Report on the Program on Homotopy Theory and Group Theory

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1. Introduction by the Organizers

The program was aiming to the connections between the homotopy theory and group theory with emphasizing on the connections between simplicial groups, braid groups and the free group automorphisms. The activities were taken place by various talks and discussions among the organizers.

A fundamental and central problem in homotopy theory is to study the homotopy groups of spheres. There have been much progress on this topic in history, but the general homotopy groups of spheres remain far unknown. As a combinatorial tool for studying homotopy theory, simplicial groups were first studied by John Moore. The classical Moore theorem states that the homotopy groups of the geometric realization of a simplicial group are given by the homology of the Moore chain complex. An important construction due to Dan Kan gave simplicial group models for the loop spaces of any simply connected spaces, and so theoretically speaking the homotopy groups of any simply connected spaces can be investigated by studying simplicial groups. In particular, the classical Adams spectral sequence arises as the associated graded by taking mod p lower central series of Kan's G-construction on reduced simplicial sets. In addition to geometry, the group structure in a simplicial group gives another nature of these objects with connections to group theory. It is possible that two different simplicial groups with same homotopy type have sharply different group structures. For instance one could have a perfect simplicial group model (that is the abelianization is trivial) for certain loop spaces using Carlsson's construction. For such a type of simplicial groups, the lower central series will not give any information as the groups are perfect, but the word length filtration provide different information.

Recent progress on simplicial groups has been successfully to give some interesting connections between the general homotopy groups of spheres and the braid groups. In particular, the general higher homotopy groups of the 2-sphere are given by the group of Brunnian braids over the sphere modulo the Brunnian braids over the disk, where a braid is called Brunnian if it becomes a trivial braid after removing any one of its strands. These results were obtained from studying the simplicial structure on braids, which gives a motivation for studying simplicial structures on generalized braid groups as further investigations.

Braid groups are important mathematical objects with applications to other sciences such as robotics. The classical Markov theorem states that any link can be obtained by closing up a braid and all links are given by the braids subject to the equivalence relation generated by the Markov moves. Some generalizations of the classical Artin braid groups such as virtual braids and welded braids have been introduced and studied recently. In particular, the presentations of virtual braid groups and welded braid groups have been given. Algebraically the welded braid groups can be embedded into the free group automorphisms. The study of the subgroups of the automorphism groups of the free groups are important in both algebra and geometry. In particular, the groups IA_n have a lot of mathematical mysteries that need to be explored.

During this 3-week program, the progress has been achieved on the following specific topics:

(1). Transfinite homotopy groups.
(2). Brunnian braids over surfaces.
(3). A universal 2-generator group.
(4). Virtual Braids and Residually Nilpotent Property of Some Groups

The summaries of our progress on these topics are given below.

2. Transfinite homotopy groups

Let X be a simplicial group. The lower central series filtration in X gives rise to the long exact sequence

$$\cdots \to \pi_{i+1}(X/\gamma_n(X)) \to \pi_i(\gamma_n(X)/\gamma_{n+1}(X)) \to \pi_i(X/\gamma_{n+1}(X)) \to \pi_i(X/\gamma_n(X)) \to \dots \tag{2.1}$$

This exact sequence defines a graded exact couple, which gives rise to the natural spectral sequence $E(X)$ with the initial terms

$$E^1_{n,m}(X) = \pi_m(\gamma_n(X)/\gamma_{n+1}(X)).$$

and differentials $d^i,\ i \geq i$

$$d^i : E^i_{n,m}(X) \to E^i_{n+i,m-1}(X).$$

This spectral sequence naturally comes into play in homotopy theory. One of major results about this sequence is the following statement due to E. Curtis [3]: Let K be a connected and simply connected simplicial set, $G = GK$ its associated Kan construction. Then the spectral sequence $E^i(G)$ converges to $E^\infty(G)$ and $\oplus_r E^\infty_{r,q}$ is the graded group associated with the filtration on $\pi_q(GK) = \pi_{q+1}(|K|)$. The groups $E^1(K)$ are homology invariants of K.

For every group G, A. Bousfield constructed (see [1] for all details) a functorial transfinite tower of group homomorphisms:

$$\begin{array}{ccccccccccc}
G & \xleftarrow{id} & G & \xleftarrow{id} & \dots & \xleftarrow{id} & G & \xleftarrow{id} & G & \longleftarrow & \dots \\
\downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & & & \downarrow{\scriptstyle \eta_\alpha} & & \downarrow{\scriptstyle \eta_{\alpha+1}} & & \\
T_1G & \xleftarrow{t_1} & T_2G & \xleftarrow{t_2} & \dots & \longleftarrow & T_\alpha G & \xleftarrow{t_\alpha} & T_{\alpha+1}G & \longleftarrow & \dots
\end{array}$$

which is called the *HZ-tower* of the group G.

