HOMOTOPY THEORY OF LIE FUNCTORS

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1. Introduction

Let Ab be the category of abelian groups. By the classical Dold-Kan theorem, the Moore normalization functor

$$N : \mathsf{SAb} \to \mathsf{Ch}$$

defines an equivalence between the category SAb of simplicial abelian groups and the category Ch of non-negatively graded chain complexes. Denote by $K : \mathsf{Ch} \to \mathsf{SAb}$ the Dold-Kan transform inverse to the Moore normalization functor. Also denote by DAb the derived category, obtained from Ch by inverting the weak equivalences.

Given a covariant functor $F : \mathsf{Ab} \to \mathsf{Ab}$ with $F(0) = 0$, one can ask about the structure of its derived functors, or generally speaking, about "the homotopy theory of F". For an element $C \in \mathsf{Ch}$, one can ask how to describe the graded abelian groups

$$H_*(LF(C)) = \pi_*(F(K(C))))$$

These homotopy groups depend only on the class of the element C in DAb, hence we can ask the following: for an element $C \in \mathsf{DAb}$, describe the groups $H_*(LF(C))$ as functors from DAb to the category of graded abelian groups.

In the present paper, we construct the bigraded functors

$$\mathcal{E}^m(-, n) : \mathsf{DAb} \to \mathsf{DAb}$$
$$\widetilde{\mathcal{E}}^m(-, n) : \mathsf{DAb} \to \mathsf{DAb}$$

such that, for all $C \in \mathsf{DAb}$, there are absract isomorphisms of graded abelian groups

$$L_*\mathcal{L}^m(C[n]) \simeq H_*(\mathcal{E}^m(C, n)) \tag{1.1}$$
$$L_*\mathcal{L}^m_s(C[n]) \simeq H_*(\widetilde{\mathcal{E}}^m(C, n)) \tag{1.2}$$

Here $\mathcal{L}$ and $\mathcal{L}_s$ are graded Lie and super-Lie functors with squares respectively (for the definition see the next section). The homotopy groups of $\mathcal{E}^m(-, n)$ and $\widetilde{\mathcal{E}}^m(-, n)$ can be described. In particular, this will give a way how to compute (abstractly) the derived functors of Lie functors and super-Lie functors in all dimensions. Moreover, we show that, in the case when C is a free abelian group, the isomorphisms (1.1), (1.2) are natural. That is, this gives a complete description of all derived functors of $\mathcal{L}$, $\mathcal{L}_s$ for free abelian groups.

For certain degrees we will give a general functorial description of derived functors of Lie and super-Lie functors (not only for free abelian groups). It is shown in [14] that if p is an odd prime then the groups $L_{n+k}\mathcal{L}^p(\mathbb{Z}, n)$ are p-torsion for all k, and in particular

$$L_{n+k}\mathcal{L}^p(\mathbb{Z}, n) = \begin{cases} \mathbb{Z}/p, \ k = 2i(p-1)-1, \ i = 1, 2, \dots, [n/2] \\ 0, \ \text{otherwise} \end{cases} \tag{1.3}$$

For example, in the simplest case, our theory gives the following functorial generalization of the above description (for $n \geq 0$):

$$L_i\mathcal{L}^p(A,2n) = \begin{cases} \mathcal{L}^p(A)\oplus \mathrm{Tor}(A,\mathbb{Z}/p),\ i=2np\\ L_j\mathcal{L}^p(A),\ i=2np+j, j=1,\dots,p-1\\ A\otimes\mathbb{Z}/p,\ i=2n+2j(p-1)-1,\ j=1,2,\dots,n\\ \mathrm{Tor}(A,\mathbb{Z}/p),\ i=2n+2j(p-1),\ j=1,2,\dots,n-1\end{cases}$$

$$L_i\mathcal{L}^p(A,2n+1) = \begin{cases} L_j\mathcal{L}^p_s(A),\ i=2np+j, j=0,\dots,p-1\\ A\otimes\mathbb{Z}/p,\ i=2n+2j(p-1),\ j=1,2,\dots,n\\ \mathrm{Tor}(A,\mathbb{Z}/p),\ i=2n+2j(p-1)+1,\ j=1,2,\dots,n\end{cases}$$

In some cases the description of derived functors is very simple. For example, for a free abelian group A, the derived functors $L_i\mathcal{L}^6(A,1)$ and $L_i\mathcal{L}^6_s(A,1)$ are the following:

$$L_i\mathcal{L}^6(A,1) = \begin{cases} \mathcal{L}^6_s(A),\ i=6\\ \Gamma_2(A)\otimes\mathbb{Z}/3,\ i=5\\ \mathcal{L}^3_s(A)\otimes\mathbb{Z}/2,\ i=4\\ 0,\ i\neq 4,5,6\end{cases} \qquad L_i\mathcal{L}^6_s(A,1) = \begin{cases} \mathcal{L}^6(A),\ i=6\\ \Lambda^2(A)\otimes\mathbb{Z}/3\oplus\mathcal{L}^3(A)\otimes\mathbb{Z}/2,\ i=5\\ \mathcal{L}^3(A)\otimes\mathbb{Z}/2,\ i=4\\ 0,\ i\neq 3,5,6\end{cases}$$

In a more general case, for a free abelian A, if m is a square free number, i.e. m is a product of different primes, then there is the following description of the derived functors:

$$\bigoplus_{i=1}^{2nm} L_i\mathcal{L}^m(A,2n)[i] \simeq \mathcal{L}^m(A)[2nm]\oplus \bigoplus_{\substack{p \text{ prime}\\ p|m}} \bigoplus_{i=1}^{\frac{mn}{p}} \mathcal{L}^{\frac{m}{p}}(A)\otimes\mathbb{Z}/p\ [\frac{2mn}{p}+(2p-2)i-1]$$

To describe the derived functors in a general case, we introduce the collection of special functors $\mathcal{N}^{k;p}, \mathcal{N}^{k;p} : \mathsf{Ab}\to\mathsf{Ab}$ (p prime, $k\geq 1$). We call the collection of these functors "hierarchies" by the following reason. For any prime p and $k \geq 1$, there are chains of natural epimorphisms

$$\mathcal{N}^{kp^i;p} \twoheadrightarrow \dots \mathcal{N}^{kp;p} \twoheadrightarrow \mathcal{N}^{k;p}$$

which split abstractly, but not naturally, moreover, every natural transformation of the type $\mathcal{N}^{k;p}\to\mathcal{N}^{kp;p}$ is the zero map. These functors play a crucial role in the description of derived functors of Lie and super-Lie functors. For example, the functor $L_9\mathcal{L}^9(A,2)$ is naturally isomorphic to the functor $\mathcal{N}^{3;3}(A)$ and can be presented in the short exact sequence

$$0\to\mathcal{L}^3(A)\otimes\mathbb{Z}/3\to L_9\mathcal{L}^9(A,2)\to A\otimes\mathbb{Z}/3\to 0$$

which splits as a sequence of abelian groups and does not split as a sequence of functors. Moreover, every natural transormation $A\otimes\mathbb{Z}/3\to L_9\mathcal{L}^9(A,2)$ is the zero map.

Notation.

$\Lambda^n : \mathsf{Ab}\to\mathsf{Ab}$ the nth exterior power;

$SP^n : \mathsf{Ab}\to\mathsf{Ab}$ the nth symmetric power;

$\Gamma_n : \mathsf{Ab}\to\mathsf{Ab}$ the nth divided power;

$\mathcal{L}^n : \mathsf{Ab} \to \mathsf{Ab}$ the nth graded Lie functor;

$\mathcal{L}^n_s : \mathsf{Ab} \to \mathsf{Ab}$ the nth super-Lie functor with squares;

$\mathbb{L} : \mathsf{Ch} \to \mathsf{Ch}$ the universal differential graded Lie algebra with squares (see (7.4));

$\mathcal{N}^{n;p} : \mathsf{Ab} \to \mathsf{Ab}$ the special functors (see (9.1), (9.2));

$J^n, Y^n : \mathsf{Ab} \to \mathsf{Ab}$ the special Schur functors (see (2.10), (2.11)) which coincide with metabelian nth Lie and super-Lie functors;

$\mathcal{E}^m(-,n),\ \widetilde{\mathcal{E}}^m(-,n) : \mathsf{DAb} \to \mathsf{DAb}$ the $\mathcal{E}$-functors (see 9.2);

2. Graded Lie rings

The tensor algebra $\otimes A$ is endowed with a $\mathbb{Z}$-Lie algebra structure, for which the bracket operation is defined by

$$[a,\, b] = a \otimes b - b \otimes a, \quad a, b \in \otimes(A).$$

One defines n-fold brackets inductively by setting

$$[a_1, \dots, a_n] := [[a_1 \dots, a_{n-1}], a_n] \tag{2.1}$$

We will denote $\otimes A$, viewed as a $\mathbb{Z}$-Lie algebra, by $\otimes(A)^{Lie}$. Let $\mathcal{L}(A) = \bigoplus_{n\geq 1} \mathcal{L}^n(A)$ be the sub-Lie ring of $\otimes(A)^{Lie}$ generated by A. Its degree 2 and 3 components are generated by the expressions

$$a \otimes b - b \otimes a \qquad \text{and} \qquad a \otimes b \otimes c - b \otimes a \otimes c + c \otimes a \otimes b - c \otimes b \otimes a \tag{2.2}$$

where $a, b, c \in A$. $\mathcal{L}(A)$ is called the *free Lie ring generated by the abelian group A*. It is universal for homomorphisms from A to $\mathbb{Z}$-Lie algebras. The grading of $\otimes A$ determines a grading on $\mathcal{L}(A)$, so that we obtain a family of endofunctors on the category of abelian groups:

$$\mathcal{L}^i : \mathsf{Ab} \to \mathsf{Ab},\ i \geq 1.$$

The universal property of the Lie functor implies that there is a natural transformation of the graded functors:

$$\mathcal{L}\mathcal{L} \to \mathcal{L}$$

which assigns to an abelian group A, the unique map

$$\mathcal{L}\mathcal{L}(A) \to \mathcal{L}(A)$$

which is the identity on $\mathcal{L}_1\mathcal{L}(A) = \mathcal{L}(A)$.

Definition 2.1. [13] *A graded Lie ring with squares (GLRS for short) is a graded abelian group $B = \bigoplus_{i=0}^{\infty} B_i$ with homomorphisms*

$$\{\ ,\ \} : B_i \otimes B_j \to B_{i+j}, \tag{2.3}$$

$$^{[2]} : B_n \to B_{2n} \text{ for } n \text{ odd} \tag{2.4}$$

such that the following conditions are satisfied (for elements $x \in B_i,\ y \in B_j,\ z \in B_k$):

$$1)\ \{x,y\} + (-1)^{ij}\{y,x\} = 0 \tag{2.5}$$
$$2)\ \{x,x\} = 0 \ \ \textit{for } i \textit{ even}$$
$$3)\ (-1)^{ik}\{\{x,y\},z\} + (-1)^{ji}\{\{y,z\},x\} + (-1)^{kj}\{\{z,x\},y\} = 0 \tag{2.6}$$
$$4)\ \{x,x,x\} = 0$$
$$5)\ (ax)^{[2]} = a^2x^{[2]} \ \textit{for}\ \ i \textit{ odd},\ a \in \mathbb{Z}$$
$$6)\ (x+y)^{[2]} = x^{[2]} + y^{[2]} + \{x,y\} \ \textit{for}\ \ i = j \textit{ odd}$$
$$7)\ \{y, x^{[2]}\} = \{y,x,x\} \ \ \textit{for } i \textit{ odd.} \tag{2.7}$$

For an abelian group A, define $\mathcal{L}_s(A)$ to be the graded Lie ring with squares freely generated by A in degree 1. It may be defined as a GLRS together with a homomorphism of abelian groups $l : A \to \mathcal{L}_s(A)$ such that for every map $f : A \to B$ with B a GLRS, there is a unique morphism of GLRS $d : \mathcal{L}_s(A) \to B$ such that $f = d \circ l$. The abelian group $\mathcal{L}_s(A)$ is naturally graded by $\mathcal{L}_s(A) = \bigoplus_{n=1}^{\infty} \mathcal{L}_s^n(A)$ and for any $x \in \mathcal{L}_s(A)$, we set $|x| = n$ whenever $x \in \mathcal{L}_s^n(A)$.

Theorem 2.1. (Schlesinger) *Let $A_1, A_2, \dots, A_n$ be free abelian groups. There is a natural isomorphism*

$$\mathcal{L}\Big(\bigoplus_{i=1}^{n} A_i\Big) = \bigoplus_{i=1}^{n} \mathcal{L}(A_i) \oplus \bigoplus_{J} \mathcal{L}(A_J),$$

where $J = (j_1, \dots, j_k),\ j_1 > j_2 \leq \dots \leq j_k,\ k \geq 2$ and $A_J = A_{j_1} \otimes A_{j_2} \otimes \dots \otimes A_{j_k}$.

The map

$$A_J \to \mathcal{L}\Big(\bigoplus_{i=1}^{n} A_i\Big),\ \ J = (j_1, \dots, j_k)$$

is given by

$$a_{j_1} \otimes \dots \otimes a_{j_k} \mapsto [a_{j_1}, a_{j_2}, \dots, a_{j_k}],\ \ a_{j_i} \in A_{j_i}.$$

For example, for $n = 2$ and free abelian groups A, B, we have the following decomposition:

$$\mathcal{L}(A \oplus B) = \mathcal{L}(A) \oplus \mathcal{L}(B) \oplus \mathcal{L}(B \otimes A) \oplus \mathcal{L}(B \otimes A \otimes A) \oplus \mathcal{L}(B \otimes A \otimes B) \oplus$$
$$\mathcal{L}(B \otimes A \otimes A \otimes A) \oplus \mathcal{L}(B \otimes A \otimes A \otimes B) \oplus \mathcal{L}(B \otimes A \otimes B \otimes B) \oplus \dots$$

This gives a way to compute all cross-effects of the graded components of Lie functors:

$$\mathcal{L}^m(A \oplus B) = \bigoplus_{d|m,\ 1 \leq d \leq m} \ \bigoplus_{C \in J_{m/d}} \mathcal{L}^d(C), \tag{2.8}$$

where $J_{m/d}$ is the set of all basic tensor products of weight m/d in A and B. For example,

$$\mathcal{L}^2(A \oplus B) = \mathcal{L}^2(A) \oplus \mathcal{L}^2(B) \oplus A \otimes B$$
$$\mathcal{L}^3(A \oplus B) = \mathcal{L}^3(A) \oplus \mathcal{L}^3(B) \oplus (A \otimes B \otimes B) \oplus (A \otimes B \otimes A)$$
$$\mathcal{L}^4(A \oplus B) = \mathcal{L}^4(A) \oplus \mathcal{L}^4(B) \oplus \mathcal{L}^2(A \otimes B) \oplus (A \otimes B \otimes B \otimes B) \oplus$$
$$(A \otimes B \otimes B \otimes A) \oplus (A \otimes B \otimes A \otimes A)$$

The number of all basic products of r modules $A_1, \dots, A_r$ of weight m is given by the following formula:

$$M_r(m) = \frac{1}{m}\sum_{d|m} \mu(d) r^{\frac{m}{d}}$$

where $\mu(d)$ is the Möbius function. The number of all basic products of r modules $A_1, \dots, A_r$ with m_i entries in A_i (i.e. $m = m_1 + \dots + m_r$) is given by the following formula:

$$M(m_1, \dots, m_r) = \frac{1}{m}\sum_{d|m_i} \mu(d) \frac{\left(\frac{m}{d}\right)!}{\left(\frac{m_1}{d}\right)! \dots \left(\frac{m_r}{d}\right)!}$$

By definition, we have

$$|J_{m/d}| = M_2(m/d)$$

in the formula (2.8).

The cross-effects of graded components of super-Lie functors with squares are the following:

Proposition 2.1. *For free abelian A and B, one has*

$$\mathcal{L}_s^m(A \oplus B) = \bigoplus_{\substack{d|m,\ 1 \le d \le m \\ m/d\ odd}} \bigoplus_{C \in J_{m/d}} \mathcal{L}_s^d(C) \oplus \bigoplus_{\substack{d|m,\ 1 \le d < m \\ m/d\ even}} \bigoplus_{C \in J_{m/d}} \mathcal{L}^d(C) \tag{2.9}$$

where C runs over all basic tensor products of weight m/d in A and B.

For example,

$$\begin{aligned}
\mathcal{L}_s^2(A \oplus B) &= \mathcal{L}_s^2(A) \oplus \mathcal{L}_s^2(B) \oplus A \otimes B \\
\mathcal{L}_s^3(A \oplus B) &= \mathcal{L}_s^3(A) \oplus \mathcal{L}_s^3(B) \oplus (A \otimes B \otimes B) \oplus (A \otimes B \otimes A) \\
\mathcal{L}_s^4(A \oplus B) &= \mathcal{L}_s^4(A) \oplus \mathcal{L}_s^4(B) \oplus \mathcal{L}^2(A \otimes B) \oplus (A \otimes B \otimes B \otimes B) \oplus \\
&\qquad (A \otimes B \otimes B \otimes A) \oplus (A \otimes B \otimes A \otimes A)
\end{aligned}$$

2.1. **Curtis decomposition.** Consider the Schur functors

$$J^n, Y^n : \mathsf{Ab} \to \mathsf{Ab},\ n \ge 2$$

defined by [1]

$$J^n(A) = ker\{A \otimes SP^{n-1}(A) \to SP^n(A)\},\ n \ge 2, \tag{2.10}$$

$$Y^n(A) = ker\{A \otimes \Lambda^{n-1}(A) \to \Lambda^n(A)\},\ n \ge 2 \tag{2.11}$$

Curtis gave in [5] a decomposition of the functors $\mathcal{L}^m(A)$ in terms of functors SP^m, J^m and their iterates. Analogous decomposition exists also in the super-Lie case (see [3]). For a free abelian group A, there are natural exact sequences

$$0 \to \widetilde{J}^m(A) \to \mathcal{L}^m(A) \overset{p_m}{\to} J^m(A) \to 0 \tag{2.12}$$

$$0 \to \widetilde{Y}^m(A) \to \mathcal{L}_s^m(A) \overset{\bar{p}_m}{\to} Y^m(A) \to 0 \tag{2.13}$$

[1]The functors $Y^n(A)$ are the $\mathbb{Z}$-forms of the Schur functors $\mathbb{S}_\lambda(V)$ associated to the partition $\lambda = (2, 1 \dots, 1)$ of the set (n) (see [9] exercise 6.11). The functors $J^n(A)$ and the $\mathbb{Z}$- forms of the Schur functors $\mathbb{S}_\mu$ associated to the partition $\mu = (n-1, 1)$ of (n), which is the conjugate partition of λ.

where

$$\begin{aligned}
p_m:\ &[a_1,\dots,a_m]\mapsto a_1\otimes a_2\dots a_m - a_2\otimes a_1a_3\dots a_m,\\
\bar p_m: &\{a_1,\dots,a_m\}\mapsto a_1\otimes a_2\wedge a_3\wedge\dots\wedge a_m + a_2\otimes a_1\wedge a_3\wedge\dots\wedge a_m\\
\bar p_2: &\, a_1^{[2]}\mapsto a_1\otimes a_1,\\
\bar p_{2m}: &\{a_1,\dots,a_m\}^{[2]}\mapsto 0,\ \text{if } m \text{ is odd}
\end{aligned}$$

for $a_i\in A$. Here $\widetilde{J}^m(A)$ and $\widetilde{Y}^m(A)$ are defined as kernels of natural projections p_m and $\bar p_m$. For low degrees the sequence (2.12) is the following:

$$\begin{aligned}
&\mathcal{L}^2(A)\overset{p_2}{\simeq} J^2(A)\\
&\mathcal{L}^3(A)\overset{p_3}{\simeq} J^3(A)\\
&0\to\Lambda^2\Lambda^2(A)\to\mathcal{L}^4(A)\overset{p_4}{\to} J^4(A)\to 0\\
&0\to\Lambda^2(A)\otimes J^3(A)\to\mathcal{L}^5(A)\overset{p_5}{\to} J^5(A)\to 0,
\end{aligned}$$

where the left-hand arrows are respectively defined by

$$\begin{aligned}
(a\wedge b)\wedge(c\wedge d)&\mapsto [[a,b],[c,d]]\\
(a\wedge b)\otimes(c,d,e)&\mapsto [[a,b],[c,d,e]]\,.
\end{aligned}$$

The super-analogs of Curtis decomposition can be constructed analogously (see [3]). In low degrees the sequence (2.13) is the following:

$$\begin{aligned}
&\mathcal{L}_s^2(A)\overset{\bar p_2}{\simeq} Y^2(A)\\
&\mathcal{L}_s^3(A)\overset{\bar p_3}{\simeq} Y^3(A)\\
&0\to\Lambda^2Y^2(A)\to\mathcal{L}_s^4(A)\overset{\bar p_4}{\to} Y^4(A)\to 0\\
&0\to Y^2(A)\otimes Y^3(A)\to\mathcal{L}_s^5(A)\overset{\bar p_5}{\to} Y^5(A)\to 0
\end{aligned}$$

3. Derived functors

Let A be an abelian group, and F an endofunctor on the category of abelian groups. Recall that for every $n\geq 0$ the derived functor of F in the sense of Dold-Puppe [7] are defined by

$$L_iF(A,n)=\pi_i(FKP_*[n]),\ i\geq 0$$

where $P_*\to A$ is a projective resolution of A, and K is the Dold-Kan transform, inverse to the Moore normalization functor

$$N:\mathrm{Simpl}(\mathsf{Ab})\to C(\mathsf{Ab})$$

from simplicial abelian groups to chain complexes. We denote by $LF(A,n)$ the object $FK(P_*[n])$ in the homotopy category of simplicial abelian groups determined by $FK(P_*[n])$, so that

$$L_iF(A,n)=\pi_i(LF(A,n))\,.$$

We set $LF(A):=LF(A,0)$ and $L_iF(A):=L_iF(A,0)$ for any $i\geq 0$.

As examples of these constructions, observe that the simplicial models $LF(L \longrightarrow M)$ of LFA and $FK((L \longrightarrow M)[1])$ of $LF(A,1)$ associated to the two-term flat resolution

$$0 \to L \xrightarrow{f} M \to A \to 0 \tag{3.1}$$

of an abelian group A are respectively of the following form in low degrees:

$$F(s_0(L) \oplus s_1(L) \oplus s_1s_0(M)) \overset{\partial_0,\partial_1,\partial_2}{\rightrightarrows} F(L \oplus s_0(M)) \overset{\partial_0,\partial_1}{\rightrightarrows} F(M) \tag{3.2}$$

where the component $F(M)$ is in degree zero, and

$$\begin{array}{c} F(s_0(L) \oplus s_1(L) \oplus s_2(L) \oplus \\ s_1s_0(M) \oplus s_2s_0(M) \oplus s_2s_1(M)) \end{array} \overset{\partial_0,\dots,\partial_3}{\rightrightarrows} F(L \oplus s_1(M) \oplus s_0(M)) \overset{\partial_0,\partial_1,\partial_2}{\rightrightarrows} F(M) \tag{3.3}$$

where the component $F(M)$ is in degree 1.

