(T_N^{n-4}) \xrightarrow{\Delta^{(3)}} \pi_{N+n}(T_N^{n-3}) \to \sum_{j=1}^{m} G_j(2),$$

$$l = \operatorname{rk} H_2(M^n, Z_2), \qquad m = \operatorname{rk} H_3(M^n, Z_{24}),$$

$$G(n) \to \pi_{N+n}(T_N^{n-4}) \to 0,$$

$$G(n) \to \pi_{N+n}(T_N^{n-3}) \to \sum G_j(3),$$

for $n \leq 7$. These exact sequences are induced by the exact sequences (33)–(34). We note that

$$G(2) = Z_2, \qquad G(3) = Z_{24} = \operatorname{Im} J,$$
$$G(6) = Z_2, \qquad G(7) = Z_{240} = \operatorname{Im} J.$$

One easily proves

**Lemma 12.6.** *For $n = 6$ the set $\tilde{A}(M^n)$ contains half as many elements as the set $A(M^n)$.*

*If $\alpha \in \tilde{A}(M^n)$ and $\beta \in G(6)$, $\beta \neq 0$, then $\alpha + \beta \in A(M^n)$, but $\alpha + \beta \notin \tilde{A}(M^n)$.*

*Proof.* We consider an admissible map

$$f_\alpha \colon S^{N+6} \to T_N(M^6)$$

such that the manifold $M^6_\alpha = f_\alpha^{-1}(M^6)$ is homotopically equivalent to $M^6$. We also consider a map

$$F_\beta \colon S^{N+6} \to S^N$$

such that

$$F_\beta^{-1}(x_0) = S^3 \times S^3,$$

where $x_0 \in S^N$. The inverse image

$$F_\beta^{-1}(x_0) = S^3 \times S^3 \subset S^{N+6}$$

is an equipped manifold, and on the cycles

$$S^3 \times x \subset S^{\times} S^3$$

and

$$y \times S^3 \subset S^3 \times S^3$$

is defined an invariant $\phi \in Z_2$, obstructing the carrying over of equipment under a Morse reconstruction (cf. §§2, 4). The sum of maps

$$(F_\beta + f_\alpha) \colon S^{N+6} \to T_N(M^6)$$

represents the element $\alpha + \beta$ and

$$(F_\beta + f_\alpha)^{-1}(M^6) = S^3 \times S^3 \cup M^6_\alpha.$$

By means of a Morse reconstruction it is possible to vary the map $(F_\alpha + f_\alpha)$ in such a way that the inverse image of the manifold $M^6$ under the new map $(\widetilde{f_\alpha + F_\beta})$, homotopic to $(F_\beta + f_\alpha)$, is an equipped connected sum

$$M^6_1 = M^6_\alpha \# S^3 \times S^3$$

in analogy with §§4 and 9. There is defined on the cycles $y \times S^3$ and $S^3 - x \subset M^6_1$ an invariant $\psi \in Z_2$, obstructing a Morse reconstruction. There is defined an invariant $\psi(\alpha + \beta) \neq 0$, obstructing a simplification of the inverse image $M^6_1$ by Morse reconstructions (in view of the obstruction $\psi$ to a carrying over of frame fields). It is easy to see that the invariant $\psi$ is defined correctly, and the class $\alpha + \beta \notin \tilde{A}(M^6)$.

The lemma is proved. □

Since $G(3) = \operatorname{Im} J$ and $G(7) = \operatorname{Im} J$, from Lemma 12.5 and the definition of the homomorphism $J$ we easily obtain the following statement.

**Lemma 12.7.** *For each element $\alpha \in \tilde{A}(M^n)$ the orbit $\pi(M^n, SO_N) \circ \alpha$ for $n \leq 7$ contains all elements of the form $\alpha + \beta$, where*

$$\beta \in \Delta_*^{(2)} \pi_{N+n}(T_N^{n-3}) \subset \pi_{N+n}(T_N^{n-2}) \subset \pi_{N+n}(T_N(M^n))$$

(*here $\Delta_*$ is an inclusion homomorphism $\Delta\colon T_N^{n-3} \subset T(N^{n-2})$ in the exact sequence* (33)).

The proof follows from the fact that the sequence

$$G(n) \to \pi_{N+n}(T_N^{n-3}) \to \sum_j G_j(3)$$

is exact for $n \leq 7$, and from Lemma 12.6 (for the case $n = 6$).

**Lemma 12.8.** *The image of the composition of homomorphisms*

$$\Lambda \cdot T^N \colon \pi_n^\nu(M^n) \to \sum_{i=1}^{l} G_i(2)$$

*coincides with the image of the homomorphism $\Lambda$.*

The proof of the lemma easily follows from the form of nonstable homotopy groups of spheres in small dimensions ($\leq 7$), die structure of the suspension homomorphism $E^N$, and the definition of the homomorphism $T^N$, having all the properties that are analogous to the properties of a suspension homomorphism (cf. §7).

Comparing the lemma and the results of §§1–7, we obtain the following statement.

**Theorem 12.9.** *For $n \leq 7$ the sets $\tilde{A}(M^n)$ and $B(M^n) \subset \tilde{A}(M^n)$ coincide.*

**Remark.** A more extensive investigation of the properties of the homomorphism $T^N$ and of the connection of the homomorphism $J$ with the action of the group $\pi(M^n, SO_N)$ will be carried out in a following paper.

## § 13. The connected sum of a manifold with a Milnor sphere

Using the results of §9, we will study the problem of determining when the manifolds $M^n$ and $M^n \# \tilde{S}^n$ are diffeomorphic with degree $+1$ (mod $\theta^n(\partial\pi)$).

According to Lemma 9.1, for this purpose it is necessary to study the structure of the homomorphism $\kappa_*\colon G(n) \to \pi_{N+n}(T_N(M^n))$, where $\kappa\colon S^N \subset T_N(M^n)$ represents the natural embedding of a fiber

$$D_x^N \subset \nu^N(M^n), \qquad x \in M^n,$$

of which the boundary $\partial D_x^N$ contracts to a point under passage to the complex $T_N(M^n)$. According to Lemma 9.1 we have

$$B(M^n \# \tilde{S}^n) = B(M^n) + \kappa_* \tilde{B}(\tilde{S}^n),$$

where $\tilde{B}(\tilde{S}^n) \subset G(n)$ represents a coset mod $\operatorname{Im} J$. There holds the following

**Lemma 13.1.** *If in the set $\tilde{B}(\tilde{S}^n)$ there exists an element $\beta \in \tilde{B}(\tilde{S}^n) \subset G(n)$ such that $\kappa_* = 0$, then the manifolds $M^n$ and $M^n \# \tilde{S}^n$ are diffeomorphic* $\operatorname{mod} \theta^n(\partial\pi)$; *in this case there exists a sphere $\tilde{S}_1^n \in \theta^n(\partial\pi)$ such that the manifolds $M^n$ and $M^n \# (\tilde{S}^n \# \tilde{S}_1^n)$ are diffeomorphic with degree* 1.

