Morin maps and their characteristic classes

M. Kazarian*
kazarian@mccme.ru

Abstract

Enumerative theory of 'curvilinear projective geometry' is traditionally studied in the framework of S. Kleiman's theory of multiple points. Motivated by our general multisingularity theory, we undertake in this paper a deep revision of Kleiman's theory. The theory is enhanced in many directions: relations between relative Chern classes are described; a generalized iteration principle is formulated suitable for the study of multisingularities; this principle is justified for general corank one maps; a generalized residue intersection formula is formulated; and most important, closed formulas for the classes of multisingularity cycles of corank one maps are derived. Numerous applications to enumerative projective geometry are discussed.

Contents

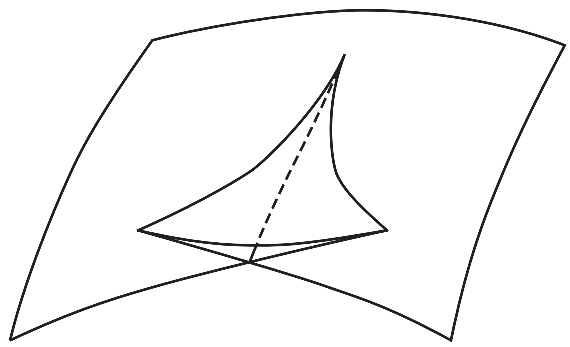
*Steklov Mathematical Institute RAS and the Poncelet Laboratory, Independent University of Moscow, partially supported by RFBR grant no. 04-01-00762

1 Introduction

A *corank one map* is a map of manifolds whose kernel derivative rank is at most 1 at any point. A typical example is the Whitney pleat $\mathbb{C}^3 \to \mathbb{C}^3$ given in coordinates by

$$(x, a, b) \mapsto (x^4 + a\,x^2 + b\,x, a, b).$$

The critical values of this map form the famous *swallowtail* singularity



Corank one singularities possess nice properties in comparison with singularities of other types: a complete classification is known (there is a single series A_k of singularities, indexed by an integer k) and all of them are stable and simple (there are no moduli in the normal form). The existence of explicit normal forms allows one a detailed topological study from both local and global viewpoint providing a nice model for the study of general singularities, see [1], [2], [4], [9], [10], [24], [27], [28], [33], [34], [35].

Points with prescribed local singularity type form a cycle in the source manifold. Its dual cohomology class is expressed as a universal polynomial combination (called *Thom* *polynomial*) of the characteristic classes of manifolds participating in the mapping [36]. Computing Thom polynomials is one of the most intriguing problems in global singularity theory. Many authors attacked this problem for the simplest possible singularity type A_k. Summarizing the experience one can affirm now that, among all singularities of the same codimension, A_k-singularities have the most complicated expressions for Thom polynomials and there is no hope to obtain an explicit closed formula for Thom polynomials of A_k singularities valid for all k, see [30], [16], [17].

A closed formula can be obtained, however, if we restrict ourselves to the case of maps admitting no singularities more complicated than those of corank one. Such an expression obtained by Porteous [28] was one of the first explicit computations in the theory of Thom polynomials. Other expressions have been obtained later by different authors [1], [9], [24]. These expressions do not always coincide. This is not surprising: the difference can always be represented by a class supported on the locus of singularities of corank at least two. Therefore, the difference vanishes for maps of corank one.

The theory of Thom polynomials provides a unified approach to many enumerative problems in projective algebraic geometry. Indeed, these problems often can be reformulated as problems of counting singularities of certain maps of algebraic varieties. In most of the enumerative problems one needs, however, an extension of Thom polynomial theory to the case of *multisingularities*. The multisingularity theory has a long and rich history, see [19], [22]. The classical formulas of Plücker, Salmon, de Jonquières are among the first examples of this theory. The modern intersection theory providing a rigorous background and new methods to this theory led to the *multiple point theory* developed by Kleiman in [20, 21], see also Colley [5, 6], and Katz [12]. This theory can be applied if the map has no singularities of corank greater than one (or if these singularities can be ignored by dimensional reasons). Two basic approaches of this theory, namely, the *iteration principle* and the *method of Hilbert schemes*, meet serious difficulties in the presence of more complicated singularities. However, even for corank one maps this theory encountered a number of important applications [5], [6], [29].

Motivated by quite different topological ideas of cobordism theory I formulated in [16] a general expression for multisingularity classes that serves as an analogue of Thom polynomial in the case of multisingularities. The topological proof of this result in [16] is quite sketchy. A technique necessary to fill up the details is well developed by Vassiliev (in his treatment of the Smale-Hirch principle [37]) and Szücs (in his theory of cobordisms of maps with prescribed singularities [31], [35]). However, I regard topological approach mostly as a motivation; I believe that the final version of the theory should be formulated in the framework of intersection theory. Such a proof is still expected. An algebro-geometric proof of the existence of Thom polynomials for local singularities is outlined in [17].

The original goal of this paper was to verify to what extent the general cohomological multisingularity theory developed in [16] agrees with Kleiman's theory of multiple points. The knowledge of the final answer resulted in a considerable revision of Kleiman's theory. Here are the principal results of the paper:

- a complete description of relations between relative Chern classes of corank one maps (Proposition 3.1);
- explicit closed formulas for the multisingularity classes (Theorems 2.6 and 2.7);

- formulation of the generalized iteration principle suitable for the study of multisingularities; its justification for maps of corank one (Theorems 2.11 and 4.3);
- a version of the residue intersection formula applicable to the generalized iteration principle (Theorems 2.12 and 2.16);
- a refined integral recursive formula for the classes of multisingularities (Theorem 6.1).