The *HZ-localization* of G is the inverse limit of this HZ-tower:

$$L : G \to L(G) := \varprojlim_\alpha T_\alpha G. \tag{2.2}$$

The functor L, constructed above, has the following properties:

(i) L induces isomorphism

$$H_1(L) : H_1(G) \to H_1(L(G))$$

and epimorphism

$$H_2(L) : H_2(G) \to H_2(L(G));$$

therefore, it induces isomorphisms

$$G/\gamma_n(G) \simeq L(G)/\gamma_n(L(G)) \tag{2.3}$$

for all finite $n \geq 1$.
(ii) $L(G)$ is transfinitely nilpotent for any group G.
(iii) There exists a canonical homomorphism

$$L(G) \to \underleftarrow{lim}_n G/\gamma_n(G),$$

where $\underleftarrow{lim}_n G/\gamma_n(G)$ is the free nilpotent completion of G, which is an epimorphism with kernel $\gamma_\omega(L(G))$ in the case of a finitely generated group G. (iv) For any ordinal number α, there is a natural isomorphisms

$$T_\alpha G = L(G)/\gamma_\alpha(L(G)). \tag{2.4}$$

The isomorphism (2.3) shows that the quotients $\gamma_\alpha(L(G))/\gamma_{\alpha+1}(L(G))$ can be viewed as transfinite extension of a usual notion of lower central quotients. The most important remark here is the fact that

$$\gamma_\omega(L(F)) \neq \gamma_{\omega+1}(L(F))$$

for a non-cyclic free group F. Hence we have a natural way how to extend the lower central quotients of a free group to the transfinite ordinals.

Now let X be a simplicial group. Consider the transfinite analog of the sequence (2.1):

$$\dots \to \pi_{i+1}(L(X)/\gamma_\alpha(L(X))) \to \pi_i(\gamma_\alpha(L(X))/\gamma_{\alpha+1}(L(X))) \to \\ \pi_i(L(X)/\gamma_{\alpha+1}(L(X))) \to \pi_i(L(X)/\gamma_\alpha(L(X))) \to \dots$$

It defines a natural exact couple spectral sequence with initial terms

$$\tilde{E}^1_{1,m}(X) = \pi_m(L(X)/\gamma_\omega(L(X)))$$
$$\tilde{E}^1_{n,m}(X) = \pi_m(\gamma_{\omega+n-1}(L(X))/\gamma_{\omega+n}(L(X))),\ n > 1$$

and differentials $d^i,\ i \geq i$

$$d^i : \tilde{E}^i_{n,m}(X) \to \tilde{E}^i_{n+i,m-1}(X).$$

The result of convergence of this spectral sequence defines a certain filtration of an object, which we don't know, but can assume that it exists. We can say that

$$E^\infty_{*,\omega+n} \Rightarrow \pi_{\omega+n}(X),\ n \geq 0$$

Now let me show one more motivation for the transfinite analog of Curtis spectral sequence.

We have a natural filtrations of the very left terms of the above spectral sequence:

$$\tilde{E}^1_{1,m}(X) = \pi_m(L(X)/\gamma_\omega(X)) \supseteq \tilde{E}^2_{1,m}(X) \supseteq \tilde{E}^3_{1,m}(X) \supseteq \dots$$

The property (iii) above tells that there is a natural homomorphism

$$L(X)/\gamma_\omega(L(X)) \to \underleftarrow{lim}_n X/\gamma_n(X)$$

hence we have the natural maps

$$\Xi_m : \pi_m(L(X)/\gamma_\omega(L(X))) \to \pi_m(\underleftarrow{lim}_n X/\gamma_n(X))$$

This defines a natural filtration

$$\pi_m(\underleftarrow{lim}_n X/\gamma_n(X)) \supseteq \Xi_m(\tilde{E}^2_{1,m}) \supseteq \Xi_m(\tilde{E}^3_{1,m}) \supseteq \dots \tag{2.5}$$

Now let R be a unital ring and X a free simplicial resolution of $GL(R)$. Then we know that the algebraic K-theory of R can be expressed as the derived functors of the nilpotent completion:

$$K_{m+1}(R) = \pi_m(\underleftarrow{lim}_n X/\gamma_n(X)),\ m \geq 1$$

Analogously, if X is a free simplicial resolution of the infinite symmetric group Σ_∞, we have the stable homotopy groups of spheres:

$$\pi^S_{m+1} = \pi_m(\underleftarrow{lim}_n X/\gamma_n(X)),\ m \geq 1.$$

Hence we obtain certain interesting filtrations defined as (2.5) of the K-theory and stable homotopy groups of spheres.