3.1. **Homotopy operations.** For a simplicial abelian group X, $i, k, l, q \geq 1$, the composition is given as a natural map

$$L_i\mathcal{L}^k(\mathbb{Z}, q) \otimes \pi_q\mathcal{L}^l(X) \to \pi_i\mathcal{L}^{kl}(X) \tag{3.4}$$

It is constructed as follows. Let $\alpha \in \pi_q\mathcal{L}^l(X)$. Take a map $f : K(\mathbb{Z}, q) \to \mathcal{L}^l(X)$ which represents α in the homotopy group π_q. This map induces the composition map

$$L\mathcal{L}^k(\mathbb{Z}, q) \to L(\mathcal{L}^k \circ \mathcal{L}^l)(X) \to L\mathcal{L}^{kl}(X)$$

Taking the i-th homotopy groups we obtain the map (3.4).

4. Allowable sets

The description of allowable sets is given in [11].

Let $n \geq 1$, $k \geq 1$. We call a sequence $(i_1, \dots, i_k)$ allowable with respect to $2n$ if

1) $i_1 \leq 2n$, $i_{j+1} \leq 2i_j$ for all $j = 1, \dots, k-1$
2) i_k is odd.

Denote the set of allowable sequences with respect to $2n$ of the length k by $\mathcal{W}_{2n,k}$ (or $\mathcal{W}_{2n,k}^{(2)} := \mathcal{W}_{2n,k}$). For $k \geq 2$, define the filtration

$$\mathcal{W}_{2n,k}^{(2)} = \mathcal{W}_{2n,k}^{(2)}(1) \supset \mathcal{W}_{2n,k}^{(2)}(2) \supset \cdots \supset \mathcal{W}_{2n,k}^{(2)}(k)$$

as follows: the subset $\mathcal{W}_{2n,k}^{(2)}(j)$ consists of allowable sequences $(i_1, \dots, i_k) \in \mathcal{W}_{2n,k}^{(2)}$ with $i_1 = 2n, i_2 = 4n, \dots, i_{j-1} = 2^{j-1}n$.

Example.

$$\begin{aligned}
\mathcal{W}^{(2)}_{2,3}(1) = \{&(1,1,1),\ (2,1,1),\ (2,2,1),\ (1,2,3),\ (1,2,1)\\
&(2,3,1),\ (2,2,3),\ (2,3,3),\ (2,4,3),\\
&(2,3,5),\ (2,4,5),\ (2,4,7),\ (2,4,1)\}\\
\mathcal{W}^{(2)}_{2,3}(2) = \{&(2,1,1),\ (2,2,1),\ (2,3,1),\ (2,2,3),\ (2,3,3),\\
&(2,4,3),\ (2,3,5),\ (2,4,5),\ (2,4,7),\ (2,4,1)\}\\
\mathcal{W}^{(2)}_{2,3}(3) = \{&(2,4,1),\ (2,4,3),\ (2,4,5),\ (2,4,7)\}
\end{aligned}$$

For $w = (i_1, \dots, i_k) \in \mathcal{V}_{2n,k}$, let $o(w)$ be the number of odd elements in w and $d(w) = i_1 + \dots + i_k$.

Let p be an odd prime. Define the set $\mathcal{W}^{(p)}_{2n,k}$ as follows. The set $\mathcal{W}^{(p)}_{2n,k}$ consists of sequences $(\nu_{i_1} \dots \nu_{i_k})$ with ν_{i_j} equal either λ_{i_j} or $\nu_{i_j} = \mu_{i_j}$, and such that $i_1 \leq n,\ i_{j+1} \leq pi_j - 1$ whenever $\nu_{i_j} = \lambda_{i_j}$ and $i_{j+1} \leq pi_j$, whenever $\nu_{i_j} = \mu_{i_j}$ and $\nu_{i_k} = \lambda_{i_k}$. For $k \geq 2$, define the filtration

$$\mathcal{W}^{(p)}_{2n,k} = \mathcal{W}^{(p)}_{2n,k}(1) \supset \mathcal{W}^{(p)}_{2n,k}(2) \supset \dots \supset \mathcal{W}^{(p)}_{2n,k}(k)$$

as follows: the subset $\mathcal{W}^{(p)}_{2n,k}(j)$ consists of allowable sequences $(\nu_{i_1}, \dots, \nu_{i_k}) \in \mathcal{W}^{(p)}_{2n,k}$ with $\nu_{i_1} = \mu_n, \nu_{i_2} = \mu_{pn}, \dots, \nu_{i_{j-1}} = \mu_{p^{j-1}n}$.

Example.

$$\begin{aligned}
\mathcal{W}^{(3)}_{2,2}(1) &= \{(\lambda_1,\lambda_1),\ (\lambda_1,\lambda_2),\ (\mu_1,\lambda_1),\ (\mu_1,\lambda_2),\ (\mu_1,\lambda_3)\}\\
\mathcal{W}^{(3)}_{2,2}(2) &= \{(\mu_1,\lambda_1),\ (\mu_1,\lambda_2),\ (\mu_1,\lambda_3)\}\\
\mathcal{W}^{(3)}_{4,2}(1) &= \{(\lambda_1,\lambda_1),\ (\lambda_1,\lambda_2),\ (\mu_1,\lambda_1),\ (\mu_1,\lambda_2),\ (\mu_1,\lambda_3),(\lambda_2,\lambda_1),\\
&\quad (\lambda_2,\lambda_2),\ (\lambda_2,\lambda_3),\ (\lambda_2,\lambda_4),(\lambda_2,\lambda_5),\\
&\quad (\mu_2,\lambda_1),\ (\mu_2,\lambda_2),\ (\mu_2,\lambda_3),\ (\mu_2,\lambda_4),\ (\mu_2,\lambda_5),\ (\mu_2,\lambda_6)\}\\
\mathcal{W}^{(3)}_{4,2}(2) &= \{(\mu_2,\lambda_1),\ (\mu_2,\lambda_2),\ (\mu_2,\lambda_3),\ (\mu_2,\lambda_4),\ (\mu_2,\lambda_5),\ (\mu_2,\lambda_6)\}
\end{aligned}$$

For $\nu \in \mathcal{W}^{(p)}_{2n,k}$, let $o(\nu)$ be the number of entries of λ_{i_j} in ν and

$$d(\nu) = (2p-2)(i_1 + \dots + i_k) - o(\nu).$$

4.1. We will need also the following notation. Let p be an odd prime

$$\widetilde{\mathcal{W}}^{(p)}_{2n,k} \subseteq \mathcal{W}^{(p)}_{2n,k}$$

consists of all sequences $(\nu_{i_1}, \dots, \nu_{i_k}) \in \mathcal{W}^{(p)}_{2n,k}$ such that $i_k > \frac{p^{k-1}-1}{2}$. For $p = 2$, $\widetilde{\mathcal{W}}^{(2)}_{2n,k}$ consists of all sequences $(i_1, \dots, i_k) \in \mathcal{W}^{(2)}_{2n,k}$ such that $i_k \geq 2^{k-1}$.

5. Derived functors $L_*\mathcal{L}(\mathbb{Z}, m)$

Theorem 5.1. (Bousfield) *For $n > 0$ and a pointed simplicial set Y, the natural map*

$$\pi_*(\mathcal{L}K(\mathbb{Z}, 2n) \otimes \mathbb{Z}[Y]) \to \pi_*\mathcal{L}(K(\mathbb{Z}, 2n) \otimes \mathbb{Z}[Y])$$

is a monomorphism onto a direct summand.

For the case of the k-dimensional simplicial sphere $Y = S^k$, one has the following:

Corollary 5.1. *For $n, k > 0$, the k-fold suspension map*

$$L_*\mathcal{L}(\mathbb{Z}, 2n) \to L_{*+k}\mathcal{L}(\mathbb{Z}, 2n + k)$$

is a monomorphism onto a direct summand.

Proof of theorem 5.1. For $m \geq 1$, consider the natural map of bisimplicial groups

$$(\mathcal{L}K(\mathbb{Z}, m) \otimes \mathbb{Z}[Y]) \to \mathcal{L}(K(\mathbb{Z}, m) \otimes \mathbb{Z}[Y])$$

By the Eilenberg-MacLane-Cartier theorem it is enough, for the analysis of the induced map on homotopy groups, to consider the corresponding map of diagonals of these bisimplicial groups:

$$\begin{array}{ccccccc} \dots \; \substack{\longrightarrow \\ \cdots \\ \rightleftarrows \\ \cdots \\ \longleftarrow} & \mathcal{L}(\mathbb{Z}, m)_3 \otimes \mathbb{Z}[Y]_3 & \substack{\rightrightarrows \\ \rightrightarrows \\ \leftleftarrows \\ \leftleftarrows} & \mathcal{L}(\mathbb{Z}, m)_2 \otimes \mathbb{Z}[Y]_2 & \substack{\rightrightarrows \\ \rightleftarrows \\ \leftarrow} & \mathcal{L}(\mathbb{Z}, m)_1 \otimes \mathbb{Z}[Y]_1 \\ & \downarrow & & \downarrow & & \downarrow \\ \dots \; \substack{\longrightarrow \\ \cdots \\ \rightleftarrows \\ \cdots \\ \longleftarrow} & \mathcal{L}(K(\mathbb{Z}, m)_3 \otimes \mathbb{Z}[Y]_3) & \substack{\rightrightarrows \\ \rightrightarrows \\ \leftleftarrows \\ \leftleftarrows} & \mathcal{L}(K(\mathbb{Z}, m)_2 \otimes \mathbb{Z}[Y]_2) & \substack{\rightrightarrows \\ \rightleftarrows \\ \leftarrow} & \mathcal{L}(K(\mathbb{Z}, m)_1 \otimes \mathbb{Z}[Y]_1) \end{array} \tag{5.1}$$

Theorem 2.1 implies that the maps

$$\mathcal{L}(\mathbb{Z}, m)_k \otimes \mathbb{Z}[Y]_k \to \mathcal{L}(K(\mathbb{Z}, m)_k \otimes \mathbb{Z}[Y]_k)$$

are monomorphisms onto direct summands, since $\mathbb{Z}[Y]_k$ is a free abelian group with a basis Y_k.

For $k \geq 1$, fix an order of elements of Y_k and denote by $\mathcal{D}_k$ the set of all sequences $(\sigma_1, \dots, \sigma_l)$, $\sigma_i \in Y_k$, $l \geq 2$ such that $\sigma_1 > \sigma_2 \leq \dots \leq \sigma_l$. Theorem 2.1 implies that

$$\mathcal{L}(K(\mathbb{Z}, m) \otimes \mathbb{Z}[Y]_k) \simeq \mathcal{L}K(\mathbb{Z}, m) \otimes \mathbb{Z}[Y]_k \oplus \mathcal{L}(\bigoplus_{(\sigma_1,\dots,\sigma_l) \in \mathcal{D}_k} K(\mathbb{Z}, m)^{\otimes l})$$

For every $(\sigma_1, \dots, \sigma_l) \in \mathcal{D}_k$, consider the simplicial abelian subgroup

$$A_{(\sigma_1,\dots,\sigma_l)}(\simeq K(\mathbb{Z}, lm)) \hookrightarrow \mathcal{L}(K(\mathbb{Z}, m) \otimes \mathbb{Z}[Y]_k)$$

generated by the element $[[i_m \otimes \sigma_1, \dots, i_m \otimes \sigma_l]]$. There is a monomorphism which is a homotopy equivalence

$$\mathcal{L}(\bigoplus_{(\sigma_1,\dots,\sigma_l) \in \mathcal{D}_k} A_{(\sigma_1,\dots,\sigma_l)}) \hookrightarrow \mathcal{L}(\bigoplus_{(\sigma_1,\dots,\sigma_l) \in \mathcal{D}_k} K(\mathbb{Z}, m)^{\otimes l})$$

Hence, there is the following monomorphism which induces a homotopy equivalence

$$(\mathcal{L}K(\mathbb{Z}, m) \otimes \mathbb{Z}[Y]_k) \oplus \mathcal{L}(\bigoplus_{(\sigma_1,\dots,\sigma_l) \in \mathcal{D}_k} A_{(\sigma_1,\dots,\sigma_l)}) \to \mathcal{L}(K(\mathbb{Z}, m) \otimes \mathbb{Z}[Y]_k)$$

Let $m = 2n$. Then the simplicial abelian group $\mathcal{L}(\bigoplus_{(\sigma_1,\dots,\sigma_l)\in\mathcal{D}_k} A_{(\sigma_1,\dots,\sigma_l)})$ may be viewed as a free simplicial Lie subring in $\mathcal{L}(K(\mathbb{Z},m)\otimes\mathbb{Z}[Y]_k)$ generated by all products of the form $[[i_m\otimes\sigma_1,\dots,i_m\otimes\sigma_l]]$ of weight >1.

Remark 5.1. *The suspension $L_*\mathcal{L}(\mathbb{Z},m)\to L_{*+1}\mathcal{L}(\mathbb{Z},m+1)$ is not a monomorphism for an odd m. For example, we have the following:*

$$\begin{array}{ccc} L_2\Lambda^2(\mathbb{Z},1) & \longrightarrow & L_3\Lambda^2(\mathbb{Z},2) \\ \| & & \| \\ \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/2 \end{array}$$

Consider the simplicial circle $S^1=\Delta[1]/\partial\Delta[1]$:

$$S^1_0=\{*\},\ S^1_1=\{*,\sigma\},\ S^1_2=\{*,s_0\sigma,s_1\sigma\},\dots,S^1_n=\{*,x_0,\dots,x_n\},$$

where $x_i=s_n\dots\hat{s}_i\dots s_0\sigma$. Take the standard abelian simplicial model of $K(\mathbb{Z},1)$:

$$\dots \;\substack{\longrightarrow\\ \cdots\\ \longleftarrow}\; \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longleftarrow\\\longleftarrow}\; \mathbb{Z}\oplus\mathbb{Z} \;\substack{\longrightarrow\\\longrightarrow\\\longleftarrow}\; \mathbb{Z}$$

with free generators $y_i=s_n\dots\hat{s}_i\dots s_0(i_1),\ i=0,\dots,n+1$ in degree $n+1$, for a generator $i_1\in K(\mathbb{Z},1)$. Consider the map (5.1) between diagonals of correspondent bisimplicial groups:

$$\begin{array}{ccccccc} \dots \substack{\longrightarrow\\ \cdots\\ \longleftarrow} & \Lambda^2(\mathbb{Z}^{\oplus 3})\otimes\mathbb{Z}[*,x_0,x_1,x_2] & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longleftarrow\\\longleftarrow} & \Lambda^2(\mathbb{Z}\oplus\mathbb{Z})\otimes\mathbb{Z}[*,x_0,x_1] & \substack{\longrightarrow\\\longrightarrow\\\longleftarrow} & \Lambda^2(\mathbb{Z})\otimes\mathbb{Z}[*,\sigma] \\ & \downarrow v_3 & & \downarrow v_2 & & \downarrow v_1 \\ \dots \substack{\longrightarrow\\ \cdots\\ \longleftarrow} & \Lambda^2(\mathbb{Z}^{\oplus 3}\otimes\mathbb{Z}[*,x_0,x_1,x_2]) & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longleftarrow\\\longleftarrow} & \Lambda^2((\mathbb{Z}\oplus\mathbb{Z})\otimes\mathbb{Z}[*,x_0,x_1]) & \substack{\longrightarrow\\\longrightarrow\\\longleftarrow} & \Lambda^2(\mathbb{Z}\otimes\mathbb{Z}[*,\sigma]) \end{array}$$

The generator of $\pi_3(\Lambda^2K(\mathbb{Z},1)\otimes\mathbb{Z}[S^1])=L_2\Lambda^2(\mathbb{Z},1)$ is given by the element

$$\alpha=[s_1s_0(i_1),s_2s_0(i_1)]\otimes s_2s_1(\sigma)-[s_1s_0(i_1),s_2s_1(i_1)]\otimes s_2s_0(\sigma)+[s_2s_0(i_1),s_2s_1(i_1)]\otimes s_1s_0(\sigma)$$

Then

$$\begin{aligned} v_3(\alpha)=[s_1s_0(i_1)\otimes s_2s_1(\sigma),s_2s_0(i_1)\otimes s_2s_1(\sigma)]-\\ [s_1s_0(i_1)\otimes s_2s_0(\sigma),s_2s_1(i_1)\otimes s_2s_0(\sigma)]+[s_2s_0(i_1)\otimes s_1s_0(\sigma),s_2s_1(i_1)\otimes s_1s_0(\sigma)] \end{aligned}$$

Theorem 5.2. (Kan) *Let $n\geq 1$. If r is odd, then the suspension map*

$$L_*\mathcal{L}^r(\mathbb{Z},2n)\to L_{*+1}\mathcal{L}^r(\mathbb{Z},2n+1)$$

is an isomorphism. Let $\alpha_{2n+1}\in\pi_{4n+2}\Lambda^2(\mathbb{Z},2n+1)=\mathbb{Z}$ and $i_{2n+1}\in K(\mathbb{Z},2n+1)_{2n+1}$ be generators, then the suspension and the composition homomorphism α_{2n+1} induce an isomorphism

$$L_*\mathcal{L}^{2r}(\mathbb{Z},2n)\oplus L_{*+1}\mathcal{L}^r(\mathbb{Z},4n+2)\simeq L_{*+1}\mathcal{L}^{2r}(\mathbb{Z},2n+1).$$

5.1.

Theorem 5.3. *$L_i\mathcal{L}^{2^k}(\mathbb{Z},2n)$ is a $\mathbb{Z}/2$-vector space indexed by all sequences $(i_1,\dots,i_k)\in\mathcal{W}_{2n,k}$ with $i=2n+i_1+\dots+i_k$.*

Recall the construction of basis elements of $L_i\mathcal{L}^{2^k}(\mathbb{Z},2n)$. Consider the simplicial group

$$\mathcal{L}^{2^k}(K(\mathbb{Z},2n)\otimes\mathbb{Z}/2)\simeq(\mathcal{L}^{2^k}K(\mathbb{Z},2n))\otimes\mathbb{Z}/2$$

For $k,l,i,n\geq 1$, the composition

$$\pi_i(\mathcal{L}^k(\mathbb{Z},q)\otimes\mathbb{Z}/2)\otimes\pi_q(\mathcal{L}^l(\mathbb{Z},n)\otimes\mathbb{Z}/2)\to\pi_i(\mathcal{L}^{kl}(\mathbb{Z},n)\otimes\mathbb{Z}/2)$$

is defined analogously with (3.4).

Proposition 5.1. *Let $w=(i_1,\dots,i_k)$ be the sequence with the following properties: 1) $i_1\leq 2n$; 2) $2i_j\geq i_{j+1}$. For $i=1,2,\dots$ let $\beta_i\in\pi_{2i}(\Lambda^2K(\mathbb{Z},i)\otimes\mathbb{Z}/2)=\mathbb{Z}/2$ be a non-zero element. Consider the following element*

$$\beta_w=\sigma^{2n-i_1}\beta_{i_1}\sigma^{2i_1-i_2}\beta_{i_2}\dots\sigma^{2i_{k-1}-i_k}\beta_{i_k}\in\pi_{2n+i_1+\dots+i_k}(\mathcal{L}^{2^k}(\mathbb{Z},2n)\otimes\mathbb{Z}/2)$$

The group $\pi_i(\mathcal{L}^{2^k}(\mathbb{Z},2n)\otimes\mathbb{Z}/2)$ is a $\mathbb{Z}/2$-vector space with a basis consisting of elements β_w and $i=2n+i_1+\dots+i_k$.

It follows that the natural projection $K(\mathbb{Z},2n)\to K(\mathbb{Z},2n)\otimes\mathbb{Z}/2$ induces a monomorphism

$$L_i\mathcal{L}^{2^k}(\mathbb{Z},2n)\to\pi_i(\mathcal{L}^{2^k}(K(\mathbb{Z},2n)\otimes\mathbb{Z}/2))$$

The image of this monomorphism is generated by elements β_w for which i_k is odd, i.e. for which $w\in\mathcal{W}^{(2)}_{2n,k}$.

Example 5.1.

$$L_i\mathcal{L}^8(\mathbb{Z},2)=\begin{cases}\mathbb{Z}/2\oplus\mathbb{Z}/2, & i=6,8,9\\ \mathbb{Z}/2, & i=5,7,10,11,12,13,15\\ 0 \text{ otherwise}\end{cases}$$

5.2.

Theorem 5.4. *The group $L_i\mathcal{L}^{p^k}(\mathbb{Z},2n)$ is a $\mathbb{Z}/p$-vector space indexed by all sequences $(\nu_{i_1},\dots,\nu_{i_k})\in\mathcal{W}^{(p)}_{2n,k}$ with $i=2n+(2p-2)(i_1+\dots+i_j)-|$number of λ_i in $(\nu_{i_1}\dots\nu_{i_j})|$.*

The elements from the sequences considered in the theorem correspond to the following elements in the derived functors:

$$\mu_i\in\pi_{2pi}(\mathcal{L}^p(\mathbb{Z},2i)\otimes\mathbb{Z}/p)\simeq\mathbb{Z}/p,$$
$$\lambda_i\in\pi_{2pi-1}(\mathcal{L}^p(\mathbb{Z},2i)\otimes\mathbb{Z}/p)\simeq\mathbb{Z}/p$$

Theorem 5.5. (Kan) *$L_*\mathcal{L}^r(\mathbb{Z},2n)=0$ if $r\neq p^j$ for any prime p and $j\geq 1$.*

5.3. Generating function. Since there is a general description of derived functors of Lie functors for $\mathbb{Z}$, one can define a generating function with dimensions of values derived functors as coefficients. The following result is due to M. Tangora [15]. Let H_n be the number of sequences of positive integers $(i_1, \dots, i_l)$ with $i_1 + \dots + i_l = n$ satisfying $i_{k+1} \leq di_k$ and $i_1 \leq m$. The function

$$H(q) = 1 + H_1 q + H_2 q^2 + \dots + H_n q^n + \dots$$

is

$$H(q) = \frac{a(q)}{1 - b(q)}$$

where

$$a(q) = \sum_{k\geq 0} \left(\frac{q^{e(k)}(1 - q^{me(k)})}{1 - q^{e(k)}} \prod_{j=0}^{k-1} \frac{-q^{e(j)}}{1 - q^{e(k)}} \right),$$

$$b(q) = \sum_{k\geq 0} \left(\frac{q^{e(k)}}{1 - q^{e(k)}} \prod_{j=0}^{k-1} \frac{-q^{e(k)}}{1 - q^{e(k)}} \right),$$

$$e(k) = \frac{d^{k+1} - 1}{d - 1}$$

6. Derived functors $L_*\mathcal{L}_s(\mathbb{Z}, m)$

Analog of theorems 5.2 and 5.5 is the following

Theorem 6.1. *1)* $L_*\mathcal{L}_s^r(\mathbb{Z}, 2n+1) = 0$ *if* $r \neq p^j$ *for some prime* p *and* $j \geq 1$*;*

2) if r *is odd, the suspension* $L_*\mathcal{L}_s^r(\mathbb{Z}, 2n+1) \to L_{*+1}\mathcal{L}_s^r(\mathbb{Z}, 2n+2)$ *is an isomorphism;*

3) if r *is odd, there is an isomorphism*

$$L_{*+1}\mathcal{L}_s^{2r}(\mathbb{Z}, 2n) \simeq L_{*+1}\mathcal{L}_s^r(\mathbb{Z}, 4n) \oplus L_*\mathcal{L}_s^{2r}(\mathbb{Z}, 2n-1)$$

Analog of theorem 5.4 is the following:

Theorem 6.2. *For an odd prime* p*,* $k \geq 1$*, the group* $L_i\mathcal{L}_s^{p^k}(\mathbb{Z}, 2n)$ *is a* $\mathbb{Z}/p$*-vector space indexed by all sequences* $(\nu_{i_1}, \dots, \nu_{i_k}) \in \widetilde{\mathcal{W}}_{2n,k}^{(p)}$ *with* $i = 2n + (2p-2)(i_1 + \dots + i_j) - |$*number of* λ_i *in* $(\nu_{i_1} \dots \nu_{i_j})|$.