Besides, I dare hope that my presentation should help non-specialists in intersection theory to clear up the residual intersections technique used in computations. As a consequence, the presented paper provides an intersection theory justification of the general multisingularity theory of [16] in the particular case of corank one maps.

Main results of the paper are formulated in Sect. 2. Sections 3–6 are devoted to the detailed exposition and the proofs.

A number of enumerative problems that can be reduced to the study of multisingularity classes of corank one maps are considered in the last Sect. 7 (these are problems of *curvilinear geometry* in the terminology of [29]). Most of problems under discussion are classical and well studied. I present a new point of view at the known results rather than new ones. It is shown that most of these results could be considered as specializations of my universal formulas.

Many authors noticed relations between various problems of curvilinear geometry by reducing one to another. The point is, however, that the very fact of the existence of a universal answer (even if it is not known explicitly) provides a new tool: it implies that information obtained in one of the problems can be used in another even if the problems themselves cannot be reduced to each other directly! For example, we shall see that the answer to the de Jonquieres' problem is contained in a more simple Hilbert's formula. Besides, the universality principle implies that the same de Jonqieres' formula describes classes of certain natural subvarieties in the symmetric powers of a complex curve of arbitrary genus, namely, the subvarieties defined by prescribed multiplicities of points in the divisors parameterizing the symmetric power. Similarly, the computation of Thom polynomials for the 'coincidence root loci' in [7] is equivalent to the classical problem of counting multisecant lines to a hypersurface in projective space [5], [25] and also covered by the main formulas of the present paper.

A more involved application of the general multisingularity theory to the study of moduli spaces of meromorphic functions is initiated in [18].

To keep the exposition more geometric we prefer to use complex varieties and cohomology. However, all results of the paper hold in the case of arbitrary algebraically closed ground field of zero characteristic and Chow groups. The ambient varieties participating in the map are always assumed nonsingular and for any algebraic subvariety its fundamental homology class is always identified with the Poincaré dual cohomology class defined by the intersection with this variety.

Acknowledgements. I am sincerely grateful to V. Arnold, L. Fehér, S. Lando, D. Mond, R. Rimányi, S. Sedykh, A. Szücs, V. Vassiliev for numerous discussions of the topic. I wont to acknowledge the hospitality of Warwick University in Spring 2002 where this project was initiated.

2 Overview

2.1 Maps of corank one

Consider a holomorphic map $f : X \to Y$ between two nonsingular complex varieties of relative dimension

$$\ell = \dim Y - \dim X \geq 0.$$

We say that f is a *corank one map* if the kernel rank of its derivative $df(x) : T_xX \to T_{f(x)}Y$ does not exceed 1 at any point $x \in X$.

Assume that f is finite. Then the local algebra of any preimage point is isomorphic to the algebra A_k of truncated polynomials

$$A_k \simeq \mathbb{C}[x]/x^{k+1}$$

for some integer k. The isomorphism type of local algebras gives rise to a stratification of the jet space of corank one maps.

Definition 2.1 A corank one map is called a *Morin map* if, in addition, it satisfies the following transversality conditions:

- the jet extension of f is transversal to the partition of the jet space into the classes of singularities A_k;
- the map f is finite and the multi-jet extension of f at the points of the total inverse image $f^{-1}(y)$, $y \in Y$, is transversal to the corresponding stratification of the multi-jet space for any $y \in Y$.

Smooth Morin maps of real differentiable manifolds form an open dense set in the space of all corank one maps. In the complex algebraic situation the transversality condition is open but not necessary dense: the lack of sufficient number of parameters may not permit to achieve the transversality. However, in many applications Morin maps are indeed generic.

Singularities of Morin maps are stable: if two germs of Morin maps of manifolds of the same dimensions have isomorphic local algebras then they can be brought to one another by a suitable change of coordinates in the source and in the target. Locally any germ of a corank 1 map with A_k-singularity (denoted also by $\Sigma^{1_k} = \Sigma^{1,\dots,1}$ in the Boardman notation) at the origin can be presented as the unfolding of a certain family of curves,

$$(\mathbb{C} \times \mathbb{C}^{m-1}, 0) \to (\mathbb{C}^{\ell+1} \times \mathbb{C}^{m-1}, 0) : (x, \lambda) \mapsto (F(x, \lambda), \lambda). \tag{1}$$

The transversality condition for this germ is equivalent to the condition that this unfolding is $\mathcal{K}$-versal for the initial curve $\mathbb{C}, 0 \to \mathbb{C}^{\ell+1}, 0 : x \mapsto F(x, 0)$. All versal unfoldings of a curve with the singularity A_k at the origin are left-right equivalent to each other and so to the map germ given by the following normal form ([3], [26]):

$$F(x, \lambda) = \begin{pmatrix} x^{k+1} \qquad\qquad + \lambda_{0,k-1}x^{k-1} + \dots + \lambda_{0,1}x \\ \lambda_{1,k}x^k + \lambda_{1,k-1}x^{k-1} + \dots + \lambda_{1,1}x \\ \dots \\ \lambda_{\ell,k}x^k + \lambda_{\ell,k-1}x^{k-1} + \dots + \lambda_{\ell,1}x \end{pmatrix}. \tag{2}$$

Respectively, for any point $y \in Y$ of the target manifold of a Morin map $f : X \to Y$ the multigerm of f at $f^{-1}(y)$ is left-right equivalent to the direct product of several copies of the standard stable Morin-type singularities.