References

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3. Brunnian Braids over Surfaces

Let M be a space and let M^n be the n-fold Cartesian product of M. The *ordered configuration space* $F(M,n)$ is defined by

$$F(M,n) = \{(x_1,\dots,x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

with subspace topology of M^n. The symmetric group Σ_n acts on $F(M,n)$ by permuting coordinates. The *braid group* $B_n(M)$ is defined to be the fundamental group $\pi_1(F(M,n)/\Sigma_n)$.

In geometry, the elements in $B_n(M)$ can be described as follows. Let $(q_1,\dots,q_n)$ be the basepoint of $F(M,n)$ and let $p\colon F(M,n) \to F(M,n)/\Sigma_n$ be the quotient map. The basepoint of $F(M,n)/\Sigma_n$ is chosen to be $p(q_1,\dots,q_n)$. Let $[\lambda]$ be an element in $\pi_1(F(M,n)/\Sigma_n)$ represented by a loop $\lambda\colon S^1 \to F(M,n)/\Sigma_n$. Since

$$p\colon F(M,n) \to F(M,n)/\Sigma_n$$

is a covering, the loop λ lifts to a unique path $\tilde{\lambda}\colon [0,1] \to F(M,n)$ starting from $\tilde{\lambda}(0) = (q_1,\dots,q_n)$ and ending with $\tilde{\lambda}(1) = (q_{\sigma(1)},\dots,q_{\sigma(n)})$ for some $\sigma \in \Sigma_n$. Let

$$\tilde{\lambda}(t) = (\tilde{\lambda}_1(t),\dots,\tilde{\lambda}_n(t)) \in F(M,n) \subseteq M^n.$$

Then $\tilde{\lambda}_i(t) \neq \tilde{\lambda}_j(t)$ for $i \neq j$ and any $0 \leq t \leq 1$. The strands $\{\tilde{\lambda}_i(t) \mid 1 \leq i \leq n\}$ in the cylinder $M \times [0,1]$ give the intuitive braided description of λ.

The *pure braid group* $P_n(M)$ is defined to the fundamental group $\pi_1(F(M,n)$. From the covering $F(M,n) \to F(M,n)/\Sigma_n$, there is a short exact sequence of groups

$$\{1\} \to P_n(M) \to B_n(M) \to \Sigma_n \to \{1\}.$$

A *Brunnian braid* means a braid that becomes trivial after removing any one of its strands. The formal definition of Brunnian braids can be given as follows. Let M be a space with a whisker, that is, there is injective continuous map $\omega\colon \mathbb{R}^+ = [0,\infty) \to M$. The basepoint $(q_1,\ldots,q_n)$ of $F(M,n)$ is chosen with $q_i = \omega(i-1)$. Let $B_n(M)$ be identified with the subset of the fundamental groupoid of $F(M,n)$ consisting of the path homotopy classes from $(q_1,\ldots,q_n)$ to a coordinate permutation of $(q_1,\ldots,q_n)$. Then the coordinate projection

$$d_i\colon F(M,n) \to F(M,n-1) \quad (x_1,\ldots,x_n) \mapsto (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$$

induces a function $d_i\colon B_n(M) \to B_{n-1}(M)$ that describes the removal of the ith strand in geometry. The function $d_i\colon B_n(M) \to B_{n-1}(M)$ is not a group homomorphism in general. But the restriction $d_i = d_i|_{P_n(M)}\colon P_n(M) \to P_{n-1}(M)$ is a group homomorphism. The Brunnian braids $\mathrm{Brun}_n(M)$ over M are defined by

$$\mathrm{Brun}_n(M) = \bigcap_{i=1}^{n} \mathrm{Ker}(d_i\colon B_n(M) \to B_{n-1}(M)) = \{\beta \in B_n(M) \mid d_i(\beta) = 1 \text{ for all } i\}.$$

It is straightforward to check that $\mathrm{Brun}_n(M)$ is a subgroup of $B_n(M)$. If $n = 1$, then $\mathrm{Brun}_1(M) = B_1(M) = \pi_1(M)$. If $n = 2$, then there is a short exact sequence of groups

$$\{1\} \to P_2(M) \cap \mathrm{Brun}_2(M) \to \mathrm{Brun}_2(M) \to \Sigma_2 \to \{1\}.$$

For $n \geq 3$, $\mathrm{Brun}_n(M) \subseteq P_n(M)$. (See [2] for details.)