Above results show that the structures of derived functors of Lie and super-Lie functors have similar points but they are not the same. The derived functors of $\mathcal{L}_s^2 = \Gamma_2$ of $\mathbb{Z}$ are well-known and follow from the decalage

$$L\Gamma_2(\mathbb{Z}, n)[2] \simeq L\Lambda^2(\mathbb{Z}, n+1)$$

In this connection the following result looks surprising:

Theorem 6.3. *For* $k \geq 2$*,* $n \geq 1$*, one has an isomorphism*

$$L_*\mathcal{L}^{2^k}(\mathbb{Z}, n) \simeq L_*\mathcal{L}_s^{2^k}(\mathbb{Z}, n)$$

Theorem 6.3 uses the specific structure of Lie and super-Lie functors of degrees powers of 2 and can not be generalized to odd prime powers.

Example 6.1.

$$L_i\mathcal{L}^9(\mathbb{Z},2) = \begin{cases} \mathbb{Z}/3, \ i=8,9,12,13,17 \\ 0 \text{ otherwise} \end{cases} \qquad L_i\mathcal{L}_s^9(\mathbb{Z},2) = \begin{cases} \mathbb{Z}/3, \ i=4,5,9 \\ 0 \text{ otherwise} \end{cases}$$

Generators of the 3-torsion of $L_*\mathcal{L}_s^9(\mathbb{Z},2)$ in degrees $4,5,9$ correspond to the elements (λ_1,λ_2), (μ_1,λ_2), (μ_1,λ_3) from $\widetilde{\mathcal{W}}_{2,2}^{(3)}$ respectively.

7. Derived functors $L_*\mathcal{L}(\mathbb{Z}/p^k,m)$

A differential graded Lie ring with squares (shortly DGLS) is a GLRS $B=\bigoplus_{i=0}^\infty B_i$ together with homomorphisms $\partial: B_i\to B_{i-1},\ i=1,2,\dots$ such that

$$\partial\circ\partial = 0 \tag{7.1}$$

$$\partial\{x,y\} = \{\partial(x),y\}+(-1)^i\{x,\partial(y)\},\ x\in B_i \tag{7.2}$$

$$\partial(x^{[2]}) = \{\partial(x),x\},\ \text{for } i \text{ odd} \tag{7.3}$$

The main example of DGLS is the following. Let X be a simplicial Lie ring. Define $\partial: X_n\to X_{n-1}$ by $\partial=\sum_{i=0}^n(-1)^i\partial_i$, the Lie bracket

$$[[\ ,\ ]]: X_i\otimes X_j\to X_{i+j}$$

define by

$$[[x,y]] = \sum_{(a;b)\in(i,j)-\text{shuffles}} sign(a,b)[s_a(x),s_b(y)], x\in X_j,\ y\in X_i$$

and

$$x^{[2]} = \sum_{(a;b)\in(n,n)-\text{shuffles}} sign(a;b)[s_a(x),s_b(x)],\ x\in X_n,\ n \text{ is odd}$$

Then X is a DGLS.

For every free abelian chain complex $C=\{C_i,\ \partial: C_{i+1}\to C_i\}$, there exists a DGLS $\mathbb{L}(C)$ and a morphism of chain complexes

$$C\to\mathbb{L}(C) \tag{7.4}$$

such that, for any DGLS R and chain map $f: C\to R$, there is a unique map of DGLS-s: $\bar{f}:\mathbb{L}(C)\to R$ such that the diagram

$$\begin{array}{ccc} C & \longrightarrow & \mathbb{L}(C) \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \bar{f}} & \\ R & & \end{array}$$

commutes (this is lemma 3.4 from [13]).

For $k\geq 0$, denote by $(n+1,n;k)$ the chain complex $(\mathbb{Z}\xrightarrow{k}\mathbb{Z})[n]$.

Theorem 7.1. (Leibowitz), [13] *Let p be a prime, $k\geq 1$, $i\geq 1$, $r\geq 1$. There is an isomorphism*

$$L_i\mathcal{L}^r(\mathbb{Z}/p^k,n)\simeq {}_p\pi_i(L\mathcal{L}^r(\mathbb{Z}[n]\oplus\mathbb{Z}[n+1]))\oplus {}_pH_i\mathbb{L}^r(n+1,n;p^k)$$

Recall the main steps of the proof of theorem 7.1 from [13]. For a prime p and $n \geq 1$, consider the standard simplicial model of $K(\mathbb{Z}/p^k, n)$:

$$A := N^{-1}((\mathbb{Z} \xrightarrow{p^k} \mathbb{Z})[n]).$$

Define the filtration

$$\mathcal{L}^r(A) \supset F^r \supset F^{r-1} \supset \cdots \supset F^1 \supset F^0$$

where F^j is the simplicial subgroup of $\mathcal{L}^r(A)$ generated by all basic commutators with at most j elements which arise from the generator of degree $(n+1)$ of A. This filtration defines a spectral sequence

$$E^1_{i,j} = \pi_{i+j}(F^i/F^{i-1}) \Rightarrow L_{i+j}\mathcal{L}^r(\mathbb{Z}/p, n). \tag{7.5}$$

The following facts are proved in [13]:

1) Consider the map of DGLS-s:

$$\mathbb{L}^r(n+1, n; p^k) \to \mathcal{L}^r(A) \tag{7.6}$$

induced by

$$y_n \mapsto x_n$$
$$y_{n+1} \mapsto x_{n+1}$$

where (y_n, y_{n+1}) are generators of dimensions n and $n+1$ of $(n+1, n; p^k)$ and (x_n, x_{n+1}) are generators of dimensions n and $n+1$ of $K(\mathbb{Z}/p^k, n)$.

There is an isomorphism

$$E^2_{i,rn} = H_{i+rn}\mathbb{L}^r(n+1, n; p^k);$$

2) for $j > rn$, $E^1_{i,j} = 0$;

3) For $j \neq rn$, $E^2_{i,j} = E^1_{i,j}$;

4) $E^\infty_{i,j} = {}_{(p)}E^2_{i,j}$, where ${}_{(p)}E^2_{i,j}$ is the p-primary component of $E^2_{i,j}$.

4) $\sum_{j=0,\ j\neq i-rn}^r {}_{(p)}E^2_{j,i-j} = {}_p\pi_i(K(\mathbb{Z}, n) \oplus K(\mathbb{Z}, n+1))$

Theorem 7.1 follows from these facts regarding the spectral sequence (7.5).

Recall other results from [13], which we will use bellow.

Proposition 7.1. *([13], Prop. 4.5)*

$$H_k\mathbb{L}^r(n+1, n; p^f) = (\mathbb{Z}/p^{f+1})^{\oplus d_k} \oplus (\mathbb{Z}/p^f)^{\oplus(M_k - d_k)}$$

where

$$M_k = {}_pRank\ B_k\mathbb{L}^r(n+1, n; 1)$$
$$d_k = {}_pRank\ H_k\mathbb{L}^r(n+1, n; 1)$$

The numbers from proposition 7.1 can be computed by the following formulas:

$$Rank\ B_k\mathbb{L}^r(n+1,n;1) = Rank\ \mathbb{L}^r_k(n+1,n;1) - Rank\ B_{k-1}\mathbb{L}^r(n+1,n;1) \tag{7.7}$$

$$Rank\ \mathbb{L}^r_k(1,0;1) = M(k,r-k) \text{ if } r \text{ is odd or if } r \text{ is even and } k\equiv 0 \mod 4 \tag{7.8}$$

$$Rank\ \mathbb{L}^r_k(1,0;1) = M(k,r-k) + Rank\ \mathbb{L}^{r/2}_{k/2}(1,0;1) \text{ if } r \text{ is even and } k\equiv 2 \mod 4 \tag{7.9}$$

$$Rank\ \mathbb{L}^r_{k+r}(2,1;1) = M(k,r-k) \text{ if } r \text{ is odd or if } r \text{ is even and } k+r\not\equiv 2 \mod 4 \tag{7.10}$$

$$Rank\ \mathbb{L}^r_{k+r}(2,1;1) = M(k,r-k) + Rank\ \mathbb{L}^{r/2}_{\frac{k+r}{2}}(2,1;1) \text{ if } r \text{ is even and } k+r\equiv 2 \mod 4 \tag{7.11}$$

Proposition 7.2. *([13], Prop. 4.7) For $n \geq 0$, $f\geq 0$, there are isomorphisms*

$$H_*\mathbb{L}^r(1,0;p^f) \to H_{*+rn}\mathbb{L}^r(n+1,n;p^f) \text{ if } n \text{ is even}$$
$$H_*\mathbb{L}^r(2,1;p^f) \to H_{*+r(n-1)}\mathbb{L}^r(n+1,n;p^f) \text{ if } n \text{ is odd}$$

Example. We collect the low-dimensional derived functors for $A=\mathbb{Z}/k$ in the following table:

i	$L_i\mathcal{L}^2(A)$	$L_i\mathcal{L}^3(A)$	$L_i\mathcal{L}^4(A)$	$L_i\mathcal{L}^5(A)$	$L_i\mathcal{L}^6(A)$	$L_i\mathcal{L}^7(A)$
6	0	0	0	0	0	0
5	0	0	0	0	$\mathbb{Z}/(3,k)$	$\mathbb{Z}/k$
4	0	0	0	0	$\mathbb{Z}/(3k,k^2)$	$(\mathbb{Z}/k)^{\oplus 2}$
3	0	0	$\mathbb{Z}/(2,k)$	$\mathbb{Z}/k$	$\mathbb{Z}/(2,k)\oplus\mathbb{Z}/k$	$(\mathbb{Z}/k)^{\oplus 3}$
2	0	0	$\mathbb{Z}/(2k,k^2)$	$\mathbb{Z}/k$	$\mathbb{Z}/(2k,k^2)\oplus\mathbb{Z}/k^{\oplus 2}$	$(\mathbb{Z}/k)^{\oplus 2}$
1	$\mathbb{Z}/k$	$\mathbb{Z}/k$	$\mathbb{Z}/k$	$\mathbb{Z}/k$	$\mathbb{Z}/k$	$\mathbb{Z}/k$

(7.12)

TABLE 1. Values of derived functors $L_i\mathcal{L}^m(\mathbb{Z}/k)$

Observe that, for a prime p, one has

$$L_i\mathcal{L}^p(\mathbb{Z}/k) = \mathbb{Z}/k^{\oplus \sum_{j=1}^{i}(-1)^{i-j}\frac{1}{p}\binom{p}{j}}$$

7.1. Functorial generalization. For $k\geq 2$, recall that we denote by J_k the set of basic tensor products of weight k in A and B. Set

$$J_k = J_k^e \sqcup J_k^o = \bar{J}_k^e \sqcup \bar{J}_k^o$$

where

J_k^e is the set of basic products from J_k with even number of entrances of A
J_k^o is the set of basic products from J_k with odd number of entrances of A
$\bar{J}_k^e$ is the set of basic products from J_k with even number of entrances of B
$\bar{J}_k^o$ is the set of basic products from J_k with odd number of entrances of B

The structure of the graded components of DGLS $\mathbb{L}(B \xrightarrow{f} A)$ is the following:

$$\mathbb{L}^m(B \xrightarrow{f} A)_l = \bigoplus_{d|m,\ d|l} \left(\bigoplus_{D \in \bar{J}^e_{(m-l)/d, l/d}} \mathcal{L}^d(D) \oplus \bigoplus_{D \in \bar{J}^o_{(m-l)/d, l/d}} \mathcal{L}^d_s(D) \right)$$

The structure of the DGLS $\mathbb{L}(C[1])$ for the shifted complex $C = (B \xrightarrow{f} A)$ is the following

$$\mathbb{L}^m((B \xrightarrow{f} A)[1])_{l+m} = \bigoplus_{d|m,\ d|l} \left(\bigoplus_{D \in J^e_{l/d, (m-l)/d}} \mathcal{L}^d(D) \oplus \bigoplus_{D \in J^o_{l/d, (m-l)/d}} \mathcal{L}^d_s(D) \right)$$

The following proposition follows immediately from the construction of DGLS $\mathcal{L}(C)$:

Proposition 7.3. *For $n \geq 1,\ m \geq 2$, one has natural isomorphisms of chain complexes*

$$\mathbb{L}^m((B \xrightarrow{f} A)[2n])) \simeq \mathbb{L}^m(B \xrightarrow{f} A)[2nm]$$
$$\mathbb{L}^m((B \xrightarrow{f} A)[2n+1])) \simeq \mathbb{L}^m((B \xrightarrow{f} A)[1])[2nm]$$

Observe that proposition 7.2 is a simple consequence of proposition 7.3.

Example 7.1.

Consider the simplest example of DGLS $\mathcal{L}(C)$. Let $C = \{B \xrightarrow{f} A\}$, i.e. we set the abelian group A in degree 0 and the abelian group B in degree 1. The DGLS $\mathcal{L}(C)$ is a graded object, with the following terms in low dimensions:

$$\mathcal{L}^2(C): \quad \Gamma_2(B) \to A \otimes B \to \Lambda^2(A)$$

This is quadratic Koszul complex associated to C. The cubical term is

$$\mathcal{L}^3(C): \quad \mathcal{L}^3_s(B) \to B \otimes A \otimes B \xrightarrow{\delta} B \otimes A \otimes A \to \mathcal{L}^3(A)$$

where

$$\delta: b_1 \otimes a \otimes b_2 \mapsto b_1 \otimes a \otimes f(b_2) + b_2 \otimes f(b_1) \otimes a - b_1 \otimes f(b_2) \otimes a, \quad b_1, b_2 \in B, a \in A. \quad (7.13)$$

The fourth degree component of $\mathcal{L}(C)$ has the following structure:

$$\mathcal{L}^4(C): \quad \mathcal{L}^4_s(B) \to B \otimes A \otimes B \otimes B \xrightarrow{\delta_2} B \otimes A \otimes A \otimes B \oplus \Gamma_2(B \otimes A) \xrightarrow{\delta_1}$$
$$B \otimes A \otimes A \otimes A \to \mathcal{L}^4(A) \quad (7.14)$$

where

$$\delta_2: \ b_1 \otimes a \otimes b_2 \otimes b_3 \mapsto b_2 \otimes a \otimes f(b_1) \otimes b_3 - b_2 \otimes f(b_1) \otimes a \otimes b_3 - b_1 \otimes a \otimes f(b_2) \otimes b_3 +$$
$$b_1 \otimes a \otimes f(b_3) \otimes b_2 + (b_2 \otimes f(b_3))(b_1 \otimes a)$$
$$\delta_1: \ b_1 \otimes a_1 \otimes a_2 \otimes b_2 \mapsto b_2 \otimes a_2 \otimes f(b_1) \otimes a_1 - b_2 \otimes a_2 \otimes a_1 \otimes f(b_1) - b_2 \otimes f(b_1) \otimes a_1 \otimes a_2 +$$
$$+ b_2 \otimes a_1 \otimes f(b_1) \otimes a_2 - b_1 \otimes a_1 \otimes a_2 \otimes f(b_2)$$
$$\delta_1: \gamma_2(b \otimes a) \mapsto b \otimes a \otimes f(b) \otimes a - b \otimes f(b) \otimes a \otimes a$$

with $b, b_1, b_2, b_3 \in B,\ a, a_1, a_2 \in A$. For example, one has

$$\mathcal{L}^2(\mathbb{Z} \xrightarrow{k} \mathbb{Z}) = (\mathbb{Z} \xrightarrow{k} \mathbb{Z})[1]$$
$$\mathcal{L}^3(\mathbb{Z} \xrightarrow{k} \mathbb{Z}) = (\mathbb{Z} \xrightarrow{k} \mathbb{Z})[1]$$
$$\mathcal{L}^4(\mathbb{Z} \xrightarrow{k} \mathbb{Z}) = (\mathbb{Z} \xrightarrow{(0,2k)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(k,0)} \mathbb{Z})[1]$$

Example 7.2.

Now consider the low-dimensional terms of DGLS $\mathcal{L}(C[1])$ where $C = \{B \xrightarrow{f} A\}$. The quadratic term is the following:

$$\mathcal{L}^2(C[1]): \qquad (\Lambda^2(B) \to B \otimes A \to \Gamma_2(A))[2]$$

This is the shifted quadratic dual de Rham complex associated to C. The cubical term is the following:

$$\mathcal{L}^3(C[1]): \qquad (\mathcal{L}^3(B) \to B \otimes A \otimes B \xrightarrow{\delta'} B \otimes A \otimes A \to \mathcal{L}^3_s(A))[3]$$

where

$$\delta': b_1 \otimes a \otimes b_2 \mapsto b_2 \otimes a \otimes f(b_1) + b_2 \otimes f(b_1) \otimes a + b_1 \otimes a \otimes f(b_2),\ a \in A,\ b_1, b_2 \in B$$

(compare with map δ in (7.13)). For example, one has

$$\mathbb{L}^2((\mathbb{Z} \xrightarrow{k} \mathbb{Z})[1]) = (\mathbb{Z} \xrightarrow{2k} \mathbb{Z})[2]$$
$$\mathbb{L}^3((\mathbb{Z} \xrightarrow{k} \mathbb{Z})[1]) = (\mathbb{Z} \xrightarrow{3k} \mathbb{Z})[3]$$

Observe that, given a two-step complex of free abelian groups $C = (B \xrightarrow{f} A)$ and $n \geq 0$, we have a natural (in the category of flat resolutions of abelian groups) map of DGLS-s

$$\mathbb{L}((C)[n]) \to \mathcal{L}(N^{-1}(C[n]))$$

which generalizes the map (7.6). For a fixed $m \geq 2$, the map

$$\mathcal{L}^m(C) \to \mathcal{L}^m(N^{-1}(C))$$

goes through the cross-effect sequence in the following way:

$$\begin{array}{ccccccc}
\mathcal{L}^m(C)_{l+1} & \longrightarrow & \mathcal{L}^m(C)_l & \longrightarrow & \dots & \longrightarrow & \mathcal{L}^m(C)_0 \\
\downarrow & & \downarrow & & & & \| \\
\mathcal{L}^m(\underbrace{B|\dots|B}_{l \text{ terms}}) \oplus \mathcal{L}^m(\underbrace{A|B|\dots|B}_{l+1 \text{ terms}}) & \longrightarrow & \mathcal{L}^m(\underbrace{B|\dots|B}_{l-1 \text{ terms}}) \oplus \mathcal{L}^m(\underbrace{A|B|\dots|B}_{l \text{ terms}}) & \longrightarrow & \dots & \longrightarrow & \mathcal{L}^m(A) \\
\downarrow & & \downarrow & & & & \| \\
\mathcal{L}^m(N^{-1}(C))_{l+1} & \longrightarrow & \mathcal{L}^m(N^{-1}(C))_l & \longrightarrow & \dots & \longrightarrow & \mathcal{L}^m(N^{-1}(C))_0
\end{array} \tag{7.15}$$

For $m = 2$ this diagram has the following form:

$$\begin{array}{ccccc} \Gamma_2(B) & \longrightarrow & B \otimes A & \longrightarrow & \Lambda^2(A) \\ \downarrow & & \downarrow & & \| \\ B \otimes B & \longrightarrow & \Lambda^2(B) \oplus B \otimes A & \longrightarrow & \Lambda^2(A) \\ \downarrow & & \downarrow & & \| \\ \Lambda^2(A \oplus B \oplus B) & \longrightarrow & \Lambda^2(A \oplus B) & \longrightarrow & \Lambda^2(A) \end{array} \tag{7.16}$$

Computing the cokernels in the upper part of diagram (7.16) we obtain the following diagram with exact rows and columns:

$$\begin{array}{ccccc} \Gamma_2(B) & \longrightarrow & B \otimes A & \longrightarrow & \Lambda^2(A) \\ \downarrow & & \downarrow & & \| \\ B \otimes B & \longrightarrow & \Lambda^2(B) \oplus B \otimes A & \longrightarrow & \Lambda^2(A) \\ \downarrow & & \downarrow & & \\ \Lambda^2(B) & = & \Lambda^2(B) & & \end{array} \tag{7.17}$$

which shows that the natural map

$$\Lambda^2(B \to A) \to \Lambda^2(N^{-1}(B \to A))$$

is a homotopy equivalence. The following proposition shows that this happens for all prime degrees.

Proposition 7.4. *For a prime p, the map*

$$\mathcal{L}^p(B \xrightarrow{f} A) \to \mathcal{L}^p(N^{-1}(B \xrightarrow{f} A))$$

is a homotopy equivalence.

Corollary 7.1. *For a prime p, abelian group A and a flat resolution $0 \to P \to Q \to A \to 0$, the truncated complex $\mathcal{L}^p(P \to Q)$*

$$Q \otimes P^{\oplus p-1} \to \dots \to \bigoplus_{J_{l,p-l}} P^{\otimes p-l} \otimes Q^{\otimes l} \to \dots \to P \otimes Q^{\otimes p-1} \to \mathcal{L}^p(Q)$$

represents the object $L\mathcal{L}^p(A)$ in the derived category.

Clearly, proposition 7.4 works only for prime degrees. For example, for the fourth degree, the natural map

$$\mathbb{L}^4(B \xrightarrow{f} A) \to \mathcal{L}^4(N^{-1}(B \xrightarrow{f} A))$$

is not a homotopy equivalence, in general.

Now consider the case of a shifted complex $C[1]$ with $C = (B \xrightarrow{f} A)$. The super-analog of the diagram (7.15) is the following diagram

$$\begin{array}{ccccccc}
\mathcal{L}^m(C[1])_{l+m+1} & \longrightarrow & \mathcal{L}^m(C[1])_{l+m} & \longrightarrow & \dots & \longrightarrow & \mathcal{L}^m(C[1])_m \\
\downarrow & & \downarrow & & & & \| \\
\mathcal{L}^m_s(\underbrace{B|\dots|B}_{l \text{ terms}}) \oplus \mathcal{L}^m_s(\underbrace{A|B|\dots|B}_{l+1 \text{ terms}}) & \longrightarrow & \mathcal{L}^m_s(\underbrace{B|\dots|B}_{l-1 \text{ terms}}) \oplus \mathcal{L}^m_s(\underbrace{A|B|\dots|B}_{l \text{ terms}}) & \longrightarrow & \dots & \longrightarrow & \mathcal{L}^m_s(A) \\
\downarrow & & \downarrow & & & & \| \\
\mathcal{L}^m_s(N^{-1}(C))_{l+1} & \longrightarrow & \mathcal{L}^m_s(N^{-1}(C))_l & \longrightarrow & \dots & \longrightarrow & \mathcal{L}^m_s(N^{-1}(C))_0
\end{array}$$

For $m = 2$ this diagram has the following form:

$$\begin{array}{ccccc}
\Lambda^2(B) & \longrightarrow & B \otimes A & \longrightarrow & \Gamma_2(A) \\
\downarrow & & \downarrow & & \| \\
B \otimes B & \longrightarrow & \Gamma_2(B) \oplus B \otimes A & \longrightarrow & \Gamma_2(A) \\
\downarrow & & \downarrow & & \| \\
\Gamma_2(A \oplus B \oplus B) & \longrightarrow & \Gamma_2(A \oplus B) & \longrightarrow & \Gamma_2(A)
\end{array} \tag{7.18}$$

There is no an analog of proposition 7.4 in the shifted case. Observe that the upper map of complexes in (7.18) is not a homotopy equivalence.