*Proof.* Let $\kappa_*\beta = 0$, where $\beta \in \tilde{B}(\tilde{S}^n)$. Then the intersection $B(M^n) \cap B(M^n \# \tilde{S}^n)$ is not empty and therefore

$$B(M^n) = B(M^n \# \tilde{S}^n).$$

Applying the results of §6, we obtain the first of the statements of the lemma.

The second statement of the lemma follows from the associativity of the operation #. The lemma is proved. □

We now attempt to find examples of manifolds $M^n$ for which the homomorphism $\kappa_*$ has a nontrivial kernel.

We consider an $SO$-bundle $\nu$ with fiber $S^m$ and with base $S^l$, where $m \geq l+1$. The bundle $\nu$ is defined by a certain element $h \in \pi_{l-1}(SO_{m+1})$. We denote by means of $M^n$ the space of the bundle $\nu$, $h = m + l$. There holds

**Lemma 13.2.** *The complex $T_N^l(M^n)$ is homotopically equivalent to the complex $D^{N+l} \cup_{Jh} S^N$, where $Jh \in G(l-1)$.*

*Proof.* We consider the bundle $j^*\nu^N(M^n)$, which is a restriction of the normal bundle on the skeleton

$$K^l(M^n) = S^l \overset{j}{\subset} M^n$$

of dimension $l$. It is easy to see that the normal bundle $j^*\nu^N(M^n)$ is defined by an invariant

$$\pm h \in \pi_{l-1}(SO_N) \approx \pi_{l-1}(SO_{m+1})$$

since $m \geq l+1$. Clearly, the complexes $T_N^l(M^n)$ and $T_N(S^l, j^*\nu^N(M^n))$ coincide, and, by the definition of Milnor [7] of the homomorphism $J$, we obtain the desired statement. The lemma is proved. □

**Lemma 13.3.** *Suppose, as above, $h \in \pi_{l-1}(SO_{m+1})$, $m \geq l+1$ and $\alpha \in G(m+1)$, where the element $\alpha \cdot Jh \notin \operatorname{Im} J$. Then there exists a Milnor sphere $\tilde{S}^{m+l}$ such that $\alpha \cdot Jh \in \tilde{B}(\tilde{S}^{m+l})$ and the manifolds $M^n$ and $M^n \# \tilde{S}^{m+l}$, $n = m + l$, are diffeomorphic with degree $+1$ modulo $\theta^n(\partial\pi)$.* [14]

*Proof.* Clearly, the element $\alpha \cdot Jh$ belongs to the kernel $\operatorname{Ker} \kappa_*$. If $n \not\equiv 2 \bmod 4$, then the lemma follows from the preceding statements and the results of Milnor and Kervaire (cf. [6, 8]). If $m \not\equiv 1 \bmod 4$, then it is also possible to compute the element $\alpha \in G(m+1)$ by an equipped smooth sphere $\tilde{S}_\alpha^{m+1}$, and the element $\alpha \cdot Jh$ by an equipped direct product $\tilde{S}^{m+1} \times S^{l-1}$; in this case it is possible by a Morse reconstruction to kill the cycles of dimensions $l-1$ and $m+1$, after which the element $\alpha \cdot Jh$ is realized by an equipped homotopy sphere, and the lemma is proved. If $m+1 \equiv 2 \bmod 4$ and $m+1 \equiv 2 \bmod 4$, then the element $\alpha$ is realized by an equipped manifold $Q^{m+1}$ such that

$$\pi_1(Q^{m+1}+1) = 1, \qquad H_i(Q^{m+1}) = 0, \qquad i \neq 0, \frac{m+1}{2}, m+1.$$

and the group

$$H_{\frac{m+1}{2}}(Q^{m+1}) = Z + Z,$$

where on the base cycles $Z_1, Z_2 \in H_{(m+1)/2}$ is defined the Kervaire invariant

$$\phi(Q^{m+1}) = \phi(\alpha) \in Z_2$$

[14] In this regard cf. also the paper [32].

(or $\psi(\alpha) \in Z_2$ if $m+1 = 6, 14$). The element $\alpha \cdot Jh$ is realized by an equipped direct product $Q^{m+1} \times S^{l-1}$. By means of Morse reconstructions we seal the cycle

$$Z_i \otimes 1 \in H_{\frac{m+1}{2}}(Q^{m+1} \times S^{l-1})$$

and then a cycle of dimension $l-1 < m+1$. Since the homologies are torsion-free, no new cycles are formed; it is possible to carry out the Morse reconstructions and the carrying over of equipment since $(m+1)/2 < [n/2]$ and $l-1 < [n/2]$. The element $\alpha \cdot Jh$ will be realized by a smooth sphere with equipment. The lemma is proved. □

In the paper [13] there is indicated a multiplication table for homotopy groups of spheres. In particular,

$$G(1) = \operatorname{Im} J = Z_2, \qquad G(8) = Z_2 + Z_2 \supset \operatorname{Im} J = Z_2,$$
$$G(9) = Z_2 + Z_2 + Z_2 \supset \operatorname{Im} J = Z_2, \qquad G(10) = Z_2 + Z_3 \supset \operatorname{Im} J = 0.$$

The product $G(1) \cdot G(8) \subset G(9)$ and the product $G(1) \cdot G(9) \subset G(10)$, where

$$G(1) \cdot G(8) = Z_2 + Z_2, \qquad G(1) \cdot G(9) = Z_2.$$

Analogously, $G(13) = Z_3$, and $G(3) = Z_{24} = \operatorname{Im} J$, where

$$G(13) = G(3) \cdot G(10), \qquad G(13) \supset \operatorname{Im} J = 0.$$

Comparing the cited information on the groups $G(i)$ and $\operatorname{Im} J \subset G(i)$ with the preceding statements, we obtain the following theorem.

**Theorem 13.4.** a) *There exist manifolds $M^n$ of dimensions $n = 9$ and $n = 10$ such that* 1) $w_2(M^n) \neq 0$ *and* 2) *there is a Milnor sphere $\tilde{S}^n \subset \theta^n(\pi)$ such that $M^n = M^n \# \tilde{S}^n$;*

b) *there exists a manifold $M^{13}$ such that* 1) $p_1(M^{13}) \not\equiv 0 \pmod 3$ *and* 2) *for every Milnor sphere $\tilde{S}^{13} \subset \theta^{13}(\pi) = Z_3$ the manifolds $M^{13}$ and $M^{13} \# \tilde{S}^{13}$ are diffeomorphic with degree $+1$.*

**Remark.** Theorem 13.4 is valid for any manifold $M^9$ (or $M^{10}$) such that $w_2 \neq 0$, $\pi_1 = 0$; analogously for dimension 13.