A classical question proposed by Makanin is to determine a set of generators for Brunnian braids over the disk. This question was answered by Johnson [7]. (Brunnian braids were called *smooth braids* in Johnson's paper.) A different approach to this question can be found in [8]. In Birman's book [3], she asked to determine the free basis for Brunnian braids over the sphere. Birman's question remains open. A connection between Brunnian braids and the homotopy groups was given in [2].

We are able to determine a set of generators for $\mathrm{Brun}_n(S)$ for $n \geq 3$ for any surface S except that cases $S = S^2$ or $\mathbb{R}\mathrm{P}^2$. The determination is sketched as follows:

Since S is a manifold, the coordinate projection Fadell-Neuwirth fibration

$$d_n\colon F(S,n) \to F(S,n-1)$$

is a fibration with fibre $S \smallsetminus Q_{n-1}$ by the Fadell-Neuwirth Theorem [6], where $Q_{n-1} = \{q_1,\ldots,q_{n-1}\}$ is the subset of $n-1$ distinct points in $S \smallsetminus \partial S$. Thus there is an exact sequence

$$\pi_2(F(S,n-1)) \to \pi_1(S \smallsetminus Q_{n-1}) \to \pi_1(F(S,n)) \to \pi_1(F(S,n-1)) \to \{1\}.$$

If $S \neq S^2, \mathbb{R}\mathrm{P}^2$, then S is a $K(\pi,1)$-space. By induction, one gets that $F(S,n)$ is a $K(\pi,1)$-space for any $n \geq 1$. It follows that there is a short exact sequence

$$\{1\} \to \pi_1(S \smallsetminus Q_{n-1}) \to \pi_1(F(S,n)) \to \pi_1(F(S,n-1)) \to \{1\}$$

and so $\mathrm{Ker}(d_n\colon P_n(S) \to P_{n-1}(S)) = \pi_1(S \smallsetminus Q_{n-1})$.

Next consider the commutative diagram

$$\begin{array}{ccc} S \smallsetminus Q_{n-1} & \hookrightarrow & F(S,n) \\ \downarrow & & \downarrow d_i \\ S \smallsetminus (Q_{n-1} \cup \{q_i\}) & \hookrightarrow & F(S,n-1) \end{array} \tag{3.1}$$

for $1 \leq i \leq n-1$. Let

$$R_i = \mathrm{Ker}(\pi_1(S \smallsetminus Q_{n-1}) \to \pi_1(S \smallsetminus (Q_{n-1} \cup \{q_i\}))$$

be the normal subgroup of $\pi_1(S \smallsetminus Q_{n-1})$ for $1 \leq i \leq n-1$. Note that R_i is the normal closure of the element x_i in $\pi_1(S \smallsetminus Q_{n-1})$ represented by a small simple loop around the point q_i. Thus the group R_i is generated by the conjugations gx_ig^{-1} for $g \in \pi_1(S \smallsetminus Q_{n-1})$. By taking the fundamental group functor to Diagram (3.1), we obtain that

$$\mathrm{Ker}(d_n \colon P_n(S) \to P_{n-1}(S)) \cap \mathrm{Ker}(d_i \colon P_n(S) \to P_{n-1}(S)) = R_i$$

and so

$$\mathrm{Brun}_n(S) = \mathrm{Brun}_n(S) \cap P_n(S) = \bigcap_{i=1}^{n-1} R_i$$

the intersection of the subgroups R_i in $\pi_1(S \smallsetminus Q_{n-1})$ for $n \geq 3$.