7.2. Spectral sequence. Now consider the functorial generalization of the spectral sequence (7.5). Let $C = \{B \xrightarrow{f} A\}$ be a complex of length two with free abelian groups A and B. Let $m \geq 2$, $n \geq 0$. Consider the standard model for $L\mathcal{L}^m(C[n])$:

$$\mathcal{L}^m(N^{-1}(C[n])): \quad \dots \mathcal{L}^m(B^{\oplus n+2} \oplus A^{\oplus \binom{n+2}{2}}) \overset{\longrightarrow}{\underset{\longleftarrow}{\vdots}} \mathcal{L}^m(B \oplus A^{\oplus n+1}) \overset{\longrightarrow}{\underset{\longleftarrow}{\vdots}} \mathcal{L}^m(A)$$

Define the natural simplicial filtration

$$\mathcal{L}^m(N^{-1}(C[n])) = I_m \supset I_{m-1} \supset \dots \supset I_0 \supset I_{-1} = \{1\} \tag{7.19}$$

where I_j is the simplicial subgroup of $\mathcal{L}^m(N^{-1}(C[n]))$ generated at each dimension by basic commutators with at most j elements which arise from B. For example, for $n = 0$,

$$I_0: \quad \dots\ \mathcal{L}^m(A) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longleftarrow\\\longleftarrow} \mathcal{L}^m(A) \substack{\longrightarrow\\\longrightarrow\\\longleftarrow} \mathcal{L}^m(A)$$

and

$$I_1: \quad \dots\ (B \oplus B) \otimes A^{\otimes m-1} \oplus \mathcal{L}^m(A) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longleftarrow\\\longleftarrow} B \otimes A^{\otimes m-1} \oplus \mathcal{L}^m(A) \substack{\longrightarrow\\\longrightarrow\\\longleftarrow} \mathcal{L}^m(A)$$

This filtration gives rise to the spectral sequence

$$E^1_{i,j}(C[n]) = \pi_{i+j}(I_i/I_{i-1}) \Rightarrow \pi_{i+j} L\mathcal{L}^m(C[n]) \tag{7.20}$$

with differentials

$$d^k_{i,j} : E^k_{i,j} \to E^k_{i-k,j+k-1}$$

Observe that, the general E^1-term can be described as follows:

$$E^1_{mn+m-l,j-mn-m+l}(C[n]) = \bigoplus_{d|m,\ d|l} \bigoplus_{D\in J_{\frac{l}{d},\frac{m-l}{d}}} L_j\mathcal{L}^d(D, \frac{mn+m-n}{d}) \qquad (7.21)$$

In particular,

$$E^1_{mn+i,0}(C[n]) = \mathcal{L}^m(C[n])_i,\ i \geq 0$$

As an example, consider initial terms of the spectral sequence for the fourth degree Lie functor. The spectral sequence for $L\mathcal{L}^4(C)$ has the following form:

$$\mathcal{L}^4(A) \xleftarrow{d^1_{1,0}} A\otimes B^{\otimes 3} \xleftarrow{d^1_{2,0}} A\otimes B\otimes B\otimes A\oplus \Gamma_2(A\otimes B) \xleftarrow{d^1_{3,0}} B\otimes A^{\otimes 3} \xleftarrow{d^1_{4,0}} \mathcal{L}^4_s(B)$$

$$A\otimes B\otimes B\otimes A\oplus \Gamma_2(A\otimes B) \overset{d^2_{4,-1}}{\dashleftarrow} \Gamma_2(B)\otimes \mathbb{Z}/2$$

where the dash arrow is defined on homology. The E^1-term of the spectral sequence for $L\mathcal{L}^4(C[1])$ has the following form:

q	$E^1_{4,q}$	$E^1_{5,q}$	$E^1_{6,q}$	$E^1_{7,q}$	$E^1_{8,q}$
0	$\mathcal{L}^4_s(A)$	$A^{\otimes 3}\otimes B$	$A\otimes B^{\otimes 2}\otimes A\oplus\Gamma_2(A\otimes B)$	$A\otimes B^{\otimes 3}$	$\mathcal{L}^4(B)$
-1	$\Gamma_2(A)\otimes\mathbb{Z}/2$	0	0	0	$\Gamma_2(B)\otimes\mathbb{Z}/2$
-2	0	0	$A\otimes B\otimes\mathbb{Z}/2$	0	0
-3	0	0	0	0	$\Gamma_2(B)\otimes\mathbb{Z}/2$
-4	0	0	0	0	$B\otimes\mathbb{Z}/2$

Table 2. The E^1-term of the spectral sequence for $\mathcal{L}^4((B\to A)[1])$

7.3. Décalage. Proposition 7.4 implies that, for an exact sequence of free abelian groups

$$0\to P\to Q\to A\to 0,$$

there is a natural long exact sequence

$$0\to\mathcal{L}^p_s(P)\to Q\otimes P^{\oplus p-1}\to\cdots\to\bigoplus_{J_{l,p-l}} P^{\otimes p-l}\otimes Q^{\otimes l}\to$$
$$\cdots\to P\otimes Q^{\otimes p-1}\to\mathcal{L}^p(Q)\to\mathcal{L}^p(A)\to 0 \quad (7.22)$$

Given a complex B of free abelian groups, we have a natural short exact sequence

$$0\to B\to C(B)\to B[1]\to 0$$

where $C(B)$ is the cone of B, which is contractible. Applying the sequence (7.22) to the cone sequence of an arbitrary element of the derived category $\mathsf{DAb}_{\leq 0}$, we obtain the following

Theorem 7.2. *For a prime p and $C\in\mathsf{DAb}_{\leq 0}$, there is a natural isomorphism in the derived category*

$$L\mathcal{L}^p(C[1])\simeq L\mathcal{L}^p_s(C)[p].$$

8. The super-analog of Leibowitz spectral sequence

Now consider the super-analog of the spectral sequence (7.20). Let $C = \{B \xrightarrow{f} A\}$ be a complex of length two with free abelian groups A and B. Let $m \geq 2$, $n \geq 0$. Consider the standard model for $L\mathcal{L}_s^m(C[n])$:

$$\mathcal{L}_s^m(N^{-1}(C[n])): \quad \dots \mathcal{L}_s^m(B^{\oplus n+2} \oplus A^{\oplus \binom{n+2}{2}}) \overset{\longrightarrow}{\underset{\longleftarrow}{\Longleftarrow}} \mathcal{L}_s^m(B \oplus A^{\oplus n+1}) \overset{\longrightarrow}{\underset{\longleftarrow}{\Longleftarrow}} \mathcal{L}_s^m(A)$$

Define the natural simplicial filtration

$$\mathcal{L}_s^m(N^{-1}(C[n])) = \bar{I}_m \supset \bar{I}_{m-1} \supset \dots \supset \bar{I}_0 \supset \bar{I}_{-1} = \{1\}$$

where $\bar{I}_j$ is the simplicial subgroup of $\mathcal{L}_s^m(N^{-1}(C[n]))$ generated at each dimension by basic commutators with at most j elements which arise from B together with elements of the type $x^{[2]}$, if $m \equiv 2 \mod 4$ and x is a basic commutator of length $m/2$ with at most $j/2$ elements which arise from B. For example, for $n = 0$,

$$\bar{I}_0: \quad \dots \ \mathcal{L}_s^m(A) \overset{\longrightarrow}{\underset{\longleftarrow}{\Longleftarrow}} \mathcal{L}_s^m(A) \overset{\longrightarrow}{\underset{\longleftarrow}{\Longleftarrow}} \mathcal{L}_s^m(A)$$

and

$$\bar{I}_1: \quad \dots \ (B \oplus B) \otimes A^{\otimes m-1} \oplus \mathcal{L}_s^m(A) \overset{\longrightarrow}{\underset{\longleftarrow}{\Longleftarrow}} B \otimes A^{\otimes m-1} \oplus \mathcal{L}_s^m(A) \overset{\longrightarrow}{\underset{\longleftarrow}{\Longleftarrow}} \mathcal{L}_s^m(A)$$

This filtration gives rise to the spectral sequence

$$\bar{E}^1_{i,j}(C[n]) = \pi_{i+j}(\bar{I}_i/\bar{I}_{i-1}) \Rightarrow \pi_{i+j} L\mathcal{L}_s^m(C[n])$$

with differentials

$$d^k_{i,j}: \bar{E}^k_{i,j} \to \bar{E}^k_{i-k,j+k-1}$$

A natural analog of the formula (7.21) for the E^1-term is the following:

$$\bar{E}^1_{mn+m-l,j-mn-m+l}(C[n]) = \bigoplus_{d|m,\ d|l} \ \bigoplus_{D \in J_{\frac{l}{d},\frac{m-l}{d}}} L_j\mathcal{L}_s^d(D, \frac{mn+m-n}{d}) \tag{8.1}$$

Example. Consider, for example, the spectral sequence for the third super-Lie functor $\mathcal{L}_s^3$:

$$\begin{array}{ccccccc} \mathcal{L}_s^3(A) & \xleftarrow{d^1_{1,0}} & A \otimes B \otimes A & \xleftarrow{d^1_{2,0}} & A \otimes B \otimes B & \xleftarrow{d^1_{3,0}} & \mathcal{L}^3(B) \\ & & & \nwarrow{\scriptstyle d^2_{3,-1}} & & & \\ & & & & & & B \otimes \mathbb{Z}/3 \end{array}$$

One can compare the spectral sequence for $\mathcal{L}^3$ and $\mathcal{L}_s^3$ applied for the complex $\mathbb{Z} \xrightarrow{3} \mathbb{Z}$:

$j=0$	0	$\mathbb{Z}$ $\overset{3}{\leftarrow}$	$\mathbb{Z}$	0
$j=-1$	0	0	0	0

$j=0$	0	$\mathbb{Z}$ $\overset{9}{\leftarrow}$	$\mathbb{Z}$	0
$j=-1$	0	0	0	$\mathbb{Z}/3$

TABLE 3. The E^1-term of spectral sequences for $L\mathcal{L}^3(\mathbb{Z}/3)$ and $L\mathcal{L}^3_s(\mathbb{Z}/3)$

In particular, one has the following values of the derived functors:

$$L_i\mathcal{L}^3(\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & i=1 \\ 0, & i\neq 1 \end{cases} \qquad L_i\mathcal{L}^3_s(\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/9, & i=1 \\ \mathbb{Z}/3, & i=2 \\ 0, & i\neq 1,2 \end{cases}$$

Here is an example of the initial terms of the spectral sequence for the case $\mathcal{L}^3_s(C[1])$:

q	$E^1_{3,q}$	$E^1_{4,q}$	$E^1_{5,q}$	$E^1_{6,q}$
0	$\mathcal{L}^3(A)$	$A\otimes A\otimes B$	$A\otimes B\otimes B$	$\mathcal{L}^3_s(B)$
-1	$A\otimes\mathbb{Z}/3$	0	0	0
-2	0	0	0	0
-3	0	0	0	$B\otimes\mathbb{Z}/3$

TABLE 4. The E^1-term of the spectral sequence for $\mathcal{L}^3_s((B\to A)[1])$

Now we can formulate the super-analog of theorem 7.1.

Theorem 8.1. *Let $A=\mathbb{Z}/p^k$, then*

$$L_i\mathcal{L}^m_s(A,n)\simeq {}_p\pi_i(\mathcal{L}^m_s(\mathbb{Z}[n]\oplus\mathbb{Z}[n+1]))\oplus {}_pH_{i+m}\mathbb{L}^m(n+2,n+1;p^k)$$

In particular, for $A=\mathbb{Z}/p^k$, one has an isomorphism

$$L_i\mathcal{L}^m_s(A)\simeq {}_p\pi_i(\mathcal{L}^m_s(\mathbb{Z}\oplus\mathbb{Z}[1]))\oplus {}_pH_{i+m}\mathbb{L}^m(2,1;p^k)$$

Example. Analogously to table (7.12) we will collect the low-dimensional derived functors for $A=\mathbb{Z}/k$ in the following table:

i	$L_i\mathcal{L}^2_s(A)$	$L_i\mathcal{L}^3_s(A)$	$L_i\mathcal{L}^4_s(A)$	$L_i\mathcal{L}^5_s(A)$	$L_i\mathcal{L}^6_s(A)$	$L_i\mathcal{L}^7_s(A)$
6	0	0	0	0	0	$\mathbb{Z}/(7,k)$
5	0	0	0	0	$\mathbb{Z}/(3,k)$	$\mathbb{Z}/(7k,k^2)$
4	0	0	0	$\mathbb{Z}/(5,k)$	$\mathbb{Z}/k$	$(\mathbb{Z}/k)^{\oplus 2}$
3	0	0	$\mathbb{Z}/(2,k)$	$\mathbb{Z}/(5k,k^2)$	$\mathbb{Z}/k^{\oplus 2}$	$(\mathbb{Z}/k)^{\oplus 3}$
2	0	$\mathbb{Z}/(3,k)$	$\mathbb{Z}/(2k,k^2)$	$\mathbb{Z}/k$	$\mathbb{Z}/(2,k)\oplus\mathbb{Z}/k^{\oplus 2}$	$(\mathbb{Z}/k)^{\oplus 2}$
1	$\mathbb{Z}/(2,k)$	$\mathbb{Z}/(3k,k^2)$	$\mathbb{Z}/k$	$\mathbb{Z}/k$	$\mathbb{Z}/(2,k)\oplus\mathbb{Z}/k$	$\mathbb{Z}/k$
0	$\mathbb{Z}/(2k,k^2)$	0	0	0	0	0

TABLE 5. Values of derived functors $L_i\mathcal{L}^m_s(\mathbb{Z}/k)$

9. New functors and Main Conjecture

9.1. Hierarchies of special functors. For a prime p and $n \geq 2$, define the functors

$$\mathcal{N}_s^{n;p}(A) := \ker\{\mathbb{L}^n(A \overset{\sim}{\to} A)_{n-1} \otimes \mathbb{Z}/p \to \mathbb{L}^n(A \overset{\sim}{\to} A)_{n-2} \otimes \mathbb{Z}/p\} \tag{9.1}$$

$$\mathcal{N}^{n;p}(A) := \ker\{\mathbb{L}^n((A \overset{\sim}{\to} A)[1])_{2n-1} \otimes \mathbb{Z}/p \to \mathbb{L}^n((A \overset{\sim}{\to} A)[1])_{2n-2} \otimes \mathbb{Z}/p\} \tag{9.2}$$

It follows from definition that, for every abelian group A, there are natural monomorphisms

$$\mathcal{N}^{n;p}(A) \hookrightarrow \otimes^n(A) \otimes \mathbb{Z}/p$$
$$\mathcal{N}_s^{n;p}(A) \hookrightarrow \otimes^n(A) \otimes \mathbb{Z}/p$$

For a free abelian group A, the spectral sequence (7.20) for $C = \{A \overset{\sim}{\to} A\}$ implies that there is a natural isomorphism

$$L_{n-1}\mathcal{L}^n(A,2) \simeq H_{n-2}\mathbb{L}^n(C)$$

The Künneth formula implies that there is the following natural exact sequence:

$$0 \to \mathcal{L}^n(A) \otimes \mathbb{Z}/p \to \mathcal{N}^{n;p}(A) \to Tor(L_{n-1}\mathcal{L}^n(A,2), \mathbb{Z}/p) \to 0 \tag{9.3}$$

Its super-analog has the following form:

$$0 \to \mathcal{L}_s^n(A) \otimes \mathbb{Z}/p \to \mathcal{N}_s^{n;p}(A) \to Tor(L_{n-1}\mathcal{L}^n(A,1), \mathbb{Z}/p) \to 0$$

The sequence (9.3) splits as a sequence of abelian groups, however, for every n, such that the Tor-term in (9.3) is not zero, the sequence (9.3) does not split as a sequence of functors. This follows from the fact that the functor

$$A \mapsto L_{n-1}\mathcal{L}^n(A,2)$$

is of degree less than n, however, for every functor F of degree less than n, any natural transformation $F(A) \to \otimes^n(A) \otimes \mathbb{Z}/p$ is zero. For any prime p and $k \geq 1$, there are chains of natural epimorphisms

$$\mathcal{N}^{kp^i;p} \twoheadrightarrow \dots \mathcal{N}^{kp;p} \twoheadrightarrow \mathcal{N}^{k;p}$$

which split abstractly, but not naturally, moreover, every natural transformation of the type $N^{k;p} \to N^{kp;p}$ is the zero map (the same is true for the functors $N_s^{kp^i;p}$). This is the reason why we call such chains of functors by *hierarchies.* For example, for $n = p, p^2, p^3, p^4$, the sequences (9.3) can be found in the following diagram of exact sequences which split abstractly but not naturally:

$$\begin{array}{ccccccc}
 & & \mathcal{L}^{p^4}(A)\otimes\mathbb{Z}/p & & & & \\
 & & \downarrow & & & & \\
 & & \mathcal{N}^{p^4;p}(A) & & \mathcal{L}^{p^2}(A)\otimes\mathbb{Z}/p & & \\
 & & \downarrow & & \downarrow & & \\
\mathcal{L}^{p^3}(A)\otimes\mathbb{Z}/p & \rightarrowtail & \mathcal{N}^{p^3;p}(A) & \twoheadrightarrow & \mathcal{N}^{p^2;p}(A) & & \\
 & & & & \downarrow & & \\
 & & \mathcal{L}^{p}(A)\otimes\mathbb{Z}/p & \rightarrowtail & \mathcal{N}^{p;p}(A) & \twoheadrightarrow & A\otimes\mathbb{Z}/p
\end{array}$$

The spectral sequence (7.20) implies that, for a free abelian group A, there are natural isomorphisms

$$\mathcal{N}^{n,p}(A) \simeq \pi_{2n}\left(L\mathcal{L}^n(A,2) \overset{L}{\otimes} \mathbb{Z}/p\right) \tag{9.4}$$

$$\mathcal{N}_s^{n,p}(A) \simeq \pi_{n}\left(L\mathcal{L}^n(A,1) \overset{L}{\otimes} \mathbb{Z}/p\right) \tag{9.5}$$

For $p=2$ the functor $\mathcal{N}^{2;2}(A)$ is exactly $\Gamma_2(A)\otimes\mathbb{Z}/2$, there is the following sequence

$$0 \to \Lambda^2(A)\otimes\mathbb{Z}/2 \to \Gamma_2(A)\otimes\mathbb{Z}/2 \to A\otimes\mathbb{Z}/2 \to 0$$

The description of cross-effects of the functors $\mathcal{N}^{n;p}$, $\mathcal{N}_s^{n;p}$ follows directly from the description of cross-effects of the Lie and super-Lie functors (see (2.8) and (2.9)). For free abelian A, B one has the following

$$\mathcal{N}^{m;p}(A\oplus B) = \bigoplus_{d|m,\ 1\leq d\leq m}\ \bigoplus_{C\in J_{m/d}} \mathcal{N}^{d;p}(C)$$

and

$$\mathcal{N}_s^{m;p}(A\oplus B) = \bigoplus_{\substack{d|m,\ 1\leq d\leq m\\ m/d \text{ odd}}}\ \bigoplus_{C\in J_{m/d}} \mathcal{N}_s^{d;p}(C) \oplus \bigoplus_{\substack{d|m,\ 1\leq d< m\\ m/d \text{ even}}}\ \bigoplus_{C\in J_{m/d}} \mathcal{N}^{d;p}(C)$$

9.2. Construction of $\mathcal{E}$-complexes. For $n,k\geq 1$, a prime p, we will use the following notation:

$$\overline{\mathcal{W}}^{(p)}_{n,k} := \begin{cases} \mathcal{W}^{(p)}_{n,k}(1)\setminus \mathcal{W}^{(p)}_{n,k}(2), & \text{if } n \text{ is even}\\ \mathcal{W}^{(p)}_{n-1,k}, & \text{if } n \text{ is odd}\end{cases}$$

For $C \in \mathsf{DAb}$, define the following objects of DAb $(n \geq 0)$

$$\mathcal{E}^m(C,2n) := L\mathcal{L}^m(C)[2mn] \oplus \bigoplus_{\substack{p \text{ prime} \\ p^k|m \\ p^{k+1}\nmid m}} \bigoplus_{\substack{i=1,\dots,k \\ w\in \overline{\mathcal{W}}^{(p)}_{\frac{2nm}{p^i},i}}} L\mathcal{N}^{\frac{m}{p^i};p}(C)[\frac{2nm}{p^i}+d(w)]$$

$$\mathcal{E}^m(C,2n+1) := L\mathcal{L}^m_s(C)[(2n+1)m]\oplus \bigoplus_{\substack{p \text{ prime} \\ p^k|m \\ p^{k+1}\nmid m}} \bigoplus_{\substack{i=1,\dots,k \\ w\in \overline{\mathcal{W}}^{(p)}_{\frac{(2n+1)m}{p^i},i}}} L\mathcal{N}_s^{\frac{m}{p^i};p}(C)[\frac{(2n+1)m}{p^i}+d(w)]$$

and their super-analogs $(n \geq 1)$

$$\widetilde{\mathcal{E}}^m(C,2n-1) := L\mathcal{L}^m(C)[(2n-1)m] \oplus \bigoplus_{\substack{p \text{ prime} \\ p^k|m \\ p^{k+1}\nmid m}} \bigoplus_{\substack{i=1,\dots,k \\ w=(w_1,\dots,w_i)\in \overline{\mathcal{W}}^{(p)}_{\frac{2nm}{p^i},i} \\ w_i>u(p;k)}} L\mathcal{N}^{\frac{m}{p^i};p}(C)[\frac{2nm}{p^i}+d(w)-m]$$

$$\widetilde{\mathcal{E}}^m(C,2n) := L\mathcal{L}^m_s(C)[2nm]\oplus \bigoplus_{\substack{p \text{ prime} \\ p^k|m \\ p^{k+1}\nmid m}} \bigoplus_{\substack{i=1,\dots,k \\ w=(w_1,\dots,w_i)\in \overline{\mathcal{W}}^{(p)}_{\frac{(2n+1)m}{p^i},i} \\ w_i>u(p;k)}} L\mathcal{N}_s^{\frac{m}{p^i};p}(C)[\frac{(2n+1)m}{p^i}+d(w)-m]$$

where

$$u(p,k) := \begin{cases} \frac{p^{k-1}-1}{2}, & \text{if } p \text{ is odd} \\ 2^{k-1}, & \text{if } p=2 \end{cases}$$

Main Conjecture. For $n,m,i \geq 1$, $C \in \mathsf{DAb}$, there are natural isomorphisms

$$\pi_i(L\mathcal{L}^m(C[n])) \simeq \pi_i(\mathcal{E}^m(C,n)) \tag{9.6}$$

$$\pi_i(L\mathcal{L}^m_s(C[n])) \simeq \pi_i(\widetilde{\mathcal{E}}^m(C,n)). \tag{9.7}$$

Here are the simplest examples of $\mathcal{E}$-complexes in low degrees:

Examples. For $m = 2$, $n \geq 1$, one has

$$\mathcal{E}^2(C, 2n) = L\Lambda^2(C)[4n] \oplus \bigoplus_{i=1}^{n} C \overset{L}{\otimes} \mathbb{Z}/2[2n + 2i - 1]$$

$$\mathcal{E}^2(C, 2n+1) = L\Gamma_2(C)[4n+2] \oplus \bigoplus_{i=1}^{n} C \overset{L}{\otimes} \mathbb{Z}/2[2n + 2i]$$

$$\widetilde{\mathcal{E}}^2(C, 2n) = L\Gamma_2(C)[4n] \oplus \bigoplus_{i=1}^{n} C \overset{L}{\otimes} \mathbb{Z}/2[2n + 2i - 2]$$

$$\widetilde{\mathcal{E}}^2(C, 2n+1) = L\Lambda^2(C)[4n+2] \oplus \bigoplus_{i=1}^{n+1} C \overset{L}{\otimes} \mathbb{Z}/2[2n + 2i - 1]$$