Thus the exhaustive classification of Morin singularities is known. It is discrete (there are no modula in the normal form) and consists of one series A_k of singularities labelled by an integer parameter k. This property simplifies the study of these singularities and all invariants can be computed explicitly. On the other hand Morin singularities carry many common features typical for general singularities. Therefore, they provide a wide class of test examples to verify general methods in singularity theory.

Example 2.2 The most common example of corank one map is a map that decomposes as

$$f : X \xrightarrow{i} W \xrightarrow{\pi} Y,$$

where i is a codimension $\ell + 1$ embedding and π is a locally trivial fibration with 1-dimensional fibers. The vectors from the kernel of df must be tangent to the fibers of π, therefore, the dimension of the kernel cannot be greater than 1.

Example 2.3 Starting with an arbitrary map $X \to Y$ we can pass to the map $X \setminus f^{-1}(\overline{\Sigma}^2) \to Y \setminus \overline{\Sigma}^2$ where $\overline{\Sigma}^2$ is the image of the points of the derivative kernel rank ≥ 2. If the initial map is sufficiently 'generic' then the new map is a Morin map. With passing from X to $X \setminus f^{-1}(\overline{\Sigma}^2)$ some information can be lost. For instance, if X is compact then it has the fundamental homology class and we can talk about characteristic numbers, but for the open manifold $X \setminus f^{-1}(\overline{\Sigma}^2)$ this is not available. However, if the map f is generic then the locus $f^{-1}(\overline{\Sigma}^2)$ has the codimension $2(\ell+2)$ and the embedding $X \setminus f^{-1}(\overline{\Sigma}^2) \to X$ induces an isomorphism of cohomology in small degrees. Thus the formulas for the classes of singularity loci of codimension smaller than $2(\ell+2)$ which are valid for Morin maps are valid also for any generic map.

2.2 Thom polynomials for local singularities

Consider a holomorphic map $f : X \to Y$. Denote by $X(k) \subset X$ the closure of the locus of the singularity A_{k-1}. It can be thought as the locus of 'k-tuples of infinitely close points in X sharing the same image'. In is known that in the case when f is Morin map the subvariety $X(k)$ is non-singular of 'expected' codimension

$$\operatorname{codim}_X X(k) = (k-1)\,(\ell+1).$$

By Thom's theorem, its dual cohomology class $[X(k)] \in H^*(X)$ can be expressed as a universal polynomial denoted R_k in the relative Chern classes

$$c_i(f) = c_i(f^*TY - TX).$$

Here we present a simple closed formula for this polynomial which is valid, however, for maps of corank one only. Namely, consider the following sequence of polynomials in the variables $c_1, \dots, c_{\ell+1}, t$ of weighted degree $\ell+1$ (with respect to the grading with $\deg c_i = i$,

$\deg t = 1$):

$$\sigma_0 = c_{\ell+1} - t\,c_\ell, \quad \sigma_1 = c_{\ell+1}, \quad \dots, \quad \sigma_k = c_{\ell+1} + (k-1)\sum_{j=0}^{\ell} k^j t^{j+1} c_{\ell-j}, \quad \dots. \qquad (3)$$

Theorem 2.4 *The Thom polynomial for the singularity A_{k-1} is given by the product*

$$[X(k)] = R_k = \sigma_1\,\sigma_2 \dots \sigma_{k-1}. \qquad (4)$$

Here the classes $c_i = c_i(f)$ are the relative Chern classes of the map, and t is the 'virtual' class that has the following meaning: one should to expand the brackets in the formula above and to replace every occurrence of $t^k c_{\ell+1}$ by $c_{\ell+1+k}$. In other words t can be interpreted as an operator that can be applied to the relative Chern classes via

$$t\,c_k = c_{k+1}, \quad k \geq \ell + 1. \qquad (5)$$

This procedure allows one to restore the Thom polynomial not uniquely but modulo the ideal generated by the expressions of the form

$$c_i c_j - c_{i'} c_{j'}, \qquad i, j, i', j' \geq \ell + 1.$$

Elements of this ideal are exactly those polynomials in the relative Chern classes that can be represented by cycles supported on Σ^2. These polynomials vanish for corank one maps so that the cohomology class R_k is well defined (see Sect. 3.1).

Let $\Sigma = X(2)$ be the critical set of the map f. Over Σ the derivative of f has a non-trivial kernel. The restriction of $-t$ to Σ is the 'true' cohomology class defined as the first Chern class $-t = c_1(\varkappa)$ of the kernel line bundle $\varkappa = \ker df$ and satisfying (5). Therefore, the class t can be thought also as the 'virtual extension' of the class $-c_1(\varkappa)$ from the critical subvariety Σ to the whole X. In general, the extension of $\varkappa = \ker df$ to the whole X is not necessary defined and t does not necessary correspond to any globally defined cohomology class on X. But in applications such an extension does often exist. For example, for the map of Example 2.2 $-t$ can be defined as the first Chern class of the relative line bundle of the fibration π.

The proof of Theorem 2.4 is presented in Sect. 3.2.