*Proof.* For the manifolds $M^9$ ($M^{10}$) it is necessary to take the space of the bundle $\nu$ of spheres of dimension 7 (or 8) over a sphere $S^2$ with $w_2(\nu) \neq 0$. Comparing Lemma 13.3 with the information on the groups $G(i)$, $\operatorname{Im} J$ cited above, we obtain the desired statement.

For dimension 13 the proof is analogous. The theorem is proved. □

In conclusion the author conjectures that for $\pi$-manifolds (and all manifolds that are homotopically equivalent to them) a connected sum with a Milnor sphere always varies the smoothness modulo $\theta(\partial\pi)$.

## § 14. Normal bundles of smooth manifolds. [15]

In exact analogy with the proofs of the theorems of §4 on the realization of the classes of the set $A(M^n) \subset \pi_{N+n}(T_N(M^n))$ one can prove the three following assertions.

[15]The results of this section were independently obtained by Browder [29].

**Theorem 14.1.** *Let $M^{2k+1}$ be a smooth simply connected manifold. In order that the $SO_N$-bundle $\nu$ over the manifold $M^{2k+1}$ be the normal bundle of a certain smooth manifold $\tilde{M}^{2k+1}$, that is homotopically equivalent to $M^{2k+1}$, it is necessary and sufficient that the Thom complex $T_N(M^{2k+1},\nu)$ possess the following property: the cycle $\phi[M^{2k+1}]$ is spherical.*

**Theorem 14.2.** *Let $M^{4k}$, $k > 1$, be a smooth simply connected manifold. In order that the $SO_N$-bundle $\nu$ be the normal bundle of some manifold $\tilde{M}^{4k}$ that is homotopically equivalent to $M^{4k}$ it is necessary and sufficient that the Thom complex $T_N(M^{4k},\nu)$ possess the following properties:*

1) *the cycle $\phi[M^n]$ is spherical;*
2) *if $p(\nu^N) = 1 + p_1(\nu^N) + \cdots + p_k(\nu^N)$ and*

$$\bar{p}(\nu^N) = p(\nu^N)^{-1} = 1 + \bar{p}_1 + \cdots + \bar{p}_k,$$

*then the Hirzebruch polynomial $L_k(\bar{p}_1,\dots,\bar{p}_k)$ is equal to the signature $\tau(M^n)$.*

**Theorem 14.3.** *Let $n = 4k+2$, $M^n$ be a smooth manifold, $\pi_1(M^n) = 0$, $\nu^N$ be a vector $SO_N$-bundle and $T_N(M^n,\nu^N)$ be the Thom complex of it. If the cycle $\phi[M^n]$ is spherical, then there exists a manifold $M_1^n$ with boundary $\partial M_1^n = \tilde{S}^{n-1} \in \theta^{n-1}(\partial\pi)$ such that there exists a map*

$$f\colon (M_1^n, \partial M_1^n) \to (M^n, x_0), \qquad x_0 \in M^n,$$

*for which the map*

$$f_*\colon \pi_i(M_1^n, \partial M_1^n) \to \pi_i(M^n, x_0)$$

*is an isomorphism when $i \le n$ and $f^*\nu^N = \nu^N(M_1^n)$.*

The proofs of these three theorems are analogous to the proofs of the theorems of §4 and make use of the properties of maps of degree 1 and the properties of Thom complexes.

**Remark.** It is possible to attach a combinatorial character to Theorems 14.1–14.3 (in the formulation of these theorems one need not require smoothness of the manifold $M^n$; namely, if $M^n$ is a combinatorial manifold in the Brower–Whitehead sense, then the Thom concept of $t$-regularity is extended to the combinatorial case, and the inverse images $f^{-1}(M^n) \subset S^{N+n}$ for a map

$$f\colon S^{N+n} \to T_N(M^n,\nu^N)$$

will be combinatorial subrnanifolds of the sphere $S^{N+n}$, situated in the sphere with a transverse field in the sense of Whitehead [25]. Therefore on the manifold $f^{-1}(M^n) \subset S^{N+n}$ there arises a canonical smooth structure, where

$$\nu^N(f^{-1}(M^n)) = f^*\nu^N.$$

Then the reasoning of §§1–4 is applied. In this way Theorems 14.1–14.3 may be considered as theorems on the determination, for a combinatorial manifold, of an analogous manifold that is smooth and homotopically equivalent to it.

## Appendix I
## Homotopy type and Pontrjagin classes

a. There are known quite a number of relations of the homotopy invariance of classes, which are cited with respect to this or that modulus (Thom, Wu), i.e., relations of the type of a congruence. In addition, for manifolds of dimension $4k$

the Thom–Rohlin–Hirzebruch formula expresses the index in terms of Pontrjagin numbers and gives to these numbers a relation of invariance for rational classes. A collection of counterexamples by Dold, Milnor and Thom shows that the Pontrjagin classes and numbers are “in general” not hornotopically invariant. Moreover, J. Milnor in a personal conversation showed the author a number of examples from which it follows that among the Pontrjagin numbers a linear subspace of homotopically invariant numbers a fortiori has a dimension not greater than half for $k \geq 2$, $n = 4k$.

b. A particular case is the class $p_1(M^5)$ or, what is more general, the class $L_k(p_1, \ldots, p_k)(M^{4k+1})$, considered as rational. Rohlin [35] proved the topological invariance of these classes. However the homotopy invariance is neither proved nor disproved. The author can show that classes are not defined in this case by any cohomology invariants. Nothing else is known here.

c. In §14 we gave a necessary and sufficient condition for an $SO$-bundle to be normal for some homotopically equivalent manifold when $n > 4$, $n \neq 4k + 2$ ($n = 6$ and $n = 14$ being allowed). Translating this result into the terminology of Atiyah and Hirzebruch (cf. [37]), we have the manifold $M_0^n$, the Atiyah–Hirzebruch–Grothendieck functors

$$K_R(M_0^n) = Z + \tilde{K}_R(M_0^n)$$

and

$$J_R(M_0^n) = Z + \tilde{J}_R(M_0^n)$$

and the natural epimorphism $J_R \colon \tilde{K}_R \to \tilde{J}_R$.