For $m = 3$, $n \geq 1$, one has

$$\mathcal{E}^3(C, 2n) = L\mathcal{L}^3(C)[6n] \oplus \bigoplus_{i=1}^{n} C \overset{L}{\otimes} \mathbb{Z}/3[2n + 4i - 1]$$

$$\mathcal{E}^3(C, 2n+1) = L\mathcal{L}^3_s(C)[6n+3] \oplus \bigoplus_{i=1}^{n} C \overset{L}{\otimes} \mathbb{Z}/3[2n + 4i]$$

$$\widetilde{\mathcal{E}}^3(C, 2n) = L\mathcal{L}^3_s(C)[6n] \oplus \bigoplus_{i=1}^{n} C \overset{L}{\otimes} \mathbb{Z}/3[2n + 4i - 3]$$

$$\widetilde{\mathcal{E}}^3(C, 2n+1) = L\mathcal{L}^3(C)[6n+3] \oplus \bigoplus_{i=1}^{n+1} C \overset{L}{\otimes} \mathbb{Z}/3[2n + 4i - 2]$$

10. The abstract isomorphism

Proposition 10.1. *For all $n, l \geq 1$, there is a homotopy equivalence*

$$L\mathcal{L}^m(\mathbb{Z}, l) \sim \mathcal{E}^n(\mathbb{Z}, l).$$

Proof. The description of derived functors from 5.1 and (9.4), (9.5) imply that

$$L\mathcal{N}^{m;p}(\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}/p, \text{ if } m = p^t, \ t \geq 0 \\ 0 \text{ otherwise} \end{cases} \tag{10.1}$$

and

$$L\mathcal{N}^{m;p}_s(\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}/p, \text{ if } m = 2p^t, \ t \geq 0 \text{ or } m = 1 \\ 0 \text{ otherwise} \end{cases} \tag{10.2}$$

The description (10.1) implies that, if m is not a power of prime, then $\mathcal{E}^m(\mathbb{Z}, 2n) \simeq 0,\ n \geq 1$. If m is a power of prime $m = p^k$, then

$$\bigoplus_{\substack{i=1,\dots,k \\ w\in \overline{\mathcal{W}}_{\frac{2nm}{p^i},i}}} L\mathcal{N}^{\frac{m}{p^i}}(\mathbb{Z})[\frac{2nm}{p^i} + d(w)] \simeq \bigoplus_{\substack{i=1,\dots,k \\ w\in \mathcal{W}^{(p)}_{\frac{2nm}{p^k},k}(i)\setminus \mathcal{W}^{(p)}_{\frac{2nm}{p^k},k}(i+1)}} L\mathcal{N}^{\frac{m}{p^{k-i+1}};p}(\mathbb{Z})[2n + d(w)]$$

$$\simeq \bigoplus_{w\in \mathcal{W}^{(p)}_{2n,k}} \mathbb{Z}/p\ [2n + d(w)] \quad (10.3)$$

and we have the needed statement for all even l. For $l = 2n + 1,\ n \geq 0$, we have

$$\mathcal{E}^2(\mathbb{Z}, 2n + 1) \simeq \mathbb{Z}[4n + 2] \oplus \bigoplus_{i=1}^{n} \mathbb{Z}/2\ [2n + 2i] \simeq L\mathcal{L}^2(\mathbb{Z}, 2n + 1)$$

The description (10.2) implies that $\mathcal{E}^m(\mathbb{Z}, 2n + 1) \simeq 0$ if m is neither a power of prime nor a double power of prime. Now assume that $m = p^k$ for an odd prime. In this case, (10.2) implies that

$$\mathcal{E}^m(\mathbb{Z}, 2n + 1) \simeq \bigoplus_{w\in \overline{\mathcal{W}}^{(p)}_{2n+1,k}} \mathbb{Z}/p[2n + 1 + d(w)] \simeq \bigoplus_{w\in \mathcal{W}^{(p)}_{2n,k}} \mathbb{Z}/p[2n + d(w) + 1] \simeq L\mathcal{L}^m(\mathbb{Z}, 2n + 1)$$

Now consider the case $m = 2p^k$ for an odd p. In this case

$$\mathcal{E}^{2p^k}(\mathbb{Z}, 2n + 1) \simeq \bigoplus_{\substack{i=1,\dots,k \\ w\in \overline{\mathcal{W}}_{2(2n+1)p^{k-i},i}}} \mathbb{Z}/p[(2n + 2)p^{k-i} + d(w)] \simeq$$

$$\bigoplus_{w\in \mathcal{W}^{(p)}_{4n+2,k}} \mathbb{Z}/p[4n + 2 + d(w)] \simeq L\mathcal{L}^{2p^k}(\mathbb{Z}, 2n + 1)$$

It remains to consider the case $m = 2^k,\ k > 1$. In this case,

$$\mathcal{E}^{2^k}(\mathbb{Z}, 2n + 1) \simeq \bigoplus_{w\in \overline{\mathcal{W}}^{(2)}_{2n+1,k}} \mathbb{Z}/2[2n + 1 + d(w)] \oplus \bigoplus_{\substack{i=1,\dots,k-1 \\ w\in \overline{\mathcal{W}}_{2(2n+1)2^{k-i},i}}} \mathbb{Z}/2[4n + 2 + d(w)] \simeq$$

$$L\mathcal{L}^{2^k}(\mathbb{Z}, 2n)[1] \oplus L\mathcal{L}^{2^{k-1}}(\mathbb{Z}, 4n + 2) \simeq L\mathcal{L}^{2^k}(\mathbb{Z}, 2n + 1)$$

The statement is proved. □

Proposition 10.2. *Let $C \in \mathsf{DAb}$ and homology of C are torsion-free. For all $n, m \geq 1$, there is a homotopy equivalence*

$$L\mathcal{L}^n(C[m]) \sim \mathcal{B}^n(C, m)$$

Proof. Let $A, B \in \mathsf{DAb}$. Suppose that we are given the homotopy equivalences

$$L\mathcal{L}^d(C[2n]) \sim \mathcal{E}^d(C, 2n) \quad (10.4)$$

for all $d < m$ and all $C \in \mathsf{DAb}$ with torsion-free homology and

$$L\mathcal{L}^m(A[2n]) \sim \mathcal{E}^m(A, 2n) \quad (10.5)$$

$$L\mathcal{L}^m(B[2n]) \sim \mathcal{E}^m(B, 2n) \quad (10.6)$$

It follows that

$$L\mathcal{L}^m(A\oplus B[2n]) = \bigoplus_{\substack{d|m,\ 1\le d\le m\\ C\in J_{m/d}}} L\mathcal{L}^d(C[\frac{2nm}{d}]) = \tag{10.7}$$

$$L\mathcal{L}^m(A[2n])\oplus L\mathcal{L}^m(B[2n])\oplus \bigoplus_{\substack{d|m,\ 1\le d< m\\ C\in J_{m/d}}} L\mathcal{L}^d(C[\frac{2nm}{d}]) \sim \tag{10.8}$$

$$\mathcal{E}^m(A,2n)\oplus \mathcal{E}^m(B,2n)\oplus \bigoplus_{\substack{d|m,\ 1\le d< m\\ C\in J_{m/d}}} \mathcal{E}^d(C,\frac{2nm}{d}) \sim \tag{10.9}$$

$$\mathcal{E}^m(A,2n)\oplus \mathcal{E}^m(B,2n)\oplus \bigoplus_{d|m,\ d<m,\ C\in J_{m/d}} L\mathcal{L}^d(C)[2nm]\oplus \tag{10.10}$$

$$\bigoplus_{\substack{d|m,\ d<m,\ p^k|d\\ C\in J_{m/d},\ p^{k+1}\nmid d}} \ \bigoplus_{\substack{i=1,\dots,k\\ w\in \overline{\mathcal{W}}^{(p)}_{\frac{2nm}{p^i},i}}} L\mathcal{N}^{\frac{d}{p^i};p}(C)[\frac{2nm}{p^k}+d(w)] \tag{10.11}$$

where each $C\in J_{m/d}$ is a basic tensor product $A_{i_1}\overset{L}{\otimes}\dots\overset{L}{\otimes}A_{i_{m/d}}$, with $A_{i_j}\in\{A,B\}$. From the other hand,

$$\mathcal{E}^m(A\oplus B,2n) := \mathcal{E}^m(A,2n)\oplus \mathcal{E}^m(B,2n)\oplus \bigoplus_{d|m,\ d<m,\ C\in J_{m/d}} L\mathcal{L}^d(C)[2nm]\oplus \tag{10.12}$$

$$\bigoplus_{\substack{p \text{ prime}\\ p^k|m\\ p^{k+1}\nmid m}} \ \bigoplus_{\substack{i=1,\dots,k\\ w\in \overline{\mathcal{W}}^{(p)}_{\frac{2nm}{p^i},i}}} \ \bigoplus_{\substack{s|\frac{m}{p^i}\\ C\in J_{\frac{m}{sp^i}}}} L\mathcal{N}^{s;p}(C)[\frac{2nm}{p^k}+d(w)] \tag{10.13}$$

We will show that the terms (10.11) and (10.13) are the same. Observe that (10.11) and (10.13) are direct sums of the shifted terms like

$$R_{s,t}(C) := L\mathcal{N}^{s;p}(C), C\in J_t$$

for s,t such that $s,t|m,\ m/st$ is a power of p. The contributions of the term $R_{s,t}(C),\ C\in J_t$ in (10.11) and (10.13) are the same, namely

$$\bigoplus_{w\in\overline{\mathcal{W}}^{(p)}_{2nst,log_p\frac{m}{st}}} R_{s,t}(C)[2nst+d(w)].$$

Hence, the conditions (10.4),(10.5) and (10.6) imply that

$$L\mathcal{L}^m(A\oplus B[2n])\sim \mathcal{E}^m(A\oplus B,2m).$$

Now consider the odd dimensions. Suppose that we have the conditions

$$L\mathcal{L}^d(C[2n+1])\sim \mathcal{E}^d(C,2n+1) \tag{10.14}$$

for all $d<m$ and arbitrary complexes C and

$$L\mathcal{L}^m(A[2n+1])\sim \mathcal{E}^m(A,2n+1) \tag{10.15}$$

$$L\mathcal{L}^m(B[2n+1])\sim \mathcal{E}^m(B,2n+1) \tag{10.16}$$

We have

$$L\mathcal{L}^m(A\oplus B[2n+1]) = L\mathcal{L}^m(A[2n+1])\oplus L\mathcal{L}^m(B[2n+1])\oplus \tag{10.17}$$

$$\bigoplus_{\substack{d|m,\ 1\leq d<m\\ C\in J_{m/d}}} L\mathcal{L}^d(C[\frac{(2n+1)m}{d}])\sim \tag{10.18}$$

$$\mathcal{E}^m(A,2n+1)\oplus\mathcal{E}^m(B,2n+1)\oplus\bigoplus_{\substack{d|m,\ 1\leq d<m\\ C\in J_{m/d}}}\mathcal{E}^d(C,\frac{(2n+1)m}{d})\sim \tag{10.19}$$

$$\mathcal{E}^m(A,2n+1)\oplus\mathcal{E}^m(B,2n+1)\oplus\bigoplus_{\substack{d|m,\ 1\leq d<m\\ C\in J_{m/d}}}\mathcal{L}^d(C)[(2n+1)m]\oplus \tag{10.20}$$

$$\bigoplus_{\substack{d|m,\ m/d\text{ even},\ p^k|d\\ C\in J_{m/d},\ p^{k+1}\nmid d}}\ \bigoplus_{\substack{i=1,\dots,k\\ w\in\overline{\mathcal{W}}^{(p)}_{\frac{(2n+1)m}{p^i},i}}} L\mathcal{N}^{\frac{d}{p^i};p}(C)[\frac{(2n+1)m}{p^i}+d(w)]\oplus \tag{10.21}$$

$$\bigoplus_{\substack{d|m,\ m/d\text{ odd},\ p^k|d\\ C\in J_{m/d},\ p^{k+1}\nmid d}}\ \bigoplus_{\substack{i=1,\dots,k\\ w\in\overline{\mathcal{W}}^{(p)}_{\frac{(2n+1)m}{p^i},i}}} L\mathcal{N}_s^{\frac{d}{p^i};p}(C)[\frac{(2n+1)m}{p^i}+d(w)] \tag{10.22}$$

where each $C\in J_{m/d}$ is a basic tensor product $A_{i_1}\overset{L}{\otimes}\dots\overset{L}{\otimes}A_{i_{m/d}}$, with $A_{i_j}\in\{A,B\}$. From the other hand,

$$\mathcal{E}^m(A\oplus B,2n+1):=\mathcal{E}^m(A,2n+1)\oplus\mathcal{E}^m(B,2n+1)\oplus\bigoplus_{d|m,\ d<m,\ C\in J_{m/d}} L\mathcal{L}^d(C)[(2n+1)m]\oplus \tag{10.23}$$

$$\bigoplus_{\substack{p\text{ prime}\\ p^k|m\\ p^{k+1}\nmid m}}\ \bigoplus_{\substack{i=1,\dots,k\\ w\in\overline{\mathcal{W}}^{(p)}_{\frac{(2n+1)m}{p^i},i}}}\Big(\bigoplus_{\substack{s|\frac{m}{p^i},\ \frac{m}{sp^i}\text{ even}\\ C\in J_{\frac{m}{sp^i}}}} L\mathcal{N}^{s;p}(C)[\frac{(2n+1)m}{p^i}+d(w)]\oplus \tag{10.24}$$

$$\bigoplus_{\substack{s|\frac{m}{p^i},\ \frac{m}{sp^i}\text{ odd}\\ C\in J_{\frac{m}{sp^i}}}} L\mathcal{N}_s^{s;p}(C)[\frac{(2n+1)m}{p^i}+d(w)]\Big) \tag{10.25}$$

Consider the contribution of the terms like

$$R_{s,t}(C):=L\mathcal{N}^{s;p}(C),\quad R'_{s,t}(C):=L\mathcal{N}_s^{s;2}(C),\ C\in J_t$$

in (10.22), (10.24), (10.25), for s,t such that $s,t|m,\ m/st$ is a power of p. For an even t, the contributions of the term $R_{s,t}(C),\ C\in J_t$ in (10.22) and (10.24) are the same, namely

$$\bigoplus_{w\in\overline{\mathcal{W}}^{(p)}_{(2n+1)st,log_p\frac{m}{st}}} R_{s,t}(C)[(2n+1)st+d(w)]$$

For an odd t, the contributions of the term $R'_{s,t}(C),\ C\in J_t$ in (10.22) and (10.24) are the same as well, namely

$$\bigoplus_{w\in\overline{\mathcal{W}}^{(p)}_{(2n+1)st,log_p\frac{m}{st}}} R'_{s,t}(C)[(2n+1)st+d(w)]$$

This comparison shows that there is a homotopy equivalence

$$\mathcal{L}^m(A\oplus B[2n+1])\sim \mathcal{E}^m(A\oplus B,2n+1).$$

Since any element of DAb is homotopy equivalent to a direct sum of elements like $\mathbb{Z}[l]$ for different l, the statement follows from proposition 10.1. □

Proposition 10.3. *Let $m,n,l\geq 1$, p a prime. There is a homotopy equivalence*

$$\mathcal{E}^m(\mathbb{Z}/p^l,n)\sim L\mathcal{L}^m(\mathbb{Z}/p^l,n).$$

Proof. Observe that, for all $s\geq 1$, there are homotopy equivalences

$$L\mathcal{N}^{s;p}(\mathbb{Z}/p^l)\sim L\mathcal{N}^{s;p}(\mathbb{Z}\oplus\mathbb{Z}[1]).$$

$$L\mathcal{N}^{s;p}_s(\mathbb{Z}/p^l)\sim L\mathcal{N}^{s;p}_s(\mathbb{Z}\oplus\mathbb{Z}[1]).$$

We have

$$\begin{aligned}\mathcal{E}^m(\mathbb{Z}/p^l,2n) =& L\mathcal{L}^m(\mathbb{Z}/p^l)[2nm]\bigoplus_{\substack{p \text{ prime}\\ p^k|m\\ p^{k+1}\nmid m}}\ \bigoplus_{\substack{i=1,\dots,k\\ \overline{\mathcal{W}}^{(p)}_{\frac{2nm}{p^i},i}}} L\mathcal{N}^{\frac{m}{p^i};p}(\mathbb{Z}/p^l)[\frac{2nm}{p^i}+d(w)]\sim\\ & L\mathcal{L}^m(\mathbb{Z}/p^l)[2nm]\bigoplus_{\substack{p \text{ prime}\\ p^k|m\\ p^{k+1}\nmid m}}\ \bigoplus_{\substack{i=1,\dots,k\\ \overline{\mathcal{W}}^{(p)}_{\frac{2nm}{p^i},i}}} L\mathcal{N}^{\frac{m}{p^i};p}(\mathbb{Z}\oplus\mathbb{Z}[1])[\frac{2nm}{p^i}+d(w)]\end{aligned}$$

We have

$$\begin{aligned}\pi_i(L\mathcal{L}^m(\mathbb{Z}/p^l)[2mn]) =&\ {}_p\pi_i(L\mathcal{L}^m(\mathbb{Z}\oplus\mathbb{Z}[1])[2mn])\oplus\ {}_pH_{i-2mn}\mathbb{L}^m(1,0;p^l)=\\ & {}_p\pi_i(L\mathcal{L}^m(\mathbb{Z}\oplus\mathbb{Z}[1])[2mn])\oplus\ {}_pH_i\mathbb{L}^m(2n+1,2n;p^l)\end{aligned}\quad (10.26)$$

by proposition 7.2. Since

$$\mathcal{E}^m(\mathbb{Z}\oplus\mathbb{Z}[1],2n)\sim L\mathcal{L}^m((\mathbb{Z}\oplus\mathbb{Z}[1])[2n])$$

by proposition 10.2, we obtain the following

$$\pi_i\mathcal{E}^m(\mathbb{Z}/p^l,2n)\simeq\ {}_p\pi_i(L\mathcal{L}^m((\mathbb{Z}\oplus\mathbb{Z}[1])[2n]))\oplus\ {}_pH_i\mathbb{L}^m(2n+1,2n;p^l)=L_i\mathcal{L}^m(\mathbb{Z}/p^l,2n)$$

by theorem 7.1.

We have

$$\mathcal{E}^m(\mathbb{Z}/p^l, 2n+1) = L\mathcal{L}_s^m(\mathbb{Z}/p^l)[(2n+1)m] \oplus \bigoplus_{\substack{p \text{ prime} \\ p^k|m \\ p^{k+1}\nmid m}} \bigoplus_{\substack{i=1,\dots,k \\ w\in \overline{\mathcal{W}}^{(p)}_{\frac{(2n+1)m}{p^i},i}}} L\mathcal{N}_s^{\frac{m}{p^i};p}(\mathbb{Z}/p^l)[\frac{(2n+1)m}{p^i}+d(w)]$$

$$L\mathcal{L}_s^m(\mathbb{Z}/p^l)[(2n+1)m] \oplus \bigoplus_{\substack{p \text{ prime} \\ p^k|m \\ p^{k+1}\nmid m}} \bigoplus_{\substack{i=1,\dots,k \\ w\in \overline{\mathcal{W}}^{(p)}_{\frac{(2n+1)m}{p^i},i}}} L\mathcal{N}_s^{\frac{m}{p^i};p}(\mathbb{Z}\oplus\mathbb{Z}[1])[\frac{(2n+1)m}{p^i}+d(w)]$$

Theorem 8.1 implies that

$$\pi_i(L\mathcal{L}_s^m(\mathbb{Z}/p^l)[(2n+1)m]) \simeq {}_p\pi_i(L\mathcal{L}_s^m(\mathbb{Z}\oplus\mathbb{Z}[1])[(2n+1)m]) \oplus {}_pH_{i-2nm}\mathbb{L}^m(2,1;p^l) \simeq$$
$${}_p\pi_i(L\mathcal{L}_s^m(\mathbb{Z}\oplus\mathbb{Z}[1])[(2n+1)m]) \oplus {}_pH_i\mathbb{L}^m(2n+2,2n+1;p^l) \quad (10.27)$$

Since

$$\mathcal{E}^m(\mathbb{Z}\oplus\mathbb{Z}[1], 2n+1) \sim L\mathcal{L}^m((\mathbb{Z}\oplus\mathbb{Z}[1])[2n+1])$$

by proposition 10.2, we obtain the following

$$\pi_i\mathcal{E}^m(\mathbb{Z}/p^l, 2n+1) \simeq {}_p\pi_i(L\mathcal{L}^m((\mathbb{Z}\oplus\mathbb{Z}[1])[2n+1])) \oplus {}_pH_i\mathbb{L}^m(2n+2,2n+1;p^l) \simeq L_i\mathcal{L}^m(\mathbb{Z}/p^l, 2n+1)$$

and the needed statement follows. □

Proposition 10.3 gives a possibility to follow the proof of the proposition 10.2,not only for elements of DAb with torsion-free homology but for all elements of DAb and prove the following

Theorem 10.1. *For every $n, m \geq 1$, $C \in$ DAb, there is an (unnatural) homotopy equivalence*

$$\mathcal{E}^m(C,n) \sim L\mathcal{L}^m(C[n]).$$

11. Semi-décalage

11.1. Semi-décalage. For any pair of free abelian groups A and B, and $m \geq 2$, we define a pair of morphisms (see [3], 7.6)

$$\chi_m : \mathcal{L}_s^m(A) \otimes \Lambda^m(B) \to \mathcal{L}^m(A\otimes B) \quad (11.1)$$

$$\bar{\chi}_m : \mathcal{L}^m(A) \otimes \Lambda^m(B) \to \mathcal{L}_s^m(A\otimes B) \quad (11.2)$$

for $a_1, \dots, a_m \in A$ and $b_1, \dots, b_m \in B$, by

$$\chi_m : \{a_1,\dots,a_m\} \otimes b_1\wedge\dots\wedge b_m \mapsto \sum_{\sigma\in\Sigma_m} \operatorname{sign}(\sigma)[a_1\otimes b_{\sigma_1},\dots,a_m\otimes b_{\sigma_m}]$$

$$\bar{\chi}_m : [a_1,\dots,a_m] \otimes b_1\wedge\dots\wedge b_m \mapsto \sum_{\sigma\in\Sigma_m} \operatorname{sign}(\sigma)\{a_1\otimes b_{\sigma_1},\dots,a_m\otimes b_{\sigma_m}\}\,.$$

For $m = 2k$ with k odd, we set

$$\chi_m : \{a_1, \dots, a_k\}^{[2]} \otimes b_1 \wedge \dots \wedge b_n \mapsto \sum_{\sigma \in A_m} [[a_1 \otimes b_{\sigma_1}, \dots, a_k \otimes b_{\sigma_k}], [a_1 \otimes b_{\sigma_{k+1}}, \dots, a_k \otimes b_{\sigma_{2k}}]],$$

Taking $X \in \mathsf{DAb}$, $B = K(\mathbb{Z}, 1)$ we obtain the natural maps

$$L\mathcal{L}^m(X) \overset{L}{\otimes} L\Lambda^m(\mathbb{Z}, 1) = L\mathcal{L}^m(X)[m] \to L\mathcal{L}^m_s(X \overset{L}{\otimes} K(\mathbb{Z}, 1)) = L\mathcal{L}^m_s(X[1]) \tag{11.3}$$

$$L\mathcal{L}^m_s(X) \overset{L}{\otimes} L\Lambda^m(\mathbb{Z}, 1) = L\mathcal{L}^m_s(X)[m] \to L\mathcal{L}^m(X \overset{L}{\otimes} K(\mathbb{Z}, 1)) = L\mathcal{L}^m(X[1]) \tag{11.4}$$

Iterating these constructions, we obtain the natural maps

$$\begin{aligned}
&L\mathcal{L}^m(X)[2nm] \to L\mathcal{L}^m(X[2n]) \\
&L\mathcal{L}^m(X)[(2n+1)m] \to L\mathcal{L}^m_s(X[2n+1]) \\
&L\mathcal{L}^m_s(X)[2nm] \to L\mathcal{L}^m_s(X[2n]) \\
&L\mathcal{L}^m_s(X)[(2n+1)m] \to L\mathcal{L}^m(X[2n+1])
\end{aligned}$$

11.2. Bousfield's pension maps. For a functor $T : \mathsf{Ab} \to \mathsf{Ab}$, such that $T(0) = 0$, there is a natural map

$$E : \mathbb{Z}[M] \otimes T(N) \to T(M \otimes N), \ M, N \in \mathsf{Ab}$$

which induces a paring

$$E_* : \tilde{H}_* K(\mathbb{Z}, n) \otimes \pi_*(LT(X)) \to \pi_*(LT(X[2])), \ n \geq 1, \ X \in \mathsf{DAb}$$

These constructions are from [2]. For $n = 2$, taking the generator $\epsilon_r \in \tilde{H}_{2r} K(\mathbb{Z}, 2) = \mathbb{Z}$, we obtain so-called pension maps

$$\epsilon_r : \pi_i(LT(X)) \to \pi_{i+2r}(LT(X[2])), \ i \geq 0.$$

Observe that, the composition of the maps (11.3) and (11.4) induces the composition map

$$\pi_i(L\mathcal{L}^m(X)) \to \pi_{i+m}(L\mathcal{L}^m_s(X[1])) \to \pi_{i+2m}(L\mathcal{L}^m(X[2]))$$

which is exactly the pension map ϵ_r for the functor $\mathcal{L}^m$.