Remark 2.5 Denote by $\nu_f = f^*TY - TX$ the (virtual) relative tangent bundle of the map. Then σ_k admits the following interpretation

$$\sigma_k = c_{\ell+1}((\nu_f + \varkappa) \otimes \varkappa^{*k}) = c_{\ell+1}(\nu_f + \varkappa - \varkappa^{*k}).$$

2.3 Characteristic classes of multisingularities

In the simplest version the loci of multisingularities are defined on the target manifold Y of a map. Namely, for any partition $\alpha = (a_1, \dots, a_r)$ of the number $|\alpha| = a_1 + \dots + a_r$ we denote by $Y(\alpha) \subset Y$ the closure of the locus consisting of points $y \in Y$ having r pairwise different preimages $x_1, \dots, x_r$ such that f acquires the singularity A_{a_i-1} at x_i. In other words, $Y(\alpha)$ parameterizes the points y in the target having $|\alpha|$ preimages such that the first a_1 of them are infinitely close to each other, the next a_2 are infinitely close to each other etc. If f is 'generic', in particular, if f is a Morin map, then $Y(\alpha)$ is a (possibly singular) subvariety of expected codimension

$$\operatorname{codim}_Y Y(\alpha) = |\alpha|\,(\ell+1) - r.$$

More refined multisingularity locus $X(\alpha)$ is defined as the closure of the locus in the cartesian product X^r parameterizing tuples $(x_1,\dots,x_r)$ of pairwise different points x_i sharing the same image $f(x_1) = \dots = f(x_r)$ and such that f acquires the singularity A_{a_i-1} at x_i for $i = 1,\dots,r$. The locus $Y(\alpha)$ is the image of $X(\alpha)$ under the natural map $q^\alpha : X(\alpha) \to Y$ induced by f. Denote also by $p^\alpha : X(\alpha) \to X$ the natural map induced by the projection $X^r \to X$ to the first factor:

$$\begin{array}{ccc} & X(\alpha) & \\ {\scriptstyle p^\alpha}\swarrow & & \searrow{\scriptstyle q^\alpha} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

Define *source* and *target multisingularity cohomology classes* $\overline{m}_\alpha \in H^*(X)$ and $\overline{n}_\alpha \in H^*(Y)$ as Poincaré duals to the subvarieties $p^\alpha(X(\alpha))$ and $q^\alpha(X(\alpha)) = Y(\alpha)$ equipped with reduced scheme structures, respectively. It is natural sometimes to consider these classes with their natural multiplicities equal to the degrees of the maps p^α and q^α to their images. Thus, we set

$$m_\alpha = p^\alpha_*(1) = |\mathrm{Aut}(\alpha')|\,\overline{m}_\alpha, \qquad n_\alpha = q^\alpha_*(1) = f_*(m_\alpha) = |\mathrm{Aut}(\alpha)|\,\overline{n}_\alpha,$$

where $|\mathrm{Aut}(\alpha)|$ is the order of the authomorphism group of the partition α, that is, the product of factorials of the numbers of repeating parts a_i's, and $\alpha' = (a_2,\dots,a_r)$. In what follows we always assume that f is proper. In this case p^α and q^α are also proper and the corresponding push-forward homomorphisms p^α_* and q^α_* are defined.

An algorithm leading to the computation of the multisingularity classes m_α and n_α for each particular multisingularity type α based on Kleiman's theory is outlined by Colley in [6]. This algorithm does not provide a closed formula valid for all multisingularity types and the algebraic structure of the final answer is not clear from this approach.

On the other hand in the paper [16] motivated by quite different topological arguments we suggested a generic form in which the final answer should be formulated. It is shown that m_α and n_α can be expressed in a universal way via certain classes of the form $f_*(R_\alpha)$ associated with various types of multisingularities α where $R_\alpha \in H^*(X)$ is given by a universal polynomial (called *residual polynomial*) in the relative Chern classes $c_i(f)$ of the map. For example, for small number of parts the expressions for the classes n_α are as follows:

$$\begin{aligned} n_p &= f_*(R_p),\\ n_{p,q} &= f_*(R_p)\,f_*(R_q) + f_*(R_{p,q}),\\ n_{p,q,r} &= f_*(R_p)\,f_*(R_q)\,f_*(R_r) + f_*(R_{p,q,r}) +\\ &\qquad f_*(R_p)\,f_*(R_{q,r}) + f_*(R_q)\,f_*(R_{p,r}) + f_*(R_r)\,f_*(R_{p,q}). \end{aligned}$$

The combinatorial expressions for the classes m_α and n_α in terms of R_α are described in [16]. S.Lando pointed out that these relations admit the following convenient presentation by means of generating series. Let us introduce infinite number of auxiliary commuting variables $\tau_1, \tau_2, \dots$. Consider the following formal series

$$\mathcal{M}_k = m_k + \sum_{i_1, i_2, \dots} m_{k, 1_{i_1}, 2_{i_2}, \dots} \frac{\tau_1^{i_1}}{i_1!} \frac{\tau_2^{i_2}}{i_2!} \cdots = m_k + \sum_\alpha \overline{m}_{k,\alpha}\, \tau_\alpha,$$

$$\mathcal{N} = 1 + \sum_{i_1, i_2, \dots} n_{1_{i_1}, 2_{i_2}, \dots} \frac{\tau_1^{i_1}}{i_1!} \frac{\tau_2^{i_2}}{i_2!} \cdots = 1 + \sum_\alpha \overline{n}_\alpha\, \tau_\alpha,$$

where by $(1_{i_1}, 2_{i_2}, \dots)$ we denote the partition having i_1 parts of length 1, i_2 parts of length 2, etc., and where for the partition $\alpha = (a_1, \dots, a_r)$ we set $\tau_\alpha = \tau_{a_1} \dots \tau_{a_r}$. The following theorem is a reformulation of the main result of [16].