We denote in terms of $\alpha \in \tilde{K}_R(M_0^n)$ the normal bundle to the same $M_0^n$ minus its degree. Our theorem reads: an element $\beta \in \tilde{K}_R(M_0^n)$ corresponds to the normal bundle of some $M_1^n$ of the homotopy type of $M_0^n$ for $n \neq 4k$, $4k + 2$ or $n = 6$, $n = 14$ if and only if $J(\beta) = J(\alpha)$ (Atiyah proved that the Thom complex $T_N(\beta)$ of the bundle $\beta + N$ is reducible if and only if $J(\beta) = J(\alpha)$, where $\alpha + N$ is a normal bundle); for $n = 4k$ one must add the Rohlin–Thom–Hirzebruch condition on the Pontrjagin classes of the element $\beta$. For actual calculations the method of Adams is recommended, its operations $\Phi_R^k$ and the “generalized characteristic classes” giving in a number of cases an exact calculation of the functor $J_R$ (cf. [28, 36]).

d. Let $X$ be a finite complex and

$$\tilde{H}^*_{(4)}(X) = \sum_{i \geq 0} \tilde{H}^{4i}(X, Z),$$

where

$$\tilde{H}^{4i}(X, Z) = H^{4i}(X, Z)/2\text{-torsion}.$$

In the ring $\tilde{H}^*_{(4)}(X)$ we consider elements of the form

$$1 + x_1 + \cdots + x_i + \ldots,$$

where $x_i \in \tilde{H}^{4i}(X, Z)$. The set of these elements forms a group $\Lambda(X)$ with respect to multiplication. There is defined a group homomorphism

$$P \colon \tilde{K}_R(X) \to \Lambda(X),$$

putting in correspondence to a stable $SO$-bundle (we consider the homomorphism $P$ only on elements of the class $w_1 = 0$) its Pontrjagin polynomial.

It is easily proved that the group $\operatorname{Im} P$ has a finite index in the group $\Lambda(X)$. The papers of Bott permit one to calculate the image $\operatorname{Im} P$ in the group $\Lambda(X)$.

e. Let $X = M_0^n$ and let $\alpha$, as above, be an element in $\tilde{K}_R$ corresponding to a normal $SO$-bundle of $M_0^n$. The kernel $\operatorname{Ker} J$ consists of $SO$-bundles. It is easy to see that the group $\operatorname{Im} P(\operatorname{Ker} J)$ has a finite index in $\Lambda(X)$. We denote it by $\Lambda'(X) = P(\operatorname{Ker} J)$. From the preceding follows

**Theorem.** *If $n$ is odd or $n = 6, 14$, then the Pontrjagin polynomials of the normal bundles of manifolds of the same homotopy type as $M_0^n$ traverse the residue class of an element $P(\alpha) \in \Lambda(X)$ with respect to a subgroup $\Lambda'(X)$ having a finite index in $\Lambda(X)$. For $n = 4k$ they do not traverse the entire residue class of an element $P(\alpha)$ but only that part of it which satisfies the Thom–Rohlin–Hirzebruch condition.*

From this theorem one may derive by analyzing a sufficiently large number of examples the fact that for simply connected manifolds of dimension $n \geq 6$, $n \neq 4k+2$ no polynomial of the Pontrjagin classes, except $L_k(M^{4k})$, is homotopically invariant.

f. The case $n = 4k+2$, $n \neq 6, 14$ is complicated. But under certain homological constraints on the manifold $M_0^n$, for example, if the group

$$H^{2k+1}(M_0^{4k+2}, Z) \otimes Z_2$$

is trivial, this case can be analyzed. In the case $n = 4k+2$ there corresponds to every element $\beta \in \tilde{K}_R(M_0^n)$ such that $J(\beta) = J(\alpha)$ an invariant $\phi(\beta) \in Z_2$, where $\phi(\beta) = 0$ if there exists a manifold $M_1^{4k+2}$ of the homotopy type of $M_0^{4k+2}$ with normal bundle $\beta + N$, and $\phi(\beta) = 1$ otherwise. We put $\beta = \alpha + \gamma$, where $\gamma \in \operatorname{Ker} J$. It is possible, by analogy with the author's paper [33], to show that

$$\phi(\alpha + \gamma_1 + \gamma_2) = \phi(\alpha) + \phi(\alpha + \gamma_1) + \phi(\alpha + \gamma_2),$$

where $\gamma_1, \gamma_2 \in \operatorname{Ker} J$. Since $\phi(\alpha) = 0$, we define a homomorphism $\bar{\phi}\colon \operatorname{Ker} J \to Z_2$, where $\bar{\phi}(\gamma) = \phi(\alpha + \gamma)$, $\gamma \in \operatorname{Ker} J$ (it is assumed that $H^{2k+1}(M_0^{4k+2}, Z) \otimes Z_2 = 0$). Thus either

$$\operatorname{Ker} \bar{\phi} = \operatorname{Ker} J,$$

or

$$\operatorname{Ker} \bar{\phi} = \frac{1}{2} \operatorname{Ker} J.$$

In the statement of the preceding subsection e one should replace the group $\Lambda'(X)$ by the group $P(\operatorname{Ker} \bar{\phi})$, which coincides with the group $\Lambda'(X)$ or has in it the index 2.

## Appendix II
## Combinatorial equivalence and Milnor's theory of microbundles

Is it possible to perform a construction in the class of combinatorial manifolds that is analogous to the construction performed by the author in the present paper in connection with the problem of a diffeomorphism of smooth manifolds (under the same restrictions on the dimensionality and under the condition of simple connectedness)?

a. First of all we require the notion of a stable normal bundle. Milnor suggested in connection with the problem of the smoothability of combinatorial manifolds that one consider "combinatorial microbundles" over complexes (cf. [31, 34]), Roughly speaking, a microbundle is a bundle over a complex, the fiber of which is the euclidean space $R^n$, and the structural group of which is the group of "microautomorphisms," i.e., piecewise linear automorphisms with a common fixed point and being identified in the event they coincide in a neighborhood of this point. Moreover, there is included in the definition the combinatorial structure of the bundle space (the description given here of the concept of a microbundle is not entirely precise). Milnor proved that the defined stable normal microbundle exists in a unique manner, even though the simple normal bundle does not always exist.

b. Thus one should consider the class of simply connected combinatorial manifolds $\{M_i^n\}$ for $n \geq 5$ of common homotopy type and with the same, as also in the smooth case, stable normal microbundle. As before, we can consider the Thom complex $T_N$ of a normal microbundle for one of the manifolds $M_0^n \in \{M_i^n\}$. A further analogy requires the concept of $t$-regularity in the combinatorial case. This concept bears a rather local character, and since the concept of transversality has meaning for combinatorial manifolds, $t$-regularity is extended without restriction. The cycle

$$\phi[M_0^n] \in H_{N+n}(T_N)$$

is spherical, as also for a smooth $M_0^n$ and therefore the inverse images

$$f^{-1}(M_0^n) \subset S^{N+n}$$

for a $t$-regular $f\colon S^{N+n} \to T_N$ will possess good properties. An analogous result holds for the inverse images under a homotopy

$$F\colon S^{N+n} \times I \to T_N.$$

c. We need to study Morse reconstructions in a new situation, desiring to kill the kernels of maps

$$M_f^n \to M_0^n,$$

where $M_f^n = f^{-1}(M_0^n)$, or

$$W_f^{n+1} \to M_0^n,$$

where $F\colon S^{N+n} \times I \to M_0^n$. Here we have a number of difficulties:

1) a sphere $S^i \subset M_f^n$ or $S^i \subset W_F^{n+1}$ does not in general have a normal microbundle in the manifold;

2) if a sphere $S^i \subset M_f^n$, $S^i \subset W_F^{n+1}$ has a normal microbundle, then it is not necessarily trivial;

3) even if a Morse reconstruction is possible, can one carry over the "equipments"?