The following statement is proved in ([2], Theorem 3.1):

Theorem 11.1. *Let $T : \mathsf{Ab} \to \mathsf{Ab}$ a polynomial functor of degree $\leq r$ ($r \geq 1$) and let $X \in DAb$, such that $H_i(X) = 0, \ i > n$ for some $n \geq 0$. Then*

$$\epsilon_r : \pi_i(LT(X)) \to \pi_{i+2r}(LT(X[2]))$$

is an isomorphism for $i > n(r-1) + 1$ and a monomorphism for $i = n(r-1) + 1$.

Corollary 11.1. *Let p be an odd prime. For $i \geq 1$, let $\beta_i, \mu_i, \lambda_i$ be nontrivial elements*

$$\begin{aligned}
&\beta_i \in \pi_{2i}(\Lambda^2(K(\mathbb{Z}, i)) \otimes \mathbb{Z}/2) = \mathbb{Z}/2 \\
&\mu_i \in \pi_{2pi}(\mathcal{L}^p(K(\mathbb{Z}, 2i)) \otimes \mathbb{Z}/p) = \mathbb{Z}/p \\
&\lambda_i \in \pi_{2pi-1}(\mathcal{L}^p(K(\mathbb{Z}, 2i)) \otimes \mathbb{Z}/p) = \mathbb{Z}/p.
\end{aligned}$$

Then $\epsilon_2(\beta_i) \neq 0$, $\epsilon_p(\mu_i) \neq 0$, $\epsilon_p(\lambda_i) \neq 0$.

There is a map

$$\beta_d : \mathcal{L}^d(B) \otimes SP^d(A) \to \mathcal{L}^d(B \otimes A)$$

given by

$$\beta_d : [b_1, \dots, b_d] \otimes a_1 \dots a_d \mapsto \sum_{\sigma \in \Sigma_d} [b_1 \otimes a_{\sigma_1}, \dots, b_d \otimes a_{\sigma_d}]$$

For $d|n$, there is a natural map

$$SP^n(A) \to \otimes^{n/d}(SP^d(A))$$

Construct the map

$$\beta_{d,n} : \mathcal{L}^d(B) \otimes SP^n(A) \to \mathcal{L}^d(B \otimes (\otimes^{n/d}(A)))$$

as a composition

$$\mathcal{L}^d(B) \otimes SP^n(A) \to \mathcal{L}^d(B) \otimes (\otimes^{n/d}(SP^d(A))) \to \mathcal{L}^d(B \otimes (\otimes^{n/d}(A)))$$

Here the last map is the n/d-th iteration of the map β_d. The following two lemmas follow straightforwardly from definition of pension maps and composition maps in derived functors of Lie functors (3.4).

Lemma 11.1. *For $d|n$ and free abelian groups A, X, Y, $X_{i_j} \in \{X, Y\}$, the following diagram*

$$\begin{array}{ccc}
\mathcal{L}^d(X_{i_1} \otimes \dots \otimes X_{i_{n/d}}) \otimes SP^n(A) & \xrightarrow{\beta_{d,n}} & \mathcal{L}^d((X_{i_1} \otimes A) \otimes \dots \otimes (X_{i_{n/d}} \otimes A)) \\
\downarrow & & \downarrow \\
\mathcal{L}^n(X \oplus Y) \otimes SP^n(A) & \xrightarrow{\beta_n} & \mathcal{L}^n((X \oplus Y) \otimes A)
\end{array}$$

is commutative.

Corollary 11.2. *For $d|n$, $C_1, C_2 \in \mathsf{DAb}$, $D_{i_j} \in \{C_1, C_2\}$, the following diagram*

$$\begin{array}{ccc}
\pi_i\left(L\mathcal{L}^d(D_{i_1} \overset{L}{\otimes} \dots \overset{L}{\otimes} D_{i_{n/d}})\right) & \xrightarrow{\epsilon_d^{n/d}} & \pi_{i+2n}\left(L\mathcal{L}^d(D_{i_1} \overset{L}{\otimes} \dots \overset{L}{\otimes} D_{i_{n/d}}[\frac{2n}{d}])\right) \\
\downarrow & & \downarrow \\
\pi_i(L\mathcal{L}^n(C_1 \oplus C_2)) & \xrightarrow{\epsilon_n} & \pi_{i+2n}(L\mathcal{L}^n((C_1 \oplus C_2)[2]))
\end{array}$$

is commutative for all $i \geq 0$.

Lemma 11.2. *For $i, k, l, q \geq 0$ and $X \in \mathsf{DAb}$, the following diagram*

$$\begin{array}{ccc}
L_i\mathcal{L}^k(\mathbb{Z}, q) \otimes \pi_q(L\mathcal{L}^l(X)) & \longrightarrow & \pi_i(L\mathcal{L}^{kl}(X))) \\
\downarrow{\scriptstyle \epsilon_k^l \otimes \epsilon_l} & & \downarrow{\scriptstyle \epsilon_{kl}} \\
L_{i+2kl}\mathcal{L}^k(\mathbb{Z}, q+2l) \otimes \pi_{q+2l}(L\mathcal{L}^l(X[2])) & \longrightarrow & \pi_{i+2kl}(L\mathcal{L}^{kl}(X[2]))
\end{array}$$

is commutative.

Now we are ready to prove the following

Theorem 11.2. *For $n \geq 1$, $C \in \mathsf{DAb}$, the map*

$$\epsilon_n : \pi_i(L\mathcal{L}^n(C)) \to \pi_{i+2n}(L\mathcal{L}^n(C[2])))$$

is injective for all $i \geq 0$.

Proof. First observe that the statement follows for $n = 1$, in this case, the considered map is an isomorphism. We will proceed by induction on n. Assume that the statement follows for all Lie powers less than n.

Theorems 5.2, 5.5, Lemma 11.2 and Corollary 11.1 together with the fact that pension maps commute with suspensions (see 2.3 [2]) imply that the map ϵ_n is injective for $C = \mathbb{Z}[l]$ for every $l \geq 0$. Corollary 11.2 imply that ϵ_n is injective for all C with torsion-free homology. The spectral sequence (7.5) and Theorem 7.1 imply that there is the following diagram

$$\begin{array}{ccccc}
{}_p\pi_i(L\mathcal{L}^n(\mathbb{Z}[m] \oplus \mathbb{Z}[m+1]) & \rightarrowtail & L_i\mathcal{L}^n(\mathbb{Z}/p^l, m) & \twoheadrightarrow & {}_pH_i\mathcal{L}^n(n+1, n; p^l) \\
\downarrow{\scriptstyle \epsilon_n} & & \downarrow{\scriptstyle \epsilon_n} & & \downarrow{\scriptstyle \simeq} \\
{}_p\pi_{i+2n}(L\mathcal{L}^n(\mathbb{Z}[m+2] \oplus \mathbb{Z}[m+3]) & \rightarrowtail & L_{i+2n}\mathcal{L}^n(\mathbb{Z}/p^l, m+2) & \longrightarrow & {}_pH_{i+2n}\mathcal{L}^n(n+1, n; p^l)
\end{array}$$

for a prime p and $l \geq 1$. The righthand map is an isomorphism by Proposition 7.2 (it can be shown straightforwardly that the maps in Proposition 7.2 are induced by the pension maps). Now we see that the middle map ϵ_n is injective and the needed inductive step follows from Corollary 11.1. □

12. Prime Lie powers

12.1. Recall the definition of the graded functor

$$\Gamma_* = \bigoplus_{n \geq 0} \Gamma_n : \mathsf{Ab} \to \mathsf{Ab}.$$

The graded abelian group $\Gamma_*(A)$ is generated by symbols $\gamma_i(x)$ of degree $i \geq 0$ satisfying the following relations for all $x, y \in A$:

1) $\gamma_0(x) = 1$
2) $\gamma_1(x) = x$
3) $\gamma_s(x)\gamma_t(x) = \binom{s+t}{s}\gamma_{s+t}(x)$
4) $\gamma_n(x+y) = \sum_{s+t=n} \gamma_s(x)\gamma_t(y),\ n \geq 1$
5) $\gamma_n(-x) = (-1)^n\gamma_n(x),\ n \geq 1.$

For $n \geq 2$, define the functor

$$\widetilde{\Gamma}_n(A) : \mathsf{Ab} \to \mathsf{Ab}$$

by setting

$$\widetilde{\Gamma}_n(A) := im\{A \otimes \Gamma_{n-1}(A) \xrightarrow{l_n} \Gamma_n(A)\}$$

where l_n is the natural map. For example, one has

$$\widetilde{\Gamma}_2(A) = SP^2(A)$$

and a natural exact sequence

$$0 \to \mathcal{L}^3(A) \to \Gamma_2(A) \otimes A \to \widetilde{\Gamma}_3(A) \to 0$$

For a prime p and $k \geq 1$, one has the natural exact sequence

$$0 \to \widetilde{\Gamma}_{p^k}(A) \to \Gamma_{p^k}(A) \to A \otimes \mathbb{Z}/p \to 0$$

This implies that, for a complex $C \in \mathsf{DAb}_{\leq 0}$, one has a triangle

$$L\widetilde{\Gamma}_{p^k}(C) \to L\Gamma_{p^k}(C) \to C \overset{L}{\otimes} \mathbb{Z}/p \to L\widetilde{\Gamma}_{p^k}(C)[1] \tag{12.1}$$

Let A be an abelian group. For $n \geq 1$, let $C^n_*(A)$ be the complex of abelian groups defined by

$$C^n_i(A) = \Lambda^i(A) \otimes \Gamma_{n-i}(A),\ 0 \leq i \leq n,$$

where the differentials $d_i : C^n_i(A) \to C^n_{i-1}(A)$ are:

$$d_i(b_1 \wedge \dots \wedge b_i \otimes X) = \sum_{k=1}^{i} (-1)^k b_1 \wedge \dots \wedge \hat{b}_k \wedge \dots \wedge b_i \otimes b_k X$$

for any $X \in \Gamma_{n-i}(A)$. The complexes $C^n(A)$ are called dual de Rham complexes, they were considered in [10].

Theorem 12.1. *For a prime p and $C \in \mathsf{DAb}_{\leq 0}$, there are natural isomorphisms*

$$\pi_i(L\Gamma_p(C[1])) \simeq \pi_i(L\widetilde{\Gamma}_p(C[1])) \oplus \pi_i\left(C \overset{L}{\otimes} \mathbb{Z}/p[1]\right) \tag{12.2}$$

for all $i \geq 0$.

The p-torsion terms $C \overset{L}{\otimes} \mathbb{Z}/p$ in theorem are basic tools for the general p-torsion terms in the derived functors of Lie and super-Lie functors from the functorial point of view.

Lemma 12.1. *For $C \in \mathsf{DAb}_{\leq 0}$, such that $H_0(C) = 0$, one has $\pi_1(L\widetilde{\Gamma}_p(C)) = 0$. If $H_i(C) = 0$ for $i \leq m$ $(m \geq 1)$, then*

$$\pi_i(L\widetilde{\Gamma}_p(C)) = 0, \quad i \leq m+2 \tag{12.3}$$

Proof. For $p = 2$ this follows from (Satz 12.1 [7]). Since, for a free abelian group A, $H_iC^p(A) = 0,\ i > 0$ (see [8], [10]), there is a natural exact sequence (a truncated part of the dual de Rham complex)

$$0 \to \Lambda^p(A) \to \Lambda^{p-1}(A) \otimes A \to \dots \to \Lambda^2(A) \otimes \Gamma_{p-1}(A) \to \widetilde{\Gamma}_p(A) \to 0 \tag{12.4}$$

Observe that $\pi_1(L\Lambda^n(C)) = 0,\ n \geq 2,\ \pi_1\left(C \overset{L}{\otimes} L\Gamma_{p-1}(C)\right) = 0$ for $C \in \mathsf{DAb}_{\leq 0}$, such that $H_0(C) = 0$, and the result follows from the exactness of the sequence (12.4). In the same way one can get (12.3) starting with a complex C with $H_i(C) = 0,\ i \leq m$, just observing that, for $n \geq 2$, $\pi_i(L\Lambda^n(C)) = 0,\ i < m+1$. □

Lemma 12.2. *For every prime p and $C \in \mathsf{DAb}_{\leq 0}$, the suspension homomorphism*

$$\pi_1(L\widetilde{\Gamma}_p(C)) \to \pi_2(L\widetilde{\Gamma}_p(C[1]))$$

is the zero map.

Proof. First consider the case $p=2$. We have the following natural diagram

$$\begin{array}{ccc} \pi_1(LSP^2(C)) & \xrightarrow{\simeq} & L_1SP^2(H_0(C)) \\ \downarrow{\scriptstyle \mathrm{susp}} & & \downarrow \\ \pi_2(LSP^2(C[1])) & \xrightarrow{\simeq} & \Lambda^2(H_0(C)) \end{array} \tag{12.5}$$

The right-hand vertical map is zero by (Corollary 6.6, [7]). Another way to see why this map is trivial is to write the cross-effect spectral sequence for $\pi_*(LSP^2(C[1])$ from [7]. The first page of this spectral sequence implies that there is an exact sequence

$$0\to L_1\Lambda^2(H_0(C))\to Tor(H_0(C),H_0(C))\to L_1SP^2(H_0(C))\to \\ \Lambda^2(H_0(C))\to H_0(C)\otimes H_0(C)\to SP^2(H_0(C))\to 0$$

where the middle map is exactly the map from (12.5) and it is zero map since the natural transformation $\Lambda^2(H_0(C))\to H_0(C)\otimes H_0(C)$ is injective.

For $p=3$, $\widetilde{\Gamma}_3=\mathcal{L}^3$ (see [3]) and one has $\pi_2(L\widetilde{\Gamma}_3(C[1]))=0$. Now consider the case $p>3$. In this case,

$$\pi_2\left(C[1]\overset{L}{\otimes}L\Gamma_{p-1}(C[1])\right)=0,$$

hence

$$\pi_2(L\widetilde{\Gamma}_p(C[1]))\simeq\pi_1(LE^p(C[1])).$$

Since $H_i(C^p(A))=0,\ i>0$ for every free abelian group A, and $L_i\Lambda^n(C[1])=0,\ i\le n$, we conclude that

$$\pi_1(LE^p(C[1]))=0$$

and hence the result. $\square$

Proof of theorem 12.1 The proof is by induction on i. Lemma 12.1 implies that there is a natural isomorphism

$$\pi_1(L\Gamma_p(C[1]))\simeq\pi_1(C\overset{L}{\otimes}\mathbb{Z}/p[1])$$

which is induced by the map $L\Gamma_p(C[1])\to C\overset{L}{\otimes}\mathbb{Z}/p[1]$ from (12.1).

Consider separately the case $i=2$. The needed statement follows from the suspension diagram

$$\begin{array}{ccccccc} \pi_3\left(C\overset{L}{\otimes}\mathbb{Z}/p[1]\right) & \longrightarrow & \pi_2(L\widetilde{\Gamma}_{p^k}(C[1])) & \longrightarrow & \pi_2(L\Gamma_{p^k}(C[1])) & \twoheadrightarrow & \pi_2\left(C\overset{L}{\otimes}\mathbb{Z}/p[1]\right) \\ \| & & \uparrow{\scriptstyle 0} & & \uparrow & \nwarrow & \| \\ \pi_2\left(C\overset{L}{\otimes}\mathbb{Z}/p\right) & \longrightarrow & \pi_1(L\widetilde{\Gamma}_{p^k}(C)) & \longrightarrow & \pi_1(L\Gamma_p(C)) & \twoheadrightarrow & \pi_1\left(C\overset{L}{\otimes}\mathbb{Z}/p\right) \end{array}$$

where the left hand vertical homomorphism is zero by lemma 12.2.

Now assume by induction, that for all $i \leq j$ (for some $j \geq 2$), there is a natural isomorphism (12.2), which is induced by (12.1). Presenting the complex C as $\dots \to C_i \overset{\partial_i}{\to} C_{i+1} \to \dots$, consider the subcomplex $Z_j(C)$ of C defined as

$$\begin{aligned}(Z_j(C))_i &= C_i,\ i \geq j-1,\\ (Z_j(C))_{j-2} &= im(\partial_{i-1}),\\ (Z_j(C))_i &= 0,\ i < j-2\end{aligned}$$

The complex $Z_j(C)$ has the following properties:

1) the natural map $Z_j(C) \to C$ induces isomorphisms

$$\pi_i\left(Z_j(C) \overset{L}{\otimes} \mathbb{Z}/p\right) \simeq \pi_i\left(C \overset{L}{\otimes} \mathbb{Z}/p\right),\ i \geq j;$$

2) $H_i(Z_j(C)) = 0,\ i \leq j-2.$

Consider the natural diagram

$$\begin{array}{ccccccc} \pi_{j+2}\left(C \overset{L}{\otimes} \mathbb{Z}/p[1]\right) & \longrightarrow & \pi_{j+1}(L\widetilde{\Gamma}_p(C[1])) & \longrightarrow & \pi_{j+1}(L\Gamma_p(C[1])) & \twoheadrightarrow & \pi_{j+1}\left(C \overset{L}{\otimes} \mathbb{Z}/p[1]\right) \\ \| & & \uparrow & & \uparrow & & \| \\ \pi_{j+2}\left(Z_j(C) \overset{L}{\otimes} \mathbb{Z}/p[1]\right) & \longrightarrow & \pi_{j+1}(L\widetilde{\Gamma}_p(Z_j(C)[1])) & \longrightarrow & \pi_{j+1}(L\Gamma_p(Z_j(C)[1])) & \twoheadrightarrow & \pi_{j+1}\left(Z_j(C) \overset{L}{\otimes} \mathbb{Z}/p[1]\right) \end{array} \tag{12.6}$$

Lemma 12.1 implies that $\pi_{j+1}(L\widetilde{\Gamma}_p(Z_j(C)[1])) = 0$. The needed splitting now follows from diagram (12.6). The inductive step is complete and the splitting (12.2) proved for all i. □

Proposition 12.1. *The sequence*

$$LSP^2(C[1]) \to L\Gamma_2(C[1]) \to C \overset{L}{\otimes} \mathbb{Z}/2[1]$$

does not split in the category $\mathsf{DAb}_{\leq 0}$.

Proof. We will prove the statement for the simplest case, when C is a free abelian group. Suppose that $L\Gamma_2(C[1]) \simeq LSP^2(C[1]) \oplus C \overset{L}{\otimes} \mathbb{Z}/2[1]$. Then

$$\pi_2\left(L\Gamma_2(C[1]) \overset{L}{\otimes} \mathbb{Z}/2\right) \simeq \pi_2\left(LSP^2(C[1]) \overset{L}{\otimes} \mathbb{Z}/2 \oplus C \overset{L}{\otimes} \mathbb{Z}/2 \overset{L}{\otimes} \mathbb{Z}/2[1]\right) \simeq \\ \Lambda^2(C) \otimes \mathbb{Z}/2 \oplus C \otimes \mathbb{Z}/2$$

However, $L\Gamma_2(C[1])$ can be presented as complex

$$(C \otimes C \otimes \mathbb{Z}/2 \to \Gamma_2(C) \otimes \mathbb{Z}/2)[1]$$

in the category $\mathsf{DAb}_{\leq 0}$ and

$$\pi_2\left(L\Gamma_2(C[1]) \overset{L}{\otimes} \mathbb{Z}/2\right) = \ker\{C \otimes C \otimes \mathbb{Z}/2 \to \Gamma_2(C) \otimes \mathbb{Z}/2\}.$$

Any natural transformation $C \otimes \mathbb{Z}/2 \to C \otimes C \otimes \mathbb{Z}/2$ is zero. Hence, the assumed splitting not possible. □

For every $m \geq 2$, there is the following décalage isomorphism in the derived category

$$L\Gamma_m(C)[2m] \simeq L\Lambda^m(C[1])[m] \simeq LSP^m(C[2])$$

which implies that the isomorphisms (12.2) can be written in the following form. For a prime p and $C \in \mathsf{DAb}_{\leq 0}$, there are natural isomorphisms

$$\pi_i(L\Lambda^p(C[2])) \simeq \pi_i(L\widetilde{\Gamma}_p(C[1])[p]) \oplus \pi_i\left(C \overset{L}{\otimes} \mathbb{Z}/p[p+1]\right) \tag{12.7}$$

for all $i \geq 0$.

Recall that, for a free abelian group A, there is the following long exact sequence which is called Koszul complex:

$$0 \to \Lambda^p(A) \to \Lambda^{p-1}(A) \otimes A \to \cdots \to A \otimes SP^{p-1}(A) \to SP^p(A) \to 0 \tag{12.8}$$

For $k = 1, \dots, p-1$, define the functor

$$V_{p,k} : \mathsf{Ab} \to \mathsf{Ab}$$

as a kernel of a map in the Koszul complex

$$V_{p,k}(A) = ker\{\Lambda^{p-k}(A) \otimes SP^k(A) \to \Lambda^{p-k+1}(A) \otimes SP^{k+1}\}.$$

Clearly, $V_{p,p-1}(A) = J_p(A),\ V_{p,1}(A) = \Lambda^p(A)$.