Theorem 2.6 *There exist universal polynomials R_α in the relative Chern classes indexed by various multisingularity types $\alpha = (a_1, \dots, a_r)$ such that for the generating series of these polynomials*

$$\mathcal{R} = \sum_{i_1, i_2, \dots} R_{1_{i_1}, 2_{i_2}, \dots} \frac{\tau_1^{i_1}}{i_1!} \frac{\tau_2^{i_2}}{i_2!} \cdots$$

the following universal relations hold

$$f_*(\mathcal{M}_k) = \frac{\partial \mathcal{N}}{\partial \tau_k}, \tag{6}$$

$$\frac{\partial \mathcal{N}}{\partial \tau_k} = \frac{\partial f_*(\mathcal{R})}{\partial \tau_k} \mathcal{N}, \tag{7}$$

$$\mathcal{N} = \exp(f_*(\mathcal{R})) \tag{8}$$

$$\mathcal{M}_k = \frac{\partial \mathcal{R}}{\partial \tau_k} f^*(\mathcal{N}) \tag{9}$$

$$= \frac{\partial \mathcal{R}}{\partial \tau_k} \exp(f^* f_*(\mathcal{R})). \tag{10}$$

Eq. (6) is equivalent to the relation $n_\alpha = f_* m_\alpha$. Eq. (7) is a recursive formula for the classes n_α in terms of residual polynomials and the equivalent closed formula (8) is the result of solving recursive relations. Eq. (7) follows from Eq. (9) by applying f_* to both sides. Therefore, all relations (7)–(10) follow formally from (9) or (10).

As for the residual polynomials, the following theorem provides a closed formula for them which holds, however, only for maps of corank one. For generic maps we need correction terms which are known explicitly up to certain degree (about 10) only, see Sect. 6.2.

If the number of local singularity types forming a given multisingularity type is one, then R_k is the corresponding Thom polynomial given by (4). In the general case, the residual polynomials R_α of multisingularities are expressed in terms of Thom polynomials R_k of monosingularities in the following explicit way.

Theorem 2.7 *The generating function for residual polynomials of Morin map singularities is given by the following series:*

$$\mathcal{R} = -\frac{t}{\sigma_0}\log\left(1+\sigma_0\sum_{i_1,i_2,\dots}(-t)^{-(i_1+i_2+\dots)}R_{i_1+2\,i_2+\dots}\frac{\tau_1^{i_1}}{i_1!}\frac{\tau_2^{i_2}}{i_2!}\cdots\right) \tag{11}$$
$$= -\frac{t}{\sigma_0}\log\left(1-\sigma_0\frac{\tau_1}{t}+\sigma_0\sigma_1\Big(\frac{\tau_1^2}{2\,t^2}-\frac{\tau_2}{t}\Big)+\sigma_0\sigma_1\sigma_2\Big(-\frac{\tau_1^3}{3!\,t^3}+\frac{\tau_1\tau_2}{t^2}-\frac{\tau_3}{t}\Big)+\dots\right),$$

where σ_k is given by (3).

The rational functions participating in the last expression are the homogeneous components of the exponent

$$e^{-\frac{1}{t}(\tau_1+\tau_2+\tau_3+\dots)} = 1-\frac{\tau_1}{t}+\Big(\frac{\tau_1^2}{2\,t^2}-\frac{\tau_2}{t}\Big)+\Big(-\frac{\tau_1^3}{3!\,t^3}+\frac{\tau_1\tau_2}{t^2}-\frac{\tau_3}{t}\Big)+\dots.$$

For partitions with small number of parts the theorem gives:

$$R_{p,q} = \frac{1}{t}(\sigma_0 R_p R_q - R_{p+q}),$$
$$R_{p,q,r} = \frac{1}{t^2}(2\,\sigma_0^2 R_p R_q R_r - \sigma_0 R_p R_{q+r} - \sigma_0 R_q R_{r+p} - \sigma_0 R_r R_{p+q} + R_{p+q+r}),$$
e.t.c.

The formulas above express R_α as a rational function in the variables $t, c_1, \dots, c_{\ell+1}$ with some power of t in the denominator. The statement of the theorem implies that the numerator is divisible by the corresponding power of t, i.e. R_α is, in fact, a polynomial for any ℓ and α. Similarly to the case of monosingularities, $c_i = c_i(f)$ is the corresponding relative Chern class and t is the 'virtual' class used to denote $c_{\ell+1+k}$ as $t^k\,c_{\ell+1}$.

In Sections 5.4 and 5.5 we give a number of equivalent to each other recursive formulas for the classes n_α of multisingularities and residue polynomials R_α resulting to Eq. (8)–(10) and (11), respectively. Some refined integer formulas concerning the classes of multisingularities of Morin maps are formulated in Sect. 6.1

The residue polynomials of small degrees for $\ell = 0, 1$ are presented below.