We remark that in solving points 2) and 3) we made considerable use of the rapid stabilization of the embeddings $SO_k \subset SO_{k+1} \subset \dots$ and the results of Bott, which do not have a combinatorial analogue. In order to resolve all of these difficulties we will introduce "local smoothnesses" and equipments in a neighborhood of the cycle being investigated. We recall that a neighborhood of this cycle may be regarded as an inverse image of a point $x_0 \in M_0^n$. Therefore it is possible to assign a smoothness and an equipment on this neighborhood. The cycle being investigated will be a smooth sphere in this smoothness. The latter remark resolves all difficulties connected with Morse reconstructions.

d. Thus all results go through without restriction. One should replace $SO_N$ by $PL$ in all statements, and also tidy up the group $\theta^n(\partial\pi)$, consisting of ordinary spheres in the combinatorial sense, which enters into certain formulations. The group $\pi^+(M_0^n, M_0^n)$ must be altered in a corresponding manner.

e. If the manifold $M_0^n$ is smooth, then one can apply to it a construction that is both smooth and combinatorial. As a result we have the possibility of studying the relation between a smooth and a combinatorial manifold by the method of Thom complexes.

f. For the application of combinatorial theory it is important to be acquainted with the homotopy groups $\pi_i(SO), \pi_i(PL)$ and the embedding

$$\pi_i(SO) \to \pi_i(PL).$$

Recently Mazur (cf. [31]) showed that

$$\pi_i(PL, SO) = \Gamma^i$$

(the Milnor–Thom groups). [16] As is known (cf. [17]), $\Gamma^i = \theta^i$ for $i \neq 3, 4$, $\Gamma^3 = 0$ and the group $\Gamma^4$ is unknown. Since the embedding $\pi_i(SO) \to \pi_i(PL)$ is monomorphic in all dimensions (Bott [1], Thom, Rohlin–Švarc, Adams), we have

$$\Gamma^i = \pi_i(PL)/\pi_i(SO).$$

We cite a table for groups $\pi_i(PL)$ and the embeddings $\pi_i(SO) \subset \pi_i(PL)$ for $i < 14$:

| $i =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9[17] | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_i(PL) =$ | 0 | $Z_2$ | 0 | $Z$ | $\Gamma^4$ | 0 | 0 | $Z + Z_4$ | $Z_2 + Z_2$ | $Z_2 + Z_2 + Q_4$ | $Z_6$ | $Z + Z_8$ | 0 | $Z_3$ | $Z_2$ |

An inclusion homomorphism $\pi_i(SO) \subset \pi_i(PL)$ for $i \leq 14$ is trivially defined by a theorem on the monomorphicity of an embedding and the structural groups $\Gamma_i$ (cf. [6]), except for the case $i = 7, 11$. Here we have:

$$\pi_7(SO) = Z, \qquad \pi_7(PL) = Z + Z_4,$$

and $u_{SO} = 7u_{PL} + v_{PL}$, where $u_{PL}$ is a generator of infinite order and $v_{PL}$ is a generator of order 4;

$$\pi_{11}(SO) = Z, \qquad \pi_{11}(PL) = Z + Z_8,$$

and $u_{SO} = 124u_{PL} + v_{PL}$, where, analogously, $v_{PL}$ is a generator of order 8.

g. The Whitehead homomorphism $J_{PL} \colon \pi_i(PL) \to \pi_{N+i}(S^N)$ [18] is an epimorphism for $i \neq 4k + 2$ or $i = 10$ and the factor group $\pi_{N+i}(S^N)/\operatorname{Im} J_{PL}$ contains two elements for $i = 2, 6, 14$ and not more than two in the remaining cases. We note that for $i = 9$

$$\operatorname{Ker} J_{PL} = Z_2 \approx \theta^9(\partial\pi).$$

**Conjecture.** *For $i = 4k - 1$ the group $\pi_i(PL)$ has the form*

$$\pi_i(PL) = Z + Z_{\lambda_k} + \pi_{N+i}(S^N)/\operatorname{Im} J_{SO},$$

*where $\lambda_k$, perhaps, is a power of* 2.

[16] This result is also obtained independently by M. Hirsch [38].

[17] $Q_4 = Z_4$ or $Z_2 + Z_2$.

[18] The definition of the homomorphism $J_{PL}$ was not given earlier, although it can be given by analogy with the ordinary $J$-homomorphism.

It is not excluded that this conjecture can be proved by an arithmetical treatment and by a comparison of the $L$-genus coefficients, the almost parallelizable Milnor manifolds $M_0^{4k}$ with index 8, the results of Bott on the divisibility of the Pontrjagin classes of $SO$-bundles over a sphere and the results of Adams on the stable $J$-homomorphism, in particular, on the singling out of the image $\operatorname{Im} J_{SO}$ as a direct summand in $\pi_{N+4k-1}(S^N)$. We assume that

$$J_{PL}(Z + Z_{\lambda_k}) = \operatorname{Im} J_{SO}$$

and that

$$\pi_{N+k-1}(S^N) = J_{PL}(Z + Z_{\lambda_k}) + \pi_{N+4k-1}(S^N)/\operatorname{Im} J_{SO}.$$

From this it would follow that the group $\theta^{4k+1}(\partial\pi) \subset \theta^{4k-1}$ is singled out as a direct summand. Moreover, the group

$$\pi_{4k-1}(SO) = Z \subset \pi_{4k-1}(PL)$$

must be embedded thus:

$$u_{SO} = \delta_k u_{PL} + v_{PL},$$

where $u_{PL}$ is a generator of infinite order and $v_{PL}$ is a generator of order $\lambda_k$. The order of the group $\theta^{4k-1}(\partial\pi)$ is then equal to $\delta_k\lambda_k$. If the conjecture is true, then one can extend Bott's theorem to the combinatorial case:

Let $a_k = 1$ if $k$ is even, and $a_k = 2$ if $k$ is odd; let

$$L_k(p_1, \dots, p_k) = \frac{t_k}{s_k} p_k + \dots,$$

where $t_k, s_k$ are relatively prime. Since $L_k(M_0^{4k}) = 8$, we have

$$p_k(M_0^{4k}) = 8\frac{s_k}{t_k}.$$

For $SO$-bundles over a sphere the class $p_k$ is divided by $a_k(2k-1)!$. We reduce the numbers $8s_k/t_k$, and $a_k(2k-1)!$ to the least common denominator $\tilde{t}_k$, which is a divisor of $t_k$ (and is equal to $t_k, t_k/2, t_k/4$ or $t_k/8$ if $t_k$ is divisible by the corresponding power of 2). After this we find the greatest common divisor $d_k$ of the numerators of the resultant irreducible fractions.