Lemma 12.3. *For $k = 1, \dots, p-1$, and $C \in \mathsf{DAb}_{\leq 0}$, there are natural splitting monomorphisms*

$$\pi_i\left(C \overset{L}{\otimes} \mathbb{Z}/p\ [p+k]\right) \hookrightarrow \pi_i(LV_{p,k}(C[2]))) \tag{12.9}$$

for all $i \geq 0$.

Proof. For $k = 1$ the needed splitting monomorphisms are given by (12.7). Assume that we have splitting monomorphisms (12.7) for a fixed $k < p-1$. Since the sequence (12.8) is exact, for $k = 1, \dots, p-1$ and a free abelian A, there is a natural short exact sequence

$$0 \to V_{p,k}(A) \to \Lambda^{p-k}(A) \otimes SP^k(A) \to V_{p,k+1}(A) \to 0$$

Observe that, for $C \in \mathsf{DAb}_{\leq 0}$, such that $H_i(C) = 0,\ i \leq m$,

$$\pi_i\left(L\Lambda^k(C[2]) \overset{L}{\otimes} LSP^{p-k}(C[2])\right) = 0,\ i < 2m + 2p - k + 2.$$

In particular, there are natural isomorphsism

$$\pi_{i+1}(LV_{p,k+1}(C[2])) \simeq \pi_i(LV_{p,k}(C[2])$$

for $i < 2p - k + 2$. This shows that the needed splitting monomorphisms

$$\pi_{i+1}(C \overset{L}{\otimes} \mathbb{Z}/p\ [p+k+1]) \simeq \pi_i\left(C \overset{L}{\otimes} \mathbb{Z}/p\ [p+k]\right) \hookrightarrow \pi_i(LV_{p,k}(C[2])) \simeq \pi_{i+1}(LV_{p,k+1}(C[2]))$$

exist for $i < 2p - k + 2$. For a fixed $j \geq 2$, consider the complex $Z_j(C)$ from the proof of theorem 12.1 with a natural map $Z_j(C) \to C$. One has the following natural diagram

$$\begin{array}{ccccc}
\pi_{j+1}(LV_{p,k+1}(Z_j(C)[2])) & \xrightarrow{\simeq} & \pi_j(LV_{p,k}(Z_j(C)[2])) & \rightleftarrows & \pi_j\left(Z_j(C) \overset{L}{\otimes} \mathbb{Z}/p[p+k]\right) \\
\downarrow & & \downarrow & & \| \\
\pi_{j+1}(LV_{p,k+1}(C[2])) & \longrightarrow & \pi_j(LV_{p,k}(C[2])) & \rightleftarrows & \pi_j\left(C \overset{L}{\otimes} \mathbb{Z}/p[p+k]\right)
\end{array}$$

Here the diagonal maps are natural monomorphisms and epimorphisms (12.9). The needed splitting monomorphism

$$\pi_{j+1}(C \overset{L}{\otimes} \mathbb{Z}/p\ [p+k+1]) \hookrightarrow \pi_{j+1}(LV_{p,k+1}(C[2]))$$

now follows from the above diagram. □

Theorem 12.2. *For a prime p and $C \in \mathsf{DAb}_{\leq 0}$, there are natural splitting monomorphisms*

$$\pi_i\left(C \overset{L}{\otimes} \mathbb{Z}/p\ [2p-1]\right) \hookrightarrow \pi_i(L\mathcal{L}^p(C[2])) \tag{12.10}$$

for all $i \geq 0$.

Proof. Recall that, for a free abelian group A, there is a natural short exact sequence (see 2.12)

$$0 \to \widetilde{J}^p(A) \to \mathcal{L}^p(A) \to J^p(A) \to 0 \tag{12.11}$$

It follows from [6] that the functor $\widetilde{J}^p$ can be decomposed as a sequence of functors of the type $F_1 \otimes \dots \otimes F_k$ for some $k \geq 2$, such that each $F_j,\ i = 1, \dots, k$ is a composition of symmetric powers and functors $J^l,\ l < p$.

Recall that, for $C \in \mathsf{DAb}_{\leq 0}$, such that $H_i(C) = 0,\ i < k$, by [7], Satz 12.1

$$\pi_i(LSP^n(C)) = 0, \begin{cases} \text{for } i < n, \text{ when } k = 1, \\ \text{for } i < k + 2n - 2, \text{ provided } k > 1. \end{cases} \tag{12.12}$$

and

$$\pi_i(LJ^n(C)) = 0, \begin{cases} \text{for } i < n+1, \text{ when } k = 1, \\ \text{for } i < k + 2n - 1, \text{ provided } k > 1. \end{cases} \tag{12.13}$$

Curtis decomposition of the functor $\widetilde{J}^p$ together with (12.12) and (12.13) imply that, for $C \in \mathsf{DAb}_{\leq 0}$, such that $H_i(C) = 0,\ i < k, (k > 1)$

$$\pi_i(L\widetilde{J}^p(C)) = 0,\ i < 2p + 2k - 2.$$

Now we construct the needed splitting monomorphism (12.10) by induction on i. The argument is the same as in the proof of theorem 12.1 and lemma (12.3). For a fixed i, we consider the natural map $Z_i(C) \to C$ and compare the sequences of functors (12.11) for the complexes $Z_i(C)[2]$ and $C[2]$. $\square$

For a prime p, $i, m \geq 1$, and $C \in \mathsf{DAb}_{\leq 0}$ consider the following map

$$\pi_i\left(L\mathcal{L}^m(C) \overset{L}{\otimes} \mathbb{Z}/p\ [2p+2m-3]\right) \to \pi_i\left(L\mathcal{L}^m(C[2])[2p-3] \overset{L}{\otimes} \mathbb{Z}/p\right) \to \pi_i\left(L\mathcal{L}^p \circ \mathcal{L}^m(C[2])\right) \to \pi_i\left(L\mathcal{L}^{pm}(C[2])\right)$$

where the last map is induced by the natural transformation $\mathcal{L}^p \circ \mathcal{L}^m \to \mathcal{L}^{pm}$. Denote this map by $w_{i,p,m}$.

Lemma 12.4. *If $(m,p)=1$ and homology of C are torsion-free, then $w_{i,p,m}$ is a monomorphism for all $i \geq 1$.*

Proof. Consider a prime decomposition of m: $m = p_1^{k_1}\dots p_s^{k_s}$, where p_j are primes and $k_j \geq 0$. We will prove the statement by induction on $k(m) = k_1 + \dots + k_s$. If $k(m) = 0$, then the statement follows from theorem 12.2.

Step 1. First we show that $w_{p,m}$ is injective if $C = \mathbb{Z}[l]$ for some l. The description of the derived functors $L_*\mathcal{L}^m(\mathbb{Z}, l)$ given in section 5 implies that

$$\pi_i\left(L\mathcal{L}^m(\mathbb{Z}, l) \overset{L}{\otimes} \mathbb{Z}/p\right) = 0,\ m \neq 2$$

Only the case which we have to consider here is $m = 2$ and an odd l. In this case

$$L\mathcal{L}^m(\mathbb{Z}, l) \overset{L}{\otimes} \mathbb{Z}/p \simeq \mathbb{Z}/p\ [2l]$$

The map

$$\mathcal{L}^m(\mathbb{Z}, l)[2m] \overset{L}{\otimes} \mathbb{Z}/p \to \mathcal{L}^m(\mathbb{Z}, l+2) \overset{L}{\otimes} \mathbb{Z}/p$$

is an equivalence in $\mathsf{DAb}_{\leq 0}$. Now observe that, by theorem 5.2, the natural map

$$\pi_i\left(L\mathcal{L}^p \circ \Lambda^2(\mathbb{Z}, l+2\right) \simeq L_i\mathcal{L}^p(\mathbb{Z}, 2l+4) \to L_i\mathcal{L}^{2p}(\mathbb{Z}, l+2)$$

is an isomorphism. Therefore, $w_{i,p,m}$ is a monomorphism for $C = \mathbb{Z}[l]$.

Step 2. Now we assume that, for complexes C_1 and C_2, the maps $w_{p,m}$ are injective. Consider the cross-effects of the functors which appear in the definition of $w_{p,m}$. We have

$$\pi_i\left(L\mathcal{L}^m(C_1[2] \mid C_2[2]) \overset{L}{\otimes} \mathbb{Z}/p\right) \simeq \bigoplus_{d|m,\ 1\leq d<m,\ D\in J_{m/d}} \pi_i\left(L\mathcal{L}^d(D_{i_1} \overset{L}{\otimes} \dots D_{i_{m/d}}[\frac{2m}{d}]) \overset{L}{\otimes} \mathbb{Z}/p\right)$$

where $\{D_1, \dots, D_{i_{m/d}}\} = \{C_1, C_2\}$. For every d, and $D \in J_{m/d}$, we have the following commutative diagram

$$\begin{array}{ccc} \pi_i\left(L\mathcal{L}^d(D_1\overset{L}{\otimes}\dots\overset{L}{\otimes}D_{i_{m/d}}[\tfrac{2m}{d}])\overset{L}{\otimes}\mathbb{Z}/p\ [2p-3]\right) & \longrightarrow & \pi_i(L\mathcal{L}^p\circ L^m(C_1[2]\mid C_2[2])) \\ \downarrow{\scriptstyle w_{i,p,d}} & & \downarrow \\ \pi_i\left(L\mathcal{L}^{pd}(D_1\overset{L}{\otimes}\dots\overset{L}{\otimes}D_{i_{m/d}}[\tfrac{2m}{d}])\right) & \rightarrowtail & \pi_i(L\mathcal{L}^{pm}(C_1[2]\mid C_2[2])) \end{array}$$

The map $w_{i,p,d}$ is a monomorphism by induction, therefore, the map $w_{i,p,m}$ induce the monomorphism

$$\pi_i\left(L\mathcal{L}^m(C_1[2]\mid C_2[2])\overset{L}{\otimes}\mathbb{Z}/p\ [2p-3]\right)\hookrightarrow\pi_i\left(L\mathcal{L}^{pm}(C_1[2]\mid C_2[2])\right))$$

Now the statement follows by induction from Step 1 and Step 2, since C is unnaturally equivalent to a direct sum of its homology considered in the corresponding dimensions. □

Proposition 12.2. *Let X be a free abelian simplicial group, then, for a prime p and $n\geq 1$, there are natural isomorphisms*

$$\pi_i(\mathcal{L}^p(\Sigma^{2n}(X))\otimes\mathbb{Z}/p)\simeq\bigoplus_{w\in\overline{\mathcal{V}}^{(p)}_{2n,1}}\pi_i(X\otimes\mathbb{Z}/p\ [2n+d(w)])\tag{12.14}$$

for $i<2np$.

13. Map κ

For an abelian A, $m,n\geq 1$, denote the graded abelian groups

$$\Theta_m(A,2n):=\bigoplus_{\substack{p\text{ prime}\\ p^k|m\\ p^{k+1}\nmid m}}\ \bigoplus_{\substack{i=1,\dots,k\\ w\in\overline{\mathcal{W}}^{(p)}_{\frac{2nm}{p^i},i}}}\mathcal{N}^{\frac{m}{p^i};p}(A)[\frac{2nm}{p^i}+d(w)]$$

$$\Theta_m(A,2n+1):=\bigoplus_{\substack{p\text{ prime}\\ p^k|m\\ p^{k+1}\nmid m}}\ \bigoplus_{\substack{i=1,\dots,k\\ w\in\overline{\mathcal{W}}^{(p)}_{\frac{(2n+1)m}{p^i},i}}}\mathcal{N}_s^{\frac{m}{p^i};p}(A)[\frac{(2n+1)m}{p^i}+d(w)]$$

and their analogs where the indexes are from $\overline{\mathcal{V}}$-sets:

$$\widetilde{\Theta}_m(A,2n):=\bigoplus_{\substack{p\text{ prime}\\ p^k|m\\ p^{k+1}\nmid m}}\ \bigoplus_{\substack{i=1,\dots,k\\ w\in\overline{\mathcal{V}}^{(p)}_{\frac{2nm}{p^i},i}}}\mathcal{N}^{\frac{m}{p^i};p}(A)[\frac{2nm}{p^i}+d(w)]$$

$$\widetilde{\Theta}_m(A,2n+1):=\bigoplus_{\substack{p\text{ prime}\\ p^k|m\\ p^{k+1}\nmid m}}\ \bigoplus_{\substack{i=1,\dots,k\\ w\in\overline{\mathcal{V}}^{(p)}_{\frac{(2n+1)m}{p^i},i}}}\mathcal{N}_s^{\frac{m}{p^i};p}(A)[\frac{(2n+1)m}{p^i}+d(w)]$$

Theorem 13.1. *For a free abelian group A and $m, n \geq 1$, there is a natural isomorphism of graded abelian groups*

$$\bigoplus_{i=1}^{nm-1} L_i\mathcal{L}^m(A,n)[i] \simeq \Theta_m(A,n)$$

Proof. *Case I: n even.* In this case we can write the dimension $2n$ instead of n. First we prove that there is a natural isomorphism of graded abelian groups

$$\bigoplus_{p|m\ p\ prime}\ \bigoplus_{i=1}^{2mn-1} \pi_i(\mathcal{L}^m N^{-1}(A[2n]) \otimes \mathbb{Z}/p)[i] \simeq \widetilde{\Theta}_m(A,2n) \tag{13.1}$$

For a prime p, define $\mathcal{V}^{(p)}_{*,0} = \overline{\mathcal{V}}^{(p)}_{*,0} = \{0\}$ and $d(0)=0$. For $w \in \overline{\mathcal{V}}^{(p)}_{2nd,i}$ $(i \geq 0)$, define the map

$$\kappa_{d,w} : \mathcal{N}^{d;p}(A) \to \pi_{2nd+d(w)}(\mathcal{L}^{dp^i} N^{-1}(A[2n]) \otimes \mathbb{Z}/p)$$

as follows. For $i=0$, this map is a natural isomorphism

$$\kappa_{d,0} : \mathcal{N}^{d;p}(A) \xrightarrow{\simeq} \pi_{2nd}(\mathcal{L}^d N^{-1}(A[2n]) \otimes \mathbb{Z}/p).$$

Suppose the map $\kappa_{d,w}$ is defined for all $w \in \overline{\mathcal{V}}^{(p)}_{2nd,l}$ for $l < i$. Let $w = (\nu_{j_1}, \dots, \nu_{j_i})$ and $v := (\lambda_{j_1}, \dots, \lambda_{j_{i-1}}) \in \overline{\mathcal{V}}^{(p)}_{2nd,i-1}$. Let

$$d(\nu_{j_i}) := \begin{cases} (2p-2)j_i - 1, & \text{if } \nu_{j_i} = \lambda_{j_i} \\ (2p-2)j_i, & \text{if } \nu_{j_i} = \mu_{j_i} \end{cases}$$

Clearly, $d(w) = d(w') + d(\nu_{j_i})$. Let $C := \mathcal{L}^{dp^{i-1}} N^{-1}(A[2n]) \otimes \mathbb{Z}/p\langle 2nd + d(w')\rangle$. Now we define the map $\kappa_{d,w}$ as the following composition map

$$\begin{array}{ccc}
\mathcal{N}^{d;p}(A) & & \\
\downarrow{\scriptstyle \kappa_{d,w'}} & & \\
\pi_{2nd+d(w')}(\mathcal{L}^{dp^{i-1}} N^{-1}(A[2n]) \otimes \mathbb{Z}/p) & \xrightarrow{\simeq} & \pi_{2nd+d(w')}(C) \\
& & \downarrow \\
& & \pi_{2nd+d(w')+d(\nu_{j_i})}(\bigoplus_{w \in \overline{\mathcal{V}}^{(p)}_{2nd+d(w'),1}} C[2nd+d(w)]) \\
& & \downarrow{\scriptstyle \simeq} \\
\pi_{2nd+d(w)}(\mathcal{L}^p \circ \mathcal{L}^{dp^{i-1}} N^{-1}(A[2n]) \otimes \mathbb{Z}/p) & \longleftarrow & \pi_{2nd+d(w)}(\mathcal{L}^p(C)) \\
\downarrow & & \\
\pi_{2nd+d(w)}(\mathcal{L}^{dp^i} N^{-1}(A[2n]) \otimes \mathbb{Z}/p) & &
\end{array}$$

Consider the case $A=\mathbb{Z}$. It follows from theorem 5.5, that it is enough to consider the case $m=p^k$ for a prime p and $k\geq 1$. In this case, we have a map of graded abelian groups

$$\bigoplus_{\substack{j=0,\dots,k-1\\ w\in\overline{\mathcal{V}}^{(p)}_{2np^j,k-j}}}\mathcal{N}^{p^j;p}(\mathbb{Z})[2np^j+d(w)]\simeq\bigoplus_{\substack{j=0,\dots,k-1\\ w\in\overline{\mathcal{V}}^{(p)}_{2np^j,k-j}}}\mathbb{Z}/p\;[2np^j+d(w)]\to$$
$$\pi_*(\mathcal{L}^{p^k}N^{-1}(\mathbb{Z}[2n])\otimes\mathbb{Z}/p)\simeq\bigoplus_{v\in\overline{\mathcal{V}}^{(p)}_{2n,k}}\mathbb{Z}/p\;[2n+d(w)]\quad(13.2)$$

defined as a sum of $\kappa_{p^j;p}$-maps for $j=1,\dots,k$. It follows immediately from the construction of the map $\kappa_{p^j;p}$ that, for $j\geq 1$, the summand $\mathbb{Z}/p$ which corresponds to $w=(w_1,\dots,w_{k-j})\in\overline{\mathcal{V}}^{(p)}_{2np^j,k-j}$, goes to the term $\mathbb{Z}/p$ which corresponds to

$$(\mu_n,\dots,\mu_{p^{j-1}n},w_1,\dots,w_{k-j})\in\mathcal{V}_{2n,k}$$

We have

$$2n+d(\mu_n,\dots,\mu_{p^{j-1}n},w_1,\dots,w_{k-j})=2np^j+d(w_1,\dots,w_{k-j})$$

Therefore, the map (13.2) is an isomorphism.

Now we compare cross-effects. Observe that, for $l|d$, $w\in\overline{\mathcal{V}}^{(p)}_{2nd,i}$, and $C_{i_1}\otimes\dots\otimes C_{i_{d/l}}\in J_{d/l}$, $C_i\in\{A,B\}$ we have the following natural diagram

$$\begin{array}{ccc}\mathcal{N}^{l;p}(C_{i_1}\otimes\dots\otimes C_{i_{d/l}}) & \xrightarrow{\kappa_{l;w}} & \pi_{2nd+d(w)}(\mathcal{L}^{lp^i}N^{-1}(C_{i_1}\otimes\dots\otimes C_{i_{d/l}}[\frac{2nd}{l}])\otimes\mathbb{Z}/p)\\ \downarrow & & \downarrow\\ \mathcal{N}^{d;p}(A\oplus B) & \xrightarrow{\kappa_{d;w}} & \pi_{2nd+d(w)}(\mathcal{L}^{dp^i}N^{-1}(A\oplus B[2n])\otimes\mathbb{Z}/p)\end{array}\quad(13.3)$$

By theorem 10.1, the graded abelian groups

$$\bigoplus_{p|m\ p\ prime}\bigoplus_{i=1}^{2mn-1}\pi_i(\mathcal{L}^mN^{-1}(A[2n])\otimes\mathbb{Z}/p)[i]\quad\text{and}\quad\widetilde{\Theta}_m(A,2n)\quad(13.4)$$

are abstractly isomorphic. The above observations show that the direct sum of maps $\kappa_{d,w}$ defines a natural surjective map from the right hand side of (13.4) to the left hand side. Components in both graded functors in (13.4) are finitely-generated for finitely-generated A and these functors commute with direct limits. Therefore, there is a natural isomorphism of graded abelian groups (13.4).

Observe that, for $i\geq 1$, there is a natural exact sequence

$$0\to L_i\mathcal{L}^m(A,2n)\otimes\mathbb{Z}/p\to\pi_i(\mathcal{L}^mN^{-1}(A[2n])\otimes\mathbb{Z}/p)\to Tor(L_{i-1}\mathcal{L}^m(A,2n),\mathbb{Z}/p)\to 0\quad(13.5)$$

For $i<2nm$, $L_i\mathcal{L}^m(A,2n)\otimes\mathbb{Z}/p$ is naturally isomorphic to the p-torsion component of $L_i\mathcal{L}^m(A,2n)$. This follows from the abstract description of the derived functors $L_i\mathcal{L}^m(A,2n)$ ($i\leq 2mn$): for a free A these are direct sums of $\mathbb{Z}/p$-vector spaces for different primes p.

Now consider the obvious natural embedding

$$p_{m,n}:\Theta_m(A,2n)\subset\widetilde{\Theta}_m(A,2n)\simeq\bigoplus_{p|m\ p\ prime}\bigoplus_{i=1}^{2mn-1}\pi_i(\mathcal{L}^mN^{-1}(A[2n])\otimes\mathbb{Z}/p)[i]$$

induced by the obvious embeddings of the indexed sets in the definition of $\Theta_m(A, 2n)$ and $\widetilde{\Theta}_m(A, 2n)$. Now observe that the image of the left hand map in (13.5) lies in the image of $p_{m,2n}$. This follows from theorem 5.5 and the diagram (13.3). Since the graded abelian groups

$$\bigoplus_{p|m\ p\ prime} \bigoplus_{i=1}^{2mn-1} L_i\mathcal{L}^m(A, 2n) \otimes \mathbb{Z}/p[i] \quad \text{and} \quad \Theta_m(A, 2n)$$

are abstractly isomorphic, they are naturally isomorphic as well, since the left hand map in (13.5) defines a natural monomorphism between them and they are isomorphic direct sums of $\mathbb{Z}/p$-vector spaces.

Case II: n odd. This case is analogous to the Case I. First we prove that the graded abelian groups

$$\bigoplus_{p|m\ p\ prime} \bigoplus_{i=1}^{2mn-1} \pi_i(\mathcal{L}^m N^{-1}(A[2n+1]) \otimes \mathbb{Z}/p)[i] \quad \text{and} \quad \widetilde{\Theta}_m(A, 2n+1) \tag{13.6}$$

are naturally isomorphic. These graded abelian groups are abstractly isomorphic by theorem 10.1, hence, as above, it is enough to construct a natural surjective map between them. For a prime p, $w \in \overline{\mathcal{V}}^{(p)}_{(2n+1)d,i}$, the construction of the map

$$\kappa'_{d,w} : \mathcal{N}^{d;p}_s(A) \to \pi_{(2n+1)d+d(w)}(\mathcal{L}^{dp^i} N^{-1}(A[2n+1]) \otimes \mathbb{Z}/p)$$

is analogous to the construction of the map $\kappa_{d,w}$. Repeating the proof from the Case I, we get a natural isomorphism of the graded abelian groups (13.6). A transition from the homotopy groups of $\mathcal{L}^m N^{-1}(A[2n+1]) \otimes \mathbb{Z}/p$ to $\mathcal{L}^m(A, 2n+1)$ is analogous to the Case I. □

Remark. Another way, how to define a natural map is to consider a composition map

$$\begin{aligned} \mathcal{N}^{d;p}(A) \hookrightarrow \bigoplus_{\mathcal{V}^{(p)}_{2nd,i}} \mathcal{N}^{d;p}(A) = \\ \pi_{2nd}(\mathcal{L}^d N^{-1}(A[2n]) \otimes \mathbb{Z}/p) \otimes \pi_{2nd+d(w)}(\mathcal{L}^{p^i} K(\mathbb{Z}, 2nd) \otimes \mathbb{Z}/p) \to \\ \pi_{2nd+d(w)}(\mathcal{L}^{dp^i} N^{-1}(A[2n]) \otimes \mathbb{Z}/p) \end{aligned} \tag{13.7}$$

Here the first inclusion is induced by the inclusion $\overline{\mathcal{V}}^{(p)}_{2nd,i} \subset \mathcal{V}^{(p)}_{2nd,i}$. However, it is not straightforward from the definition that this map is a homomorphism on A.