$\ell = 0$	$\ell = 1$
$R_1 = 1$	$R_{1,1} = -c_1$
$R_2 = c_1$	$R_2 = c_2$
$R_{2,2} = -2\,c_1(2\,c_1+t)$	$R_{1,1,1} = 2\,(c_1^2+c_2)$
$= -2\,(2\,c_1^2+c_2)$	$R_{1,2} = -2\,c_2(c_1+t) = -2\,(c_1c_2+c_3)$
$R_3 = c_1\,(c_1+t)$	$R_{1_4} = -6\,(c_1^3+3\,c_1c_2+2\,tc_2)$
$= c_1^2+c_2$	$= -6\,(c_1^3+3\,c_1c_2+2\,c_3)$
$R_{2,2,2} = 8\,c_1(5\,c_1^2+7\,t\,c_1+3\,t^2)$	$R_3 = c_2(c_2+t\,c_1+2\,t^2)$
$= 8\,(5\,c_1^2+7\,c_1c_2+3\,c_3)$	$= c_2^2+c_1c_3+2\,c_4$
$R_{2,3} = -6\,c_1(c_1+t)^2$	$R_{1,1,2} = 2\,c_2(3\,c_1^2+2\,c_2+7\,t\,c_1+6\,t^2)$
$= -6\,(c_1^3+2\,c_1c_2+c_3)$	$= 2\,(3\,c_1^2c_2+2\,c_2^2+7\,c_1c_3+6\,c_4)$
$R_4 = c_1(c_1+t)(c_1+2\,t)$	$R_{1_5} = 24\,(c_1^4+6\,c_1^2\,c_2+2\,c_2^2+9\,t\,c_1c_2+6\,t^2c_2)$
$= c_1^3+3\,c_1c_2+2\,c_3$	$= 24\,(c_1^4+6\,c_1^2c_2+2\,c_2^2+9\,c_1c_3+6\,c_4)$

Remark 2.8 The residue polynomial $R_{\alpha_1,\dots,\alpha_r}$ is independent of the order of the entries a_i. It is independent also of the particular dimensions of the manifolds X, Y, but does depend on the relative dimension $\ell = \dim Y - \dim X$. L. Feher and R. Rimanyi made the following observation: if we set $d_i = c_{\ell+1-i}$, then the Thom polynomials of some singularities (e.g. A_1, A_2, A_3) can be written as an infinite series in d_i whose coefficients are independent of ℓ. In the case of corank one maps this statement holds for all residual polynomials. It follows from the fact that the polynomials σ_k can be represented in such form:

$$\sigma_k = d_0 + (k-1)\sum_{j=0}^{\infty} k^j t^{j+1} d_{-1-j}.$$

For any particular ℓ this series has only finite number of non-zero terms since $d_i = 0$ for $i < -\ell - 1$.

To compute residual polynomials various methods can be applied. A very simple proof of (11) based on the existence of the polynomials R_α is given in Sect. 3.3. On the other hand, the methods of intersection theory provide an alternative direct derivation of the formulas for the classes of multisingularities. In this approach the equalities (7)–(11) appear as the result of quite lengthy direct computations. The basic idea of this approach is due to S.Kleiman. The following two sections contain our vision of Kleiman's theory. A more detailed explanation is given in Sect. 5

Example 2.9 Multiple point formula. The original Kleiman's theory [20], [21] deals with the classes of k-multiple points $\overline{m}_{1_k} \in H^*(X)$ and $\overline{n}_{1_k} \in H^*(Y)$ for corank one maps of positive relative dimension $\ell > 0$. The principle recursive relation of the theory can be written in the form

$$\overline{m}_{1_k} = f^*\overline{n}_{1_{k-1}} + \sum_{i=1}^{k} (-1)^i p_i\, \overline{m}_{k-i}, \tag{12}$$

where p_i are certain polynomials in the relative Chern classes. Katz [12] noticed that the polynomials p_i are independent of k and gave an algorithm for their computation. A closed formula for these polynomials is provided by Theorem 2.7. If we introduce the following generating series

$$\begin{aligned} m(t) &= 1 + \overline{m}_{1,1}\,\tau + \overline{m}_{1,1,1}\,t^2 + \dots, \\ n(t) &= 1 + \overline{n}_1\,\tau + \overline{n}_{1,1}\,t^2 + \dots, \\ p(t) &= 1 + p_1\tau - p_2\tau^2 + p_3\tau^3 - \dots, \end{aligned}$$

then (12) can be rewritten as

$$m(t)\,p(t) = n(t).$$

This equality is a particular case of the relation (9) of Theorem 2.6 with

$$\frac{1}{p(\tau_1)} = \frac{\partial \mathcal{R}}{\partial \tau_1}\Big|_{\tau_{>1}=0}.$$

Differentiating the expression for $\mathcal{R}$ of Theorem 2.7, we obtain, explicitly,

$$\begin{aligned} p(t) &= \frac{1 - \sigma_0\frac{\tau}{t} + \sigma_0\sigma_1(\frac{\tau}{t})^2 - \sigma_0\sigma_1\sigma_2(\frac{\tau}{t})^3 + \dots}{1 - \sigma_1\frac{\tau}{t} + \sigma_1\sigma_2(\frac{\tau}{t})^2 - \sigma_1\sigma_2\sigma_3(\frac{\tau}{t})^3 + \dots} \\ &= 1 + \tfrac{\sigma_1-\sigma_0}{t}\tau - \sigma_1\tfrac{\sigma_0-2\sigma_1+\sigma_2}{2t^2}\tau^2 + \sigma_1\tfrac{-3\sigma_0\sigma_1+6\sigma_1^2+2\sigma_0\sigma_2-6\sigma_1\sigma_2+\sigma_2\sigma_3}{6t^3}\tau^3 - \dots, \end{aligned}$$

where σ_i is given by (3). Remark that t cancels in the denominator of p_i for any ℓ and i, therefore, p_i is a polynomial (with integer coefficients) in the variables $t, c_1, \ldots, c_{\ell+1}$. In order to express it in terms of c_i only, one should to replace, as usual, $c_{\ell+1}t^k$ by $c_{\ell+1+k}$.