If $n = 3$, by using Brown-Loday Theorem [4], $(R_1 \cap R_2)/[R_1, R_2] \cong \pi_2(S) = 0$, where $[R_1, R_2]$ is the commutator subgroup. Thus

$$\mathrm{Brun}_3(S) = R_1 \cap R_2 = [R_1, R_2]$$

In general case, using the recent result in [5], one gets

$$\frac{\bigcap_{i=1}^{n-1} R_i}{[[R_1, R_2, \ldots, R_{n-1}]]} = \pi_{n-1}(S) = 0,$$

where $[[R_1, R_2, \ldots, R_{n-1}]]$ is the subgroup of $\pi_1(S \smallsetminus Q_{n-1})$ generated by all possible iterated commutators

$$[r_{i_1}, r_{i_2}, \ldots, r_{i_t}] \tag{3.2}$$

with $t \geq n-1$, $r_j \in R_j$ and the set of indices $\{i_1, \ldots, i_t\} = \{1, 2, \ldots, n-1\}$, that is the elements in each R_i must occurs at least once in the commutator bracket. (See [1, 8] for details on the terminology of $[[R_1, \ldots, R_n]]$.) It follows that

$$\mathrm{Brun}_n(S) = [[R_1, R_2, \ldots, R_{n-1}]] \tag{3.3}$$

for $n \geq 3$. A set of generators for $\mathrm{Brun}_n(S)$ is then described as in Equation (3.2).

If $S = D$, Equation (3.3) is the main result in Johnson's paper [7].

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4. A universal 2-generator group

Let G be a group and let $N_1, N_2, \ldots, N_k$ be normal subgroups of G. Let $[[N_1, N_2, \ldots, N_k]]$ denote the subgroup of G generated by all of the iterated commutators $\beta^t(h^{(1)}_{i_1}, h^{(2)}_{i_2}, \ldots, h^{(t)}_{i_t})$ with $t \geq k$, where

(i) $1 \leq i_s \leq k$;
(ii) all integers in $\{1, 2, \ldots, k\}$ appear as at least one of the integers i_s;
(iii) $h^{(s)}_j \in H_j$;
(1). for each $t \geq k$, β^t runs over all of the bracket arrangements of weight t.

(See [1] for details.)

An important example of such a construction of iterated commutator subgroups is as follows: Let F_n be the free group of rank n with a basis $y_1, y_2, \ldots, y_n$. For a subset $S \subseteq F_n$, let $\langle S\rangle^{F_n}$ denote the normal closure of S in F_n, that is the smallest normal subgroup of F_n containing S. Let $R_i = \langle y_i\rangle^{F_n}$ and let $R_0 = \langle y_1y_2 \cdots y_n\rangle^{F_n}$ be the normal closure of the product element $y_1y_2 \cdots y_n$ in F_n. By [1, Colorary 1.5], $\pi_{n+1}(S^2)$ is the center of the group $F_n/[[R_0, R_1, R_2, \ldots, R_n]]$. Motivated by this result, the ideas on this topic are to explore a possible universal 2-generator group which contains all homotopy groups of S^2. Our proposed construction is as follows. Consider the projection

$$p\colon F(y_1, y_2) \to F(y_1) = \mathbb{Z}$$

be the projection of free group of rank 2 to the free group of rank 1 with $p(y_1) = y_1$ and $p(y_2) = 1$. Then $\mathrm{Ker}(p)$ is the free group of infinite rank with a basis by the elements $y_2^n y_1 y_2^{-n}$ for $n \in \mathbb{Z}$. Denote by

$$x_n = y_2^n y_1 y_2^{-n}.$$

Then there is a semi-direct product $F_2 = F_\infty \rtimes \mathbb{Z}$, where the action of $\mathbb{Z} = \langle t\rangle$ on

$$F_\infty = \langle x_n \mid n \in \mathbb{Z}\rangle$$

is given by

$$t^m \cdot x_n = x_{n+m}.$$

Let $R_i = \langle x_i\rangle^{F_\infty}$ for each integer i. For a pair of integers $a \leq b$, let $R_{a,b}$ be the normal closure of the element $x_a x_{a+1} \cdots x_b$ in F_∞ and define

$$\mathrm{Bd}_{a,b} = [[R_a, R_{a+1}, \ldots, R_b, R_{a,b}]].$$

Note that $\mathrm{Bd}_{a,a} = \{1\}$. Let

$$\bar{H} = F_\infty / \langle \mathrm{Bd}_{a,b} \mid -\infty < a \leq b < +\infty\rangle^{F_\infty}$$

Note that

$$t^m(\mathrm{Bd}_{a,b}) = \mathrm{Bd}_{a+m,b+m}.$$

The action of $\mathbb{Z} = \langle t\rangle$ on F_∞ induces an $\mathbb{Z}$-action on $\bar{H}$. Define

$$\mathcal{G} = \bar{H} \rtimes \mathbb{Z}.$$

Then $\mathcal{G}$ is a 2-generator group because $\mathcal{G}$ is a quotient group of $F(y_1,y_2)=F_\infty\rtimes\mathbb{Z}$.