**Conjecture.** *The Pontrjagin class of a stable microbundle over a sphere $S^{4k}$ is a multiple of the number $d_k/\tilde{t}_k$, and there exists a microbundle with such a class.*

In particular, for k $= 2, 3$ this conjecture is proved by the author:

$$\frac{d_2}{\tilde{t}_2} = \frac{6}{7}, \qquad \frac{d_3}{\tilde{t}_3} = \frac{2\cdot 5!}{124}.$$

Thus we have the proved

**Corollary.** *The Pontrjagin classes of microbundles over the spheres $S^8$ and $S^{12}$ are multiples of the numbers $6/7$ and $2\cdot 5!/124$ respectively, and there exist microbundles with such classes.*

**Remark.** The results of §11 naturally connect up with point e of this appendix, concerning the problems of the relationship between smooth and combinatorial manifolds (under the condition that the normal bundles coincide). In particular, an analysis of the example $S^2 \times S^6$, showing the nontriviality of combinatorial theory, is essential. But this is connected with the fact that $G(6)/\operatorname{Im} J_{PL} = Z_2$.

**Conjecture.** *If the simply connected manifolds* $M_1^n$ *and* $M_2^n$, $n > 7$, *are such that they have the same homotopy type and normal bundle and* $H_{4k+2}(M_i^n, Z_2) = 0$, $2 \leq 4k+2 < n$, *then they are combinatorially equivalent* (*it is perhaps sufficient to require only that* $k = 1, 3$).

## Appendix III
## On the groups $\theta^{4k-1}(\partial\pi)$

a. Starting from the formula of Hirzebruch and the results of Milnor and Kervaire [6], the order of the group $\theta^{4k-1}(\partial\pi)$ can be expressed in terms of the order of the image of the Whitehead homomorphism

$$J_{SO} \colon \pi_{4k-1}(SO_N) \to \pi_{N+4k-1}(S^N).$$

In recent papers Adams has calculated the image $\operatorname{Im} J_{SO}$ completely for even $k$ and to within a factor, equal to 1 or 2, which in all known cases is equal to 1, for even $k+1$. Moreover, from the papers of Adams it follows that the order of the image $\operatorname{Im} J_{SO}$ is completely determined by the integral properties of the $A$-genus of Borel and Hirzebruch [30] (to within the indicated factor). From a comparison of the papers of Milnor and Kervaire [5] and Adams [28] it is seen that the odd factor of the order of $\operatorname{Im} J_{SO}$ is completely determined by the $L$-genus of Hirzebruch. Combining these results, one can obtain the following assertion.

**Theorem 1.** *The odd part of the group* $\theta^{4k-1}(\partial\pi) \subset \theta^{4k-1}$ *is singled out as a direct summand in* $\theta^{4k-1}$.

For the proof it is necessary to construct a homomorphism

$$h \colon \theta^{4k-1} \to \bar{\theta}^{4k-1}(\partial\pi),$$

where $\bar{\theta}^{4k-1}(\partial\pi)$ is the odd part of the group $\theta^{4k-1}(\partial\pi)$. The homomorphism $h$ is constructed sufficiently simply. It is necessary to stretch the membrane $W^{4k}$ onto the sphere $\tilde{S}^{4k-1} \subset \theta^{4k-1}$, to fill the boundary $\partial W^{4k} = \tilde{S}^{4k-1}$ by a ball and of the obtained combinatorial manifold $W_0^{4k}$ to take the value of the combinatorial class $p_k(W_0^{4k}) \bmod 1$. If

$$\tilde{S}^{4k-1} \subset \theta^{4k-1}(\partial\pi),$$

then the constructed homomorphism can identify only those elements whose order is of the form $2^s$; this follows from the results of Adams.

b. A study of the even part $\theta_2^{4k-1}(\partial\pi) \subset \theta^{4k-1}(\partial\pi)$ is more complicated. In this regard we consider the homomorphism

$$p \circ q \colon \theta^{4k-1} \xrightarrow{q} \pi_{N+k-1}(S^N)/\operatorname{Im} J_{SO} \xrightarrow{p} V_{\text{spin}}^{4k-1},$$

where $q$ is a Milnor homomorphism and $p$ is a homomorphism for the "removal of the equipment" of homotopy groups of spheres in "spinor cobordisms," constructed only on simply connected manifolds with the condition $W_2 = 0$. It is evident that

$$\theta^{4k-1}(\partial\pi) \subset \operatorname{Ker}(p \circ q).$$

Putting together the results of Adams, one can prove the following assertion.

**Theorem 2.** *If* $k$ *is even, then the subgroup* $\theta_{(2)}^{4k-1}(\partial\pi) \subset \operatorname{Ker}(p \circ q)$ *is singled out as a direct summand. If* $k$ *is odd, then either*

$$\theta_2^{4k-1}(\partial\pi) \subset \operatorname{Ker}(p \circ q)$$

*is singled out as a direct summand or*

$$\theta_2^{4k-1}(\partial\pi)/Z_2 \subset \mathrm{Ker}(p \circ q)/Z_2$$

*is singled out as a direct summand.*

The proof is analogous to Theorem 1, but one must stretch the membranes with $W_2 = 0$ and in place of the class $p_k$ one must take an $A$-genus for even $k$ and an $(A/2)$-genus for odd $k$ (modulo 1). We note that for dimensions 9 and 10 (as well as 17, 18) the image of the homomorphism $p \circ q$ is nontrivial (cf. [33]).

**Conjecture.** *For dimensions of the form $4k-1$ the homomorphism $p \circ q$ is trivial.*

c. A study of the action of the group $\theta^{4k-1}(\partial\pi)$ on manifolds constitutes a difficult problem that is not amenable to our usual methods. We shall discuss some comparatively simple cases and thereby shed some light on this problem. Suppose the manifold $M^{4k-1}$ (not necessarily simply connected) is such that the groups $H^{4l}(M^{4k-1}, Q)$ are trivial ($l = 1, 2, \dots$ and $Q$ is the field of rational numbers).

**Theorem 3.** [19] *If the sphere $\tilde{S}^{4k-1} \in \theta^{4k-1}(\partial\pi)$ has odd order in the group $\theta^{4k-1}(\partial\pi)$, then the manifolds $M^{4k-1}$ and $M^{4k-1} \# \tilde{S}^{4k-1}$ are not diffeomorphic with degree $+1$.*

For the proof of the theorem we adopt the following plan:

1. A membrane $W^{4k}$, $\partial W^{4k} = (-M^{4k-1}) \cup (M^{4k-1} \# \tilde{S}^{4k-1})$, is constructed such that

$$H_i(W^{4k}, M^{4k-1}) = 0, \qquad i \neq 2k.$$

and a retraction $F\colon W^{4k} \to M^{4k-1}$ is given such that

$$F^*\nu^N(M^{4k-1}) = \nu^N(W^{4k}),$$

where $\nu^N(M)$ is the normal bundle of the manifold $M$.