2.4 Iteration principle

The multisingularity varieties $X(\alpha)$ defined in the previous section are related by natural mappings. In particular, with any partition $\alpha = (a_1, \ldots, a_r)$ we associate the following diagram

$$\begin{array}{ccc} X(1,\alpha) & \xrightarrow{f^\alpha} & X(\alpha) \\ {\scriptstyle p^{1,\alpha}}\downarrow & & \downarrow{\scriptstyle q^\alpha} \\ X & \xrightarrow{f} & Y \end{array}$$

where $(1,\alpha) = (1, a_1, \ldots, a_r)$, and f^α is given by forgetting the first point of the tuple.

Definition 2.10 We call f^α the *derived map* of f associated with the partition α.

Theorem 2.11 *Assume that $f : X \to Y$ is a proper Morin map of holomorphic manifolds of relative dimension $\ell = \dim Y - \dim X$. Then the following statements hold:*

1. The multisingularity variety $X(\alpha)$ is non-singular for any α.

2. The derived map f^α is a proper Morin map of the same relative dimension ℓ.

3. The subvariety $X(k,\alpha) \subset X(1,\alpha)$ is identified with the locus of the singularity A_{k-1} for the derived map f^α.

4. For any other partition $\beta = (b_1, \ldots, b_s)$ the multisingularity variety of the derived map f^α associated with the partition β is naturally identified with the multisingularity variety $X(\beta,\alpha)$ of the original map f associated with the partition $(\beta,\alpha) = (b_1, \ldots, b_s, a_1, \ldots, a_r)$.

This theorem is referred to as the (generalized) *iteration principle*. In the original Kleiman's multiple point theory this principle have been applied (in the case when the partition has the form $\alpha = (1, \ldots, 1)$) with no satisfactory justification; its statement have been actually put as the assumption of the main theorem. In the paper [23] a similar statement is proved in the particular case of maps of relative dimension $\ell = 1$. However, without reference to singularity theory the proof looks somewhat awkward. The fact that the multisingularity variety $X(\alpha)$ is smooth for any multisingularity type is proved in [27]. A version of this theorem for Morin maps of real differentiable manifolds have been used by V.Sedykh [34].

The assumption that the map has only corank one singularities is crucial. With the presence of more complicated singularities all statements fail. This difficulty makes it complicated to extend methods of intersection theory to the study of multisingularities of generic maps.

By this theorem, the cycles of more complicated singularities can be interpreted as cycles of more simple ones for the derived map. The residue intersection formulas formulated in the next section allows us to relate characteristic classes associated with the derived map to the characteristic classes associated with the original one. This leads to the computation of multisingularity classes.

The homological part of the iteration principle relates the relative Chern classes of the derived map $f^\alpha : X(1,\alpha) \to X(\alpha)$ associated with some partition $\alpha = (a_1, \dots, a_r)$ to the relative Chern classes of f via the natural map $p^{1,\alpha} : X(1,\alpha) \to X$ induced by the projection $X^{r+1} \to X$ to the first factor. Remark that $p^{1,\alpha}$ maps the critical set $X(2,\alpha) \subset X(1,\alpha)$ of the derived map to the critical set $X(2) \subset X$ of the original one. Moreover, the kernel line bundle for the derivative df^α is isomorphic to the pull-back of the kernel line bundle for the derivative df. Having this in mind we use the same notation $\varkappa$ for both line bundles and we set $t = -c_1(\varkappa)$.

Let $\nu_f = f^*(TY) - TX$ and $\nu_{f^\alpha} = f^{\alpha *}TX(\alpha) - TX(1,\alpha)$ be the relative tangent bundles of the maps f and f^α, respectively. Besides the relative Chern classes $c_i(f) = c_i(\nu_f)$ (pulled back to $X(1,\alpha)$) and $c_i(f^\alpha) = c_i(\nu_{f^\alpha})$ we shall use one extra divisorial class on $X(1,\alpha)$. Namely, denote by $D_i \subset X(1,\alpha)$ the subvariety given by the intersection with the diagonal in X^{r+1} determined by the equation $x_0 = x_i$. It follows from the proof of Theorem 2.11 that D_i is a smooth hypersurface that can be naturally identified with $X(a_i + 1, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r)$. Denote by ζ the line bundle on $X(1,\alpha)$ associated with the divisor

$$D = a_1 D_1 + \dots + a_r D_r,$$

and set

$$\theta = -c_1(\zeta) = -[D].$$

Theorem 2.12 1. *The relative Chern classes of the derived map f^α are expressed as polynomials in the classes $c_i = c_i(p^{1,\alpha *}\nu_f) = p^{1,\alpha *}c_i(f)$, $t = -c_1(\varkappa)$, and $\theta = -c_1(\zeta)$:*

$$\begin{aligned} c(\nu_{f^a}) &= c((p^{1,\alpha *}\nu_f + \varkappa) \otimes \zeta^* - \varkappa) \\ &= \frac{(1+\theta)^{\ell+1}}{1-t}\left(1 + \frac{c_1 - t}{1+\theta} + \frac{c_2 - t\,c_1}{(1+\theta)^2} + \frac{c_3 - t\,c_2}{(1+\theta)^3} + \dots\right). \end{aligned} \tag{13}$$