14. Homotopy groups

Given a simplicial set K with simply connected geometric realization $|K|$, the Kan loop group construction GX has the following property: there is an equivalence of fiber sequences:

$$\begin{array}{ccccc} |[GK, GK]| & \longrightarrow & |GK| & \longrightarrow & |(GK)_{ab}| \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \Omega\Gamma|K| & \longrightarrow & \Omega|K| & \longrightarrow & \Omega SP^\infty|K| \end{array}$$

In particular, the Hurewicz homomorphism

$$\pi_n(|K|) \to H_n(|K|),\ n \geq 2$$

is given as $\pi_{n-1}(GK) \to \pi_{n-1}((GK)_{ab})$.

The lower central series filtration of GK gives rise to the long exact sequence

$$\dots \to \pi_{i+1}(GK/\gamma_r(GK)) \to \pi_i(\gamma_r(GK)/\gamma_{r+1}(GK)) \to \pi_i(GK/\gamma_{r+1}(GK)) \to \pi_i(GK/\gamma_r(GK)) \to \dots$$

This exact sequence defines a graded exact couple, which gives rise to a natural spectral sequence $E(K)$ with the initial terms

$$E^1_{r,q}(K) = \pi_q(\gamma_r(GK)/\gamma_{r+1}(GK))$$

and differentials

$$d^i_{p,q} : E^i_{r,q}(K) \to E^i_{r+i,\,q-1}(K). \tag{14.1}$$

This spectral sequence $E^i(K)$ converges to $E^\infty(K)$ and $\oplus_r E^\infty_{p,q}$ is the graded group associated to the filtration on $\pi_q(GK) = \pi_{q+1}(|K|)$. The groups $E^1(K)$ are homology invariants of K. By the Magnus-Witt isomorphism, the spectral sequence can be rewritten as

$$E^1_{r,q}(K) = \pi_q(\mathcal{L}^r(\widetilde{\mathbb{Z}}K, -1)) \Longrightarrow \pi_{q+1}(|K|). \tag{14.2}$$

since the abelianization $GK_{\mathrm{ab}} := GK/\gamma_2(GK)$ of GK corresponds to the reduced chains $\widetilde{\mathbb{Z}}K$ on K, with degree shifted by 1. When $K = M(A,n)$, $\widetilde{\mathbb{Z}}K$ corresponds to an Eilenberg-Mac Lane space $K(A,n)$ so that the spectral sequence is simply of the form

$$E^1_{r,q} = L_q\mathcal{L}^r(A, n-1) \Longrightarrow \pi_{q+1}(M(A,n))\,. \tag{14.3}$$

In particular,

$$E^1_{1,q} = \pi_q(K(A, n-1)) = \begin{cases} A, & q = n-1 \\ 0, & q \neq n-1 \end{cases}$$

q	$E^1_{1,q}$	$E^1_{2,q}$	$E^1_{3,q}$	$E^1_{4,q}$	$E^1_{5,q}$	$E^1_{6,q}$	$E^1_{7,q}$	$E^1_{8,q}$	$E^1_{12,q}$	$E^1_{16,q}$	$E^1_{32,q}$
7	0	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2^2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2^4$	$\mathbb{Z}/2$	$\mathbb{Z}/2^7$	$\mathbb{Z}/2^5$
6	0	0	0	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2^4$	0	$\mathbb{Z}/2^4$	$\mathbb{Z}/2$
5	0	0	0	$\mathbb{Z}/2^2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2^3$	0	$\mathbb{Z}/2$	0
4	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2^2$	0	0	0	$\mathbb{Z}/2$	0	0	0
3	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	0	0	0	0	0	0
2	0	$\mathbb{Z}/4$	0	0	0	0	0	0	0	0	0
1	$\mathbb{Z}/2$	0	0	0	0	0	0	0	0	0	0

TABLE 6. The spectral sequence for $\Sigma\mathbb{R}P^2 = M(\mathbb{Z}/2, 2)$

q	$E^1_{1,q}$	$E^1_{2,q}$	$E^1_{3,q}$	$E^1_{4,q}$	$E^1_{5,q}$	$E^1_{6,q}$	$E^1_{7,q}$
10	0	0	0	0	$\mathcal{L}^5(A)$	0	0
9	0	0	0	0	$A\otimes\mathbb{Z}/5$	$\mathcal{L}^3(A)\otimes\mathbb{Z}/2$	0
8	0	0	0	$\mathcal{L}^4(A)$	0	0	0
7	0	0	0	$\Gamma_2(A)\otimes\mathbb{Z}/2$	0	$\Lambda^2(A)\otimes\mathbb{Z}/3\oplus\mathcal{L}^3(A)\otimes\mathbb{Z}/2$	0
6	0	0	$\mathcal{L}^3(A)$	0	0	0	0
5	0	0	$A\otimes\mathbb{Z}/3$	$\Gamma_2(A)\otimes\mathbb{Z}/2$	0	0	0
4	0	$\Lambda^2(A)$	0	$A\otimes\mathbb{Z}/2$	0	0	0
3	0	$A\otimes\mathbb{Z}/2$	0	0	0	0	0
2	A	0	0	0	0	0	0

TABLE 7. The spectral sequence for $M(A,3)$ (A free)

q	$E^1_{8,q}$	$E^1_{9,q}$	$E^1_{12,q}$	$E^1_{16,q}$	$E^1_{24,q}$
10	$\Gamma_2(A)\otimes\mathbb{Z}/2$	0	$\mathcal{L}^3(A)\otimes\mathbb{Z}/2$	$(\Gamma_2(A)^{\oplus 3}\oplus A^{\oplus 2})\otimes\mathbb{Z}/2$ $\oplus\mathcal{N}^{4;2}(A)$	$(\mathcal{L}^3(A)\otimes\mathbb{Z}/2)^{\oplus 2}$
9	$\Gamma_2(A)\otimes\mathbb{Z}/2\oplus\mathcal{N}^{4;2}(A)$	$\mathcal{N}^{3;3}(A)$	$\mathcal{L}^3(A)\otimes\mathbb{Z}/2$	$(\Gamma_2(A)^{\oplus 2}\oplus A^{\oplus 2})\otimes\mathbb{Z}/2$	$\mathcal{L}^3(A)\otimes\mathbb{Z}/2$
8	$(\Gamma_2(A)\oplus A)\otimes\mathbb{Z}/2$	$A\otimes\mathbb{Z}/3$	$\mathcal{L}^3(A)\otimes\mathbb{Z}/2$	$(\Gamma_2(A)^{\oplus 2}\oplus A)\otimes\mathbb{Z}/2$	0
7	$\Gamma_2(A)\otimes\mathbb{Z}/2$	0	0	$(\Gamma_2(A)\oplus A^{\oplus 2})\otimes\mathbb{Z}/2$	0
6	$(\Gamma_2(A)\oplus A)\otimes\mathbb{Z}/2$	0	0	$A\otimes\mathbb{Z}/2$	0
5	$A\otimes\mathbb{Z}/2$	0	0	0	0
4	0	0	0	0	0

q	$E^1_{32,q}$	$E^1_{48,q}$	$E^1_{64,q}$	$E^1_{128,q}$	$E^1_{256,q}$
10	$(\Gamma_2(A)^{\oplus 4}\oplus A^{\oplus 4})\otimes\mathbb{Z}/2$	$\mathcal{L}^3(A)\otimes\mathbb{Z}/2$	$(\Gamma_2(A)^{\oplus 4}\oplus A^{\oplus 6})\otimes\mathbb{Z}/2$	$(\Gamma_2(A)\oplus A^{\oplus 5})\otimes\mathbb{Z}/2$	$A\otimes\mathbb{Z}/2$
9	$(\Gamma_2(A)^{\oplus 3}\oplus A^{\oplus 3})\otimes\mathbb{Z}/2$	0	$(\Gamma_2(A)\oplus A^{\oplus 4})\otimes\mathbb{Z}/2$	$A\otimes\mathbb{Z}/2$	0
8	$(\Gamma_2(A)\oplus A^{\oplus 3})\otimes\mathbb{Z}/2$	0	$A\otimes\mathbb{Z}/2$	0	0
7	$A\otimes\mathbb{Z}/2$	0	0	0	0
6	0	0	0	0	0

14.1. Generalized spectral sequence and bifunctors. Let X be a simply connected simplicial set and Y a finite dimensional simplicial set. There is a spectral sequence with initial term

$$E^1_{r,q}=\bigoplus_i H^i(\Sigma^q Y,\pi_i(\gamma_r(GX)/\gamma_{r+1}(GX)))$$

which depends only on homology of X,Y and converges to the graded group associated with filtration of the group $[\Sigma^{q+1}Y,X]$. Here GX is the Kan loop group of X. The case when Y is a sphere is the classical Curtis spectral sequence, which converges to homotopy groups of X.

Consider a simple space Y from the point of view of homology, say $Y=M(A,1)$. Then

$$H^i(\Sigma^q M(A,1),\pi_i(\gamma_r(GX)/\gamma_{r+1}(GX)))=\begin{cases}Hom(A,\pi_q(\gamma_r(GX),\gamma_{r+1}(GX))), & i=q-1\\ Ext(A,\pi_q(\gamma_r(GX),\gamma_{r+1}(GX))), & i=q\end{cases}$$

Take $X=M(B,2)$ for an abelian group B. Then

$$H^i(\Sigma^q M(A,1),\pi_i(\gamma_r(GX)/\gamma_{r+1}(GX)))=\begin{cases}Hom(A,L_q\mathcal{L}^r(B,1)) & i=q-1\\ Ext(A,L_q\mathcal{L}^r(B,1)) & i=q\end{cases}$$

k	${}_3\pi_{k+3}S^3$	${}_3\pi_{k+4}S^4$	${}_3\pi_{k+5}S^5$
12	0	0	0
11	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/9$
10	$\mathbb{Z}/3$	$\mathbb{Z}/3\oplus\mathbb{Z}/3$	$\mathbb{Z}/9$
9	0	0	0
8	0	0	0
7	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$
6	$\mathbb{Z}/3$	$\mathbb{Z}/3\oplus\mathbb{Z}/3$	0
5	0	0	0
4	0	0	0
3	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$
2	0	0	0
1	0	0	0

TABLE 8. 3-torsion in $\pi_{n+k}S^n$, $n=3,4,5$

k	${}_3\pi_k M(A,3)$	${}_3\pi_k M(A,4)$	${}_3\pi_k M(A,5)$
12	0	$\mathcal{L}^4(A)\otimes\mathbb{Z}/3$	0
11	$\mathcal{N}^{3;3}(A)\oplus\mathcal{L}^5(A)\otimes\mathbb{Z}/3$	$A\otimes\mathbb{Z}/3$	$\Lambda^2(A)\otimes\mathbb{Z}/3\oplus A\otimes\mathbb{Z}/9$
10	$A\otimes\mathbb{Z}/3$	$(\Gamma_2(A)\oplus A)\otimes\mathbb{Z}/3$	$A\otimes\mathbb{Z}/9$
9	$(\mathcal{L}^4(A)\oplus\Lambda^2(A))\otimes\mathbb{Z}/3$	$\mathcal{L}^3_s(A)\otimes\mathbb{Z}/3$	0
8	0	0	0
7	$\mathcal{N}^{3;3}(A)$	$A\otimes\mathbb{Z}/3$	$(\Lambda^2(A)\oplus A)\otimes\mathbb{Z}/3$
6	$A\otimes\mathbb{Z}/3$	$(\Gamma_2(A)\oplus A)\otimes\mathbb{Z}/3$	0
5	$\Lambda^2(A)\otimes\mathbb{Z}/3$	0	0
4	0	0	0
3	$A\otimes\mathbb{Z}/3$	$A\otimes\mathbb{Z}/3$	$A\otimes\mathbb{Z}/3$
2	0	0	0
1	0	0	0

TABLE 9. 3-torsion in $\pi_{n+k}M(A,n)$ for A free

The initial terms are the following:

q	$E^1_{1,q}$	$E^1_{2,q}$	$E^1_{3,q}$	$E^1_{4,q}$
4	0	0	$Hom(A,L_1\mathcal{L}^3_s(A))$	$Hom(A,L_4\mathcal{L}^4(B,1))$
3	0	$Hom(A,L_1\Gamma_2(B))$	$Hom(A,\mathcal{L}^3_s(B))$	$Ext(A,L_4\mathcal{L}^4(B,1)$
2	0	$Hom(A,\Gamma_2(B))$	$Ext(A,\mathcal{L}^3_s(B))$	0
1	$Hom(A,B)$	$Ext(A,\Gamma_2(B))$	0	0

We have immediately the sequence of Barratt:

$$0\to Ext(A,\Gamma_2(B))\to[M(A,2),M(B,2)]\to Hom(A,B)\to 0$$

which is not split. At the next step, we have an exact sequence

$$[M(A,4),M(B,2)]\to Hom(A,L_1\Gamma_2(B))\to Ext(A,\mathcal{L}^3_s(B))\to$$
$$[M(A,3),M(B,2)]\to Hom(A,\Gamma_2(B))\to 0$$

15. Appendix A: Tables of derived functors

1. Derived functors $L_i\mathcal{L}^4(A,2), L_i\mathcal{L}^8(A,2)$ for A free, together with sets of allowable words:

	$L_i\mathcal{L}^4(A,2)$	$\mathcal{W}_{2,2}$	$L_i\mathcal{L}^8(A,2)$	$\mathcal{W}_{2,3}$
16	0	0	$\mathcal{L}^8$	0
15	0	0	$\mathcal{N}^{4;2}$	$(2,4,7)$
14	0	0	0	0
13	0	0	$\mathcal{N}^{4;2}$	$(2,4,5)$
12	0	0	$\Gamma_2\otimes\mathbb{Z}/2$	$(2,3,5)$
11	0	0	$\mathcal{N}^{4;2}$	$(2,4,3)$
10	0	0	$\Gamma_2\otimes\mathbb{Z}/2$	$(2,3,3)$
9	0	0	$\Gamma_2\otimes\mathbb{Z}/2\oplus\mathcal{N}^{4;2}$	$(2,2,3),(2,4,1)$
8	$\mathcal{L}^4$	0	$\Gamma_2\otimes\mathbb{Z}/2\oplus A\otimes\mathbb{Z}/2$	$(2,3,1),(1,2,3)$
7	$\Gamma_2\otimes\mathbb{Z}/2$	$(2,3)$	$\Gamma_2\otimes\mathbb{Z}/2$	$(2,2,1)$
6	0	0	$\Gamma_2\otimes\mathbb{Z}/2\oplus A\otimes\mathbb{Z}/2$	$(2,1,1),(1,2,1)$
5	$\Gamma_2\otimes\mathbb{Z}/2$	$(2,1)$	$A\otimes\mathbb{Z}/2$	$(1,1,1)$
i=4	$A\otimes\mathbb{Z}/2$	$(1,1)$	0	0

2. 3-torsion in derived functors $L_i\mathcal{L}^n(A,2)$, for $i\leq 21, n\leq 27$ for A free:

	$n=6$	9	12	15	18	21	24	27
21	0	0	0	$\mathcal{L}^5\otimes\mathbb{Z}/3$	0	$\mathcal{L}^7\otimes\mathbb{Z}/3$	0	$\mathcal{N}^{9;3}\oplus(A/3A)^{\oplus 2}$
20	0	0	0	0	0	0	0	$\mathcal{N}^{3;3\oplus 3}\oplus A/3A$
19	0	0	$\mathcal{L}^4\otimes\mathbb{Z}/3$	0	$\mathcal{N}^{6;3}\oplus\Lambda^2\otimes\mathbb{Z}/3$	0	$\mathcal{L}^8\otimes\mathbb{Z}/3$	$A/3A$
18	0	0	0	0	$\Lambda^2\otimes\mathbb{Z}/3$	0	0	0
17	0	$\mathcal{N}^{3;3}$	0	$\mathcal{L}^5\otimes\mathbb{Z}/3$	0	$\mathcal{L}^7\otimes\mathbb{Z}/3$	0	$\mathcal{N}^{3;3\oplus 2}$
16	0	0	0	0	0	0	0	$\mathcal{N}^{3;3\oplus 2}\oplus(A/3A)^{\oplus 2}$
15	0	0	$\mathcal{L}^4\otimes\mathbb{Z}/3$	0	$\mathcal{N}^{6;3}\oplus\Lambda^2\otimes\mathbb{Z}/3$	0	0	$(A/3A)^{\oplus 2}$
14	0	0	0	0	$(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 2}$	0	0	0
13	0	$\mathcal{N}^{3;3}$	0	$\mathcal{L}^5\otimes\mathbb{Z}/3$	0	0	0	$\mathcal{N}^{3;3}$
12	0	$A/3A$	0	0	0	0	0	$\mathcal{N}^{3;3}\oplus A/3A$
11	$\Lambda^2\otimes\mathbb{Z}/3$	0	$\mathcal{L}^4\otimes\mathbb{Z}/3$	0	$\Lambda^2\otimes\mathbb{Z}/3$	0	0	$A/3A$
10	0	0	0	0	$\Lambda^2\otimes\mathbb{Z}/3$	0	0	0
9	0	$\mathcal{N}^{3;3}$	0	0	0	0	0	0
8	0	$A/3A$	0	0	0	0	0	0
7	$\Lambda^2\otimes\mathbb{Z}/3$	0	0	0	0	0	0	0

3. 3-torsion in derived functors $L_i\mathcal{L}^n(A,2)$, for $i \le 21, n \le 135$:

	$n=36$	45	54	63	81	108	135
21	0	$(\mathcal{L}^5\otimes\mathbb{Z}/3)^{\oplus 2}$	$(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 3}$	$\mathcal{L}^7\otimes\mathbb{Z}/3$	$\mathcal{N}^{3;3\oplus 3}$	$(\mathcal{L}^4\otimes\mathbb{Z}/3)^{\oplus 3}$	$\mathcal{L}^5\otimes\mathbb{Z}/3$
20	0	$(\mathcal{L}^5\otimes\mathbb{Z}/3)^{\oplus 2}$	0	$\mathcal{L}^7\otimes\mathbb{Z}/3$	$\mathcal{N}^{3;3\oplus 6}\oplus(A/3A)^{\oplus 3}$	0	$(\mathcal{L}^5\otimes\mathbb{Z}/3)^{\oplus 2}$
19	$(\mathcal{L}^4\otimes\mathbb{Z}/3)^{\oplus 2}$	0	$\mathcal{N}^{6;3}\oplus(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 2}$	0	$\mathcal{N}^{3;3\oplus 4}\oplus(A/3A)^{\oplus 5}$	$\mathcal{L}^4\otimes\mathbb{Z}/3$	$\mathcal{L}^5\otimes\mathbb{Z}/3$
18	$(\mathcal{L}^4\otimes\mathbb{Z}/3)^{\oplus 2}$	0	$\mathcal{N}^{6;3}\oplus(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 5}$	0	$(A/3A)^{\oplus 3}$	$(\mathcal{L}^4\otimes\mathbb{Z}/3)^{\oplus 2}$	0
17	0	$\mathcal{L}^5\otimes\mathbb{Z}/3$	$(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 3}$	0	$\mathcal{N}^{3;3}$	$\mathcal{L}^4\otimes\mathbb{Z}/3$	0
16	0	$\mathcal{L}^5\otimes\mathbb{Z}/3$	0	0	$\mathcal{N}^{3;3}\oplus 2\oplus A/3A$	0	0
15	$\mathcal{L}^4\otimes\mathbb{Z}/3$	0	$\Lambda^2\otimes\mathbb{Z}/3$	0	$\mathcal{N}^{3;3}\oplus(A/3A)^{\oplus 2}$	0	0
14	$\mathcal{L}^4\otimes\mathbb{Z}/3$	0	$(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 2}$	0	$A/3A$	0	0
13	0	0	$\Lambda^2\otimes\mathbb{Z}/3$	0	0	0	0

4. 3-torsion in derived functors $L_i\mathcal{L}^n(A,2)$, for $i \le 21, n > 135$:

	$n=162$	243	324	486	729
21	$\mathcal{N}^{6;3}\oplus(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 11}$	$\mathcal{N}^{3;3}\oplus(A/3A)^{\oplus 4}$	$(\mathcal{L}^4\otimes\mathbb{Z}/3)^{\oplus 3}$	$(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 6}$	$\mathcal{N}^{3;3}\oplus(A/3A)^{\oplus 4}$
20	$(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 4}$	$\mathcal{N}^{3;3\oplus 3}\oplus A/3A$	$\mathcal{L}^4\otimes\mathbb{Z}/3$	$(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 4}$	$A/3A$
19	$\Lambda^2\otimes\mathbb{Z}/3$	$\mathcal{N}^{3;3\oplus 3}\oplus(A/3A)^{\oplus 3}$	0	$\Lambda^2\otimes\mathbb{Z}/3$	0
18	$(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 3}$	$\mathcal{N}^{3;3}\oplus(A/3A)^{\oplus 3}$	0	0	0
17	$(\Lambda^2\otimes\mathbb{Z}/3)^{\oplus 3}$	$A/3A$	0	0	0
16	$\Lambda^2\otimes\mathbb{Z}/3$	0	0	0	0

References

[1] A. K. Bousfield: Homogeneous functors and their derived functors, preprint.
[2] A. K. Bousfield: Operations on derived functors of non-additive functors, preprint.
[3] L. Breen and R. Mikhailov: Derived functors of non-additive functors and homotopy theory, *Alg. Geom. Top.* **11** (2011), 327-415; arXiv: 0910.2817.
[4] L. Breen and R. Mikhailov: Derived functors of divided powers, preprint.
[5] E. B. Curtis: Lower central series of semi-simplicial complexes, *Topology* **2** (1963), 159–171.
[6] E. B. Curtis: Some relations between homotopy and homology. *Ann. of Math.* **82** (1965), 386–413.
[7] A. Dold and D. Puppe: Homologie nicht-additiver Funtoren; Anwendugen. *Ann. Inst. Fourier* **11** (1961) 201–312.
[8] V. Franjou: Cohomologie de de Rham entière, preprint arXiv:math/0404123.
[9] W. Fulton and J. Harris: Representation theory, A first course, *Graduate Texts in Mathematics* **129**, Springer-Verlag (1991).
[10] F. Jean: Foncteurs dérivés de l'algébre symétrique: Application au calcul de certains groupes d'homologie fonctorielle des espaces $K(B,n)$, Doctoral thesis, University of Paris 13, 2002, available at:
http://www.maths.abdn.ac.uk/ bensondj/html/archive/jean.html
[11] D. Kan: Handwritten manuskript (1965).
[12] B. Köck: Computing the homology of Koszul complexes, *Trans. Amer. Math. Soc.* **353** (2001), 3115-3147.

[13] D. Leibowitz: The E^1 term of the lower central series spectral sequence for the homotopy of spaces, Brandeis University Ph.D. thesis (1972).
[14] J. Schlesinger: The semi-simplicial free Lie ring, *Trans. Amer. Math. Soc.* **122** (1966), 436-442.
[15] M. Tangora: Level number sequences of trees and Lambda algebra, *Eur. J. Combin.* **12** (1991), 433-443.