2. Let a diffeomorphism

$$h\colon M^{4k-1} \to M^{4k-1} \# \tilde{S}^{4k-1}$$

of degree $+1$ be given. We identify the boundary of the membrane $W^{4k-1}$ according to the diffeomorphism $h$. The resultant orientable closed manifold is denoted by $V^{4k}$.

3. It is possible to show that the groups $H^{4l}(V^{4k}, Q) = 0$, $l = 1, \dots, k-1$, $l \neq k/2$, but when $l = k/2$ the group

$$H^{2k}(V^{4k}, Q) = H^{2k}(W^{4k}, M^{4k-1}, Q) + B, \qquad I(B) = 0.$$

4. If the sphere $\tilde{S}^{4k-1} \subset \theta^{4k-1}(\partial\pi)$ has odd order, then the class $p_k(V^{4k})$ will be fractional by analogy with Theorem 1. The resultant contradiction proves the theorem.

If in addition $H_1(M^{4k-1}) = 0$ and $W_2(M^{4k-1}) = 0$, then for $\tilde{S}^{4k-1} \subset \theta^{4k-1}(\partial\pi)$ it is possible by analogy with Theorems 2 and 3 to prove, using the $A$-genus of Hirzebruch and theorems of Adams, that the sphere $\tilde{S}^{4k-1}$ varies the smoothness after an addition to $M^{4k-1}$ (it should be noted that $W_2(W^{4k}) = 0$ and $W_2(V^{4k}) = 0$, where instead of the class $p_k$ one must take $A_k[V^{4k}]$ if $k$ is even and $A_k[V^{4k}]/2$ if $k$ is odd).

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[19] An example by the author shows that Theorem 3 is already inapplicable if $H^4(M^7, Q) \neq 0$ and $p_1 \neq 0$.

d. If $n = 4k+1$, then, as was mentioned above, the image of the homomorphism

$$p \circ q\colon \theta^{4k-1} \to V^{4k+1}_{\text{spin}}$$

may be nontrivial. For example, when $k = 2$ the image $\operatorname{Im} p \circ q = Z_2$. Moreover (cf. Appendix II), the group $\pi_9(PL) = Z_2 + Z_2 + Q_4$, where $Q_4 = Z_4$ or $Z_2 + Z_2$. By making use of analogous arguments it is possible to show that the group

$$\operatorname{Ker} p \circ q = Z_2 + Z_2 \subset \theta^9$$

and the group $\theta^9(\partial\pi) \subset \operatorname{Ker}(p \circ q)$ is singled out as a direct summand. Moreover, one can show that

$$\operatorname{Im} J_{PL} = G(9),$$

where $G(i) = \pi_{N+i}(S^N)$, while

$$J_{PL}(Z_2 + Z_2) = G(1)G(8) = Z_2 + Z_2$$

and

$$J_{PL}(Q_4) = G(9)/G(1)G(8) = Z_2,$$

$$\operatorname{Im} J_{PL} = Z_2 = \theta^9(\partial\pi) \subset \pi_9(PL)$$

(namely, $J^{-1}_{PL} J_{SO} = Z_2 + Z_2$ and $\theta^9(\partial\pi) = J^{-1}_{PL} J_{SO}/\pi_9(SO)$). Since

$$\theta^9 / \operatorname{Ker}(p \circ q) = Z_2,$$

an attempt to prove the singling out of the group $\theta^9(\partial\pi)$ as a direct summand will be unsuccessful.

**Conjecture.** $\theta^9 = Z_2 + Z_4$ *and* $\pi_9(PL) = Z_2 + Z_2 + Z_4$.

## Appendix IV
## The embedding of homotopy spheres in euclidean space and the standard suspension homomorphism

As is known, an ordinary sphere $S^n$ is situated in a standard way in euclidean space $R^{n+1}$. Moreover, from the papers of Smale it follows that a homotopy sphere $\tilde{S}^n$ for $n \neq 3, 4$ is diffeomorphic to a standard sphere $S^n$ if and only if it can be smoothly embedded in $R^{n+1}$. From papers of Milnor, Kervaire and Hirsch [6, 19] it follows that a homotopy sphere $\tilde{S}^n$ is the boundary of a $\pi$-manifold if and only if it can be embedded in $R^{n+2}$. On the other hand, Haefliger showed that any homotopy sphere $\tilde{S}^n$ is approximately inserted into the space $R^{n+j}$ for $j > n/2+1$. [20] We will only consider the embeddings of homotopy spheres $\tilde{S}^n \subset R^{n+k}$ for $2 \leq k \leq n-1$ that have a trivial normal bundle, viz., the "$\pi$-embeddings." It is easy to extract certain necessary conditions for the existence of a $\pi$-embedding $\tilde{S}^n \subset R^{n+k}$ from the homotopy groups of spheres.

Let us consider a set $\tilde{B}(\tilde{S}^n) \subset \pi_{N+n}(S^N)$, representing a coset modulo $J\pi_n(SO_N)$.

**Lemma 1.** *If there exists a $\pi$-embedding $\tilde{S}^n \subset R^{n+k}$, then there is an element $\alpha \in \tilde{B}(\tilde{S}^n)$ such that $\alpha \in E^{N-k}(\pi_{n+k}(S^k))$, where $E$ is a suspension.*

The proof of the lemma trivially follows from an interpretation of a suspension homomorphism in terms of equipped manifolds. As to the sufficiency of the condition of Lemma 1, there holds the following

[20] A normal bundle $\alpha \in \pi_{n-1}(SO_j)$ has order $2^h$ for $j > n/2+1$.

**Theorem 1.** *If there exists an element $\alpha \in \tilde{B}(S^n)$ such that $\alpha \in \operatorname{Im} E^{N-k}$, then there exists a $\pi$-embedding $S^n \subset R^{n+k+1}$.*

The proof of the theorem is extracted from the results of §11 concerning differentiable structures on a direct product of spheres and follows from Lemmas 1, 2, 3 of this appendix.

**Lemma 2.** *Under the conditions of Theorem 1 the sets $B(S^n \times S^k)$ and*

$$B(\tilde{S} \times S^k) \subset \tilde{A} \subset \pi_{N+n}(T_N(S^n \times S^k))$$

*coincide to within $\operatorname{Im} \kappa_*$, where $\kappa\colon S^n \subset T_N$.*

**Lemma 3.** *If the sets $B(M_1^m)$ and $B(M^m) \subset \tilde{A}$ coincide modulo $\operatorname{Im} \kappa_*$, then the manifolds $M_1^m$ and $M^m$ are diffeomorphic modulo $\theta^m(\pi)$.*

The proof of Lemma 3 is given in §9 in all cases except $m \equiv 2 \pmod 4$. For the proof of Lemma 3 when $m \equiv 2 \pmod 4$ cf. the paper [33].