Given an interval $[a,b]$ with $a,b\in\mathbb{Z}$, let $F_{[a,b]}$ denote the (free) subgroup of F_∞ generated by x_i for $a\leq i\leq b$. Recall from [1, Corollary 1.5] that

$$\pi_n(S^2)=Z(F_{[1,n-1]}/(\mathrm{Bd}_{1,n-1}\cap F_{[1,n-1]})),$$

the center of the $F_{[1,n-1]}/(\mathrm{Bd}_{1,n-1}\cap F_{[1,n-1]})$ for $n\geq 2$. Let

$$i\colon \pi_n(S^2)\longrightarrow F_{[1,n-1]}/(\mathrm{Bd}_{1,n-1}\cap F_{[1,n-1]})$$

be the inclusion. Now the inclusion $F_{[1,n-1]}\rightarrowtail F_\infty$ induces a group homomorphism

$$f_n\colon F_{[1,n-1]}/(\mathrm{Bd}_{1,n-1}\cap F_{[1,n-1]})\longrightarrow \bar{H}.$$

Let ϕ_n be the composite

$$\pi_n(S^2)\xrightarrow{\ i\ } F_{[1,n-1]}/(\mathrm{Bd}_{1,n-1}\cap F_{[1,n-1]})\xrightarrow{\ f_n\ }\bar{H}\hookrightarrow\mathcal{G}.$$

Question. Is $\phi_n\colon\pi_n(S^2)\to\mathcal{G}$ a monomorphism for each $n\geq 2$? The question is equivalent to whether the composite

$$\psi_n\colon\pi_n(S^2)\xrightarrow{\ i\ } F_{[1,n-1]}/(\mathrm{Bd}_{1,n-1}\cap F_{[1,n-1]})\xrightarrow{\ f_n\ }\bar{H}$$

is a monomorphism.

When $n=2$, $\pi_2(S^2)=F_1=\mathbb{Z}$. Clearly $\psi_2\colon\pi_2(S^2)\to\bar{H}$ is a monomorphism because $\mathrm{Bd}_{a,b}\leq[F_\infty,F_\infty]$ for any $a\leq b$.

When $n=3$, $F_{[1,2]}/\mathrm{Bd}_{1,2}=F_{[1,2]}/\gamma_3(F_{[1,2]}$. Note that

$$\mathrm{Bd}_{a,b}\subseteq\gamma_3(F_\infty)$$

for any $a\leq b$. The quotient map $F_\infty\to F_\infty/\gamma_3(F_\infty)$ factors through $\bar{H}$. Since the inclusion $F_{[1,2]}\to F_\infty$ induces a monomorphism

$$F_{[1,2]}/\gamma_3(F_{[1,2]})\hookrightarrow F_\infty/\gamma_3(F_\infty),$$

the map $f_3\colon F_{[1,2]}/(\mathrm{Bd}_{1,2}\cap F_{[1,2]})\longrightarrow\bar{H}$ is a monomorphism. Thus ϕ_3 is a monomorphism.

References

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5. Virtual Braids and Residually Nilpotent Property of Some Groups

Let G_1 and G_2 be some residually nilpotent groups and $G=G_1\leftthreetimes G_2$ a semi-direct product of these groups. The following question is naturally arises: when G is residually nilpotent? If G_2 acts trivially on the abelianization of G_1 then the answer is positive. In general, one can construct a non-residually nilpotent group G that is a semi-direct product of two free groups. An interesting example of residually nilpotent groups is as follows.

Theorem 5.1. *The 3-strand pure virtual braid group VP_3 is residually torsion free nilpotent.*

We use the notations for virtual braid groups in [2].

Proof. We know that

$$VP_3 = V_2^* \rtimes V_1,$$

where $V_1 = \langle \lambda_{12}, \lambda_{21} \rangle$ is a free 2-generated group and V_2^* is the normal closure of group $V_2 = \langle \lambda_{13}, \lambda_{23}, \lambda_{31}, \lambda_{32} \rangle$ in VP_3. The group V_2 is a free of rank 4. We can represent generators λ_{ij} as virtual braids (see [2]) and use the operations of doubling of string we construct new generators of VP_3 from generators of V_1. We have

$$a_1 = s_0(\lambda_{12}) = \lambda_{13}\lambda_{23}, \ b_1 = s_1(\lambda_{12}) = \lambda_{13}\lambda_{12},$$

$$b_2 = s_0(\lambda_{21}) = \lambda_{21}\lambda_{31}, \ a_2 = s_1(\lambda_{21}) = \lambda_{32}\lambda_{31},$$

and it is easy to check that

$$VP_3 = \langle a_1, a_2, b_1, b_2, \lambda_{13}, \lambda_{31} \rangle.$$

Write the old generators as words in new generators:

$$\begin{array}{rcl} \lambda_{12} & = & \lambda_{13}^{-1} b_1, \\ \lambda_{21} & = & b_2 \lambda_{31}^{-1}, \\ \lambda_{23} & = & \lambda_{13}^{-1} a_1, \\ \lambda_{32} & = & a_2 \lambda_{31}^{-1}. \end{array}$$