**Lemma 4.** *If the manifold $M^{n+k}$ is diffeomorphic to $S^n \times S^k \bmod \theta^{n+k}$, where $M^{n+k} = \tilde{S}^n \times S^k$, then the homotopy sphere $\tilde{S}^n$ admits a $\pi$-embedding in $R^{n+k+1}$.*

The proof of Lemma 4 is trivial.

Let us consider the special case $k = 3$. There holds

**Lemma 5.** [21] *If a sphere $S^n$ is $\pi$-embedded in a sphere $S^{n+3}$, then it bounds a manifold $W^{n+1} \subset S^{n+3}$, the normal bundle of which is an $SO_2$-bundle with Chern class $c_1 \in H^2(W^{n+1})$ such that $c_1^2 = 0$.*

*Proof.* We give on a sphere $\tilde{S}^n$ a frame field $\tau_3$, that is normal to the sphere in $S^{n+3}$, and we copy it onto the boundary $S^2 \times \tilde{S}^n$ of a tubular neighborhood with the use of the first vector of this frame field. The resultant manifold $\tilde{S}^n \subset S^2 \times \tilde{S}^n$ is homologous to zero in the complement

$$S^{n+3} \setminus \operatorname{Int} D^3 \times \tilde{S}^n,$$

and one can compute the membrane, stretched onto it, by the manifold $W^{n+1}$ with boundary $\tilde{S}^n \subset S^2 \times \tilde{S}^n = \partial(S^{n+3} \setminus \operatorname{Int} D^3 \times \tilde{S}^n)$. Incidentally, from the paper of Smale [19] it trivially follows that

$$S^{n+3} \setminus \operatorname{Int} D^3 \times \tilde{S}^n$$

is diffeomorphic to $S^2 \times D^{n+1}$. The membrane $W^{n+1}$ realizes a base cycle of the group

$$H_{n+1}(S^2 \times D^{n+1}, \partial(S^2 \times D^{n+1})) = Z.$$

The normal bundle of the membrane $W^{n+1}$ in $S^{n+3}$ is an $SO_2$-bundle and is defined by a Chern class $c_1 \in H^2(W^{n+1})$. Let us show that $c_1^2 = 0$. We will assume that $n > 3$. Then

$$H_{n-1}(S^2 \times D^{n+1}) = 0.$$

The self-intersection

$$W^{n+1} \cdot W^{n+1} \subset S^{n+3} \setminus \operatorname{Int} D^3 \times \tilde{S}^n$$

defines an $(n-1)$-dimensional cycle modulo the boundary and is a submanifold $V^{n-1} \subset W^{n-1}$. Since

$$\partial W^{n+1} = \tilde{S}^n \subset S^2 \times \tilde{S}^n,$$

[21] The idea for the proof of Lemma 5 is taken from a paper of V. A. Rohlin.

one can assume that $V^{n-1}$ lies strictly inside $W^{n+1}$ and is closed (it is easily seen that in the dimension $n-1$ we have $H_{n-1}(S^2 \times D^{n+1}) = H_{n-1}(S^2 \times D^{n+1}, \partial(S^2 \times D^{n+1})) = 0$).

We denote in terms of

$$D_M \colon H_j(M, \partial M) \to H^{l-j}(M)$$

the isomorphism of the Poincaré duality and in terms of $i$ we denote the embedding

$$W^{n+1} \subset S^{n+3} \setminus \operatorname{Int} D^3 \times \tilde{S}^n.$$

Then

$$c_1^2 = i^*\{D_M i^*[W^{n+1}]\}^2 = i^* D_M\{i^*[W^{n+1}] \cdot i_*[W^{n+1}]\} = i^* D_M i^*[V^{n-1}] = 0,$$

where $M = S^{n+3} \setminus \operatorname{Int} D^3 \times \tilde{S}^n$,

The lemma is proved. □

From the lemma it immediately follows that the connected submanifold

$$V^{n-1} = W^{n+1} \cdot W^{n+1},$$

where $V^{n-1} \subset W^{n+1}$, has a trivial normal bundle in the manifold $W^{n+1}$. Moreover, if we give on the boundary $\tilde{S}^n \subset S^2 \times \tilde{S}^n$ a 2-frame field $\tau_2$, that is normal to $\tilde{S}^n$ in $S^2 \times \tilde{S}^n$, and extend it inside the manifold $W^{n+1}$, then under a suitable choice of the field and the extension (which we also denote by $\tau_2$) the manifold of the singularities of the field $\tau_2$ inside $W^{n+1}$ coincides with the manifold $V^{n-1} \subset W^{n+1}$. The tubular neighborhood $D \times V^{n-1}$ of the manifold $V^{n-1}$ in $W^{n+1}$ has the boundary $S^1 \times V^{n-1}$, on which this field $\tau_2$ is defined and is nonsingular. To the field $\tau_2/S^1 \times V^{n-1}$ we add a radius vector of the interior of the ball $D^2$ so that it is normal to the boundary $S^1 = \partial D^2$ at each point. We obtain a 3-field $\tilde{\tau}_3$ on $S^1 \times V^{n-1}$.

The following lemma is obvious.

**Lemma 6.** *The equipped manifolds* $(\tilde{S}^n, \tau_3)$ *and* $(S^1 \times V^{n+1}, \tilde{\tau}_3)$ *define one and the same element of the group* $\pi_{n+3}(S^3)$ (*for the membrane connecting these equipped manifolds one must take* $W^{n+1} \setminus \operatorname{Int} D^2 \times V^{n-1}$).

**Conjecture.** *If a sphere* $\tilde{S}^n$ *is* $\pi$*-embedded in a space* $S^{n+3}$*, then there exists on this sphere a normal frame field* $\tau_3$ *such that the equipped manifold* $(\tilde{S}^n, \tau_3)$ *defines an element of the group* $\pi_{n+3}(S^3)$*, which decomposes into a composition* $\beta \circ \alpha$*, where* $\alpha \in \pi_{n+3}(S^4)$ *and* $\beta \in \pi_4(S^3) = Z_2$.

**Corollary.** *In a group* $G_n$ *the set* $\tilde{B}(\tilde{S}^n) \subset G_n$ *contains an element* $\alpha\beta$*, where* $\alpha \in G_{n-1}$*,* $\beta \in G_1$ (*so that the element* $\alpha\beta$ *has an order not greater than two*)*, if* $\tilde{S}^n$ *is* $\pi$*-inserted into* $S^{n+3}$.

Since the image of a suspension that is far removed from the groups $\pi_{n+3}(S^3)$ contains elements of odd order $p$, not belonging to the group $J\pi_n(SO_N)$, it follows that for $k = 2$ and $k = 3$ in Theorem 1 one cannot get rid of the differences in identity between the necessary condition (Lemma 1) and the sufficient condition (Theorem 1).

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