Then VP_3 is defined by the following set of relations:

$$\begin{array}{rcl} \lambda_{12}(\lambda_{13}\lambda_{23}) & = & (\lambda_{23}\lambda_{13})\lambda_{12}), \\ \lambda_{21}(\lambda_{23}\lambda_{13}) & = & (\lambda_{13}\lambda_{23})\lambda_{21}), \\ \lambda_{13}(\lambda_{12}\lambda_{32}) & = & (\lambda_{32}\lambda_{12})\lambda_{13}), \\ \lambda_{31}(\lambda_{32}\lambda_{12}) & = & (\lambda_{12}\lambda_{32})\lambda_{31}), \\ \lambda_{23}(\lambda_{21}\lambda_{31}) & = & (\lambda_{31}\lambda_{21})\lambda_{23}), \\ \lambda_{32}(\lambda_{31}\lambda_{21}) & = & (\lambda_{21}\lambda_{31})\lambda_{32}). \end{array}$$

These relations in new generators have the following form

$$\begin{array}{rcl} a_1 b_1 & = & b_1 a_1, \\ b_1^{a_2} & = & b_1^{\lambda_{13}\lambda_{31}}, \\ b_2^{a_1^{-1}} & = & b_2^{(\lambda_{13}\lambda_{31})^{-1}}, \\ a_1^{b_2} & = & a_1^{\lambda_{13}\lambda_{31}}, \\ a_2^{b_1^{-1}} & = & a_2^{(\lambda_{13}\lambda_{31})^{-1}}, \\ a_2 b_2 & = & b_2 a_2, \end{array}$$

where $y^x = x^{-1} y x$.

Let $c_1 = \lambda_{13}\lambda_{31}$ and $c_2 = \lambda_{13}$. Then

$$VP_3 = \left\langle a_1, a_2, b_1, b_2, c_1, c_2 \;\middle\|\; \begin{array}{rcl} [a_1, b_1] & = & [a_2, b_2] = 1, \\ b_1^{c_1} & = & b_1^{a_2}, \\ a_1^{c_1} & = & a_1^{b_2}, \\ b_2^{c_1} & = & b_2^{a_1 b_2}, \\ a_2^{c_1} & = & a_2^{b_1 a_2} \end{array} \right\rangle$$

Hence $VP_3 = G_3 * \langle c_2 \rangle$ and $G_3 = H_3 \rtimes \langle c_1 \rangle$, where H_3 is a subgroup of G_3 with the set of generators a_1, b_1, a_2, b_2 and with infinite set of relations

$$[a_i, b_i]^{c_1^k} = 1$$

for $i = 1, 2$ and $k \in \mathbb{Z}$.

There exist a homomorphism

$$\varphi\colon VP_3 \longrightarrow Cb_3, \quad \varphi(\lambda_{ij}) = \varepsilon_{ij}, \quad 1 \leq i \neq j \leq 3.$$

We will denote the images of a_i, b_i, c_i, by α_i, β_i, γ_i, $i = 1, 2$, correspondingly.

In Cb_3 hold the set of defining relations from VP_3 and relations

$$\begin{array}{rcl} \varepsilon_{13}\varepsilon_{23} & = & \varepsilon_{23}\varepsilon_{13}, \\ \varepsilon_{12}\varepsilon_{32} & = & \varepsilon_{32}\varepsilon_{12}, \\ \varepsilon_{21}\varepsilon_{31} & = & \varepsilon_{31}\varepsilon_{21}. \end{array}$$

Since

$$\begin{array}{rcl} \varepsilon_{13} & = & \gamma_2, \\ \varepsilon_{31} & = & \gamma_2^{-1}\gamma_1, \\ \varepsilon_{12} & = & \gamma_2^{-1}\beta_1, \\ \varepsilon_{21} & = & \beta_2\gamma_1^{-1}\gamma_2, \\ \varepsilon_{23} & = & \gamma_2^{-1}\alpha_1, \\ \varepsilon_{32} & = & \alpha_2\gamma_1^{-1}\gamma_2, \end{array}$$

this set of defining relations has the form

$$\begin{array}{rcl} \alpha_1 & = & \gamma_2^{-1}\alpha_1\gamma_2, \\ \gamma_2^{-1}\beta_1\alpha_2\gamma_1^{-1}\gamma_2 & = & \alpha_2\gamma_1^{-1}\beta_1, \\ \beta_2 & = & \gamma_2^{-1}\gamma_1\beta_2\gamma_1^{-1}\gamma_2. \end{array}$$

Rewrite this set as:

$$\begin{array}{rcl} \alpha_1^{\gamma_2} & = & \alpha_1, \\ (\beta_1\alpha_2\gamma_1^{-1})^{\gamma_2} & = & \alpha_2\gamma_1^{-1}\beta_1, \\ (\gamma_1\beta_2\gamma_1^{-1})^{\gamma_2} & = & \beta_2. \end{array}$$

We need the following lemma.

Lemma 5.2.

$$Cb_3 = \langle G_3, \gamma_2 \,||\, \gamma_2^{-1}\, A\, \gamma_2 = B, \psi \rangle$$

is an HNN-extension with associated subgroups

$$\begin{array}{rcl} A & = & \langle \alpha_1, \beta_1\alpha_2\gamma_1^{-1}, \gamma_1\beta_2\gamma_1^{-1} \rangle, \\ B & = & \langle \alpha_1, \alpha_2\gamma_1^{-1}\beta_1, \beta_2 \rangle, \end{array}$$

and the isomorphism $\psi\colon A \longrightarrow B$ *is defined by the rule*

$$\psi\colon \left\{ \begin{array}{lcl} \alpha_1 & \longrightarrow & \alpha_1, \\ \beta_1\alpha_2\gamma_1^{-1} & \longrightarrow & \alpha_2\gamma_1^{-1}\delta_1, \\ \gamma_1\beta_2\gamma_1^{-1} & \longrightarrow & \beta_2, \end{array} \right.$$

where the subgroup G_3 *is from* VP_3.

We know that Cb_3 is residually torsion free nilpotent (see [1]) hence its subgroup G_3 is residually torsion free nilpotent too. But $VP_3 = G_3 * \mathbb{Z}$ and using result of Malcev [3] we have the result. □

Proof of Lemma 5.2. We have to prove that $A \simeq B$. Find sets of generators of B and A in old generators of Cb_3. We have

$$B = \langle \varepsilon_{13}\varepsilon_{23}, \quad \varepsilon_{32}\varepsilon_{12}, \quad \varepsilon_{21}\varepsilon_{31} \rangle$$

and we see that $\varepsilon_{13}\varepsilon_{23}$ is an automorphism of F_3 which are conjugation by x_3, $\varepsilon_{32}\varepsilon_{12}$ is an automorphism of F_3 which are conjugation by x_2, $\varepsilon_{21}\varepsilon_{31}$ is an automorphism of F_3 which are conjugation by x_1. Hence

$$B = \langle \widehat{x_1}, \widehat{x_2}, \widehat{x_3} \rangle \simeq F_3,$$

where $\widehat{y}$ is an inner automorphism of F_3 which is conjugation by y.

Similar,

$$A = \langle \widehat{x_2}, \widehat{x_3}, \widehat{x_3 x_1 x_3^{-1}} \rangle \simeq F_3,$$

and $A \simeq B$. □

Since $\mathbb{Z} \times \mathbb{Z} = \langle a_1, b_1 \rangle \leq VP_3$ then we have

Corollary 5.3. *VP_3 is not word hyperbolic.*

Corollary 5.4. $\mathrm{H}_1(VP_3) = \mathrm{H}_2(VP_3) = \mathbb{Z}^{\oplus 6}$ *and* $\mathrm{H}_n(VP_3) = 0$ *for* $n > 2$.

Question: Is VP_n residually nilpotent for any $n \geq 4$?

References

[1] S. Andreadakis, On the automorphisms of free groups and free nilpotent groups, *Proc. London Math. Soc.*, Ser 3, 1965, **15**, 239–268.

[2] V. Bardakov, The virtual and universal braids, *Fund. Math.*, **181**(2004), 1–18.

[3] A. I. Malcev, Generalized nilpotent algebras and their associated groups, *Mat. Sbornik N.S.*, **25**(1949), 347–366.