2. *In the case when $r = 1$ that is for the derived map $f^k : X(1,k) \to X(k)$ the normal bundle of the divisor $D_1 \simeq X(k+1) \subset X(1,k)$ is isomorphic to the kernel line bundle $\varkappa = \ker df$ of the map f so that the additional relation holds in this case*

$$\theta = k\,t.$$

Let us denote by

$$\Psi : \mathbb{Z}[c_1, \dots, c_{\ell+1}, t] \to \mathbb{Z}[c_1, \dots, c_{\ell+1}, t, \theta]$$

the homomorphism that sends c_i to the ith homogeneous component in the series (13), then for the classes $\sigma_k = c_{\ell+1}((\nu_f + \varkappa) \otimes \varkappa^{*k})$ given by (3) one has

$$\Psi(\sigma_k) = \sigma(\theta + k\,t),$$

where

$$\sigma(\theta) = c_{\ell+1}((\nu_f + \varkappa) \otimes \zeta^*) = c_{\ell+1} + (\theta - t)\sum_{i=0}^{\ell} \theta^i c_{\ell-i}. \tag{14}$$

Since the subvariety $X(k,\alpha) \subset X(1,\alpha)$ is the locus of the singularity A_{k-1} for the derived map f^α, we obtain

Corollary 2.13 *The cohomology class $\widetilde{R}_k(\theta)$ dual to the subvariety $X(k,\alpha)\subset X(1,\alpha)$ is given by*

$$\widetilde{R}_k(\theta)=\sigma(\theta+t)\,\sigma(\theta+2t)\dots\sigma(\theta+(k{-}1)t).$$

Remark 2.14 The bundle $\varkappa$ and the class t are not defined on the whole $X(1,\alpha)$. To assign a rigorous meaning to the right hand side of (13) we use the following agreement. Let E be a (virtual) rank n vector bundle over some base B and ζ be a line bundle associated with a divisor $D\subset B$. Then the Chern classes of the difference $E-E\otimes\zeta$ can be computed as homogeneous components in the expansion of the following quotient[1]

$$c(E-E\otimes\zeta)=\frac{\sum_{i=0}^{\infty}c_i}{\sum_{i=0}^{\infty}(1-\theta)^{n-i}c_i},\qquad c_i=c_i(E),\quad \theta=-c_1(\zeta).$$

Remark that each term of positive degree on the right hand side is divisible by θ. Since $\theta=-[D]$, we obtain the equality

$$c_k(E-E\otimes\zeta)=\Delta_{k-1}\frown[D], \tag{15}$$

where Δ_{k-1} is a polynomial in the classes θ, c_i defined formally by the equality

$$\Delta_{k-1}=-\frac{1}{\theta}\left\{\frac{\sum_{i=0}^{\infty}c_i}{\sum_{i=0}^{\infty}(1-\theta)^{n-i}c_i}\right\}_k,$$

$$\Delta_0=-n,\quad \Delta_1=c_1-\tbinom{n+1}{2}\theta,\quad \Delta_2=2\,c_2-c_1^2+(n{+}1)\,c_1\,\theta-\tbinom{n+2}{3}\theta^2,\quad \dots.$$

Therefore, *the Chern classes of the difference $E-E\otimes\zeta$ are supported on D and are determined uniquely by the restriction of E to D. Moreover, the equality* (15) *provides a rigorous meaning to the class $c_k(E-E\otimes\zeta)\in H^*(B)$ even if the bundle E is defined on D only and does not extend necessary to B.*

2.5 Residue intersection formula

The classes $c_i=p^{1,\alpha*}c_i(f)$, $t=-c_1(\varkappa)$, and $\theta=-c_1(\zeta)$ on $X(1,\alpha)$ introduced in the previous section can be pulled back to $X(a_0,\alpha)$ for arbitrary integer a_0. The residue intersection formula formulated in this section deals with the classes of the form

$$P(\theta)\frown[X(a_0,\alpha)]=P(\theta)\,\widetilde{R}_{a_0}(\theta)\frown[X(1,\alpha)], \tag{16}$$

where $\widetilde{R}_{a_0}(\theta)$ is as in Corollary 2.13 and $P(\theta)$ is arbitrary polynomial in the classes c_i,t,θ. The class represented in this form is considered as an element of either the cohomology group $H^*(X(1,\alpha))$ or the group $H^*(X^{r+1})$. In the latter case it is still useful to remember that *this class is supported on* $X(1,\alpha)\subset X^{r+1}$ i.e. it belongs to the image of the homomorphism $H^*(X^{r+1},X^{r+1}\setminus X(1,\alpha))\to H^*(X^{r+1})$ induced by the embedding.

[1] We use the fact that for any virtual vector bundle E of virtual rank n with the total Chern class $c(E)=1+c_1+\dots$ and a line bundle L with $c_1(L)=t$ the total Chern class of $E\otimes L$ is given by

$$c(E\otimes L)=(1{+}t)^n\left(1+\frac{c_1}{1{+}t}+\frac{c_2}{(1{+}t)^2}+\dots\right)=(1{+}t)^n+(1{+}t)^{n-1}c_1+\dots+c_n+\frac{c_{n+1}}{1{+}t}+\dots,$$

where the sum could be infinite. We shall often use this equality in the paper. It can easily be proved using, for instance, the splitting principle.

Remark 2.15 If the multiindices $\alpha = (a_1, \dots, a_r)$ and $\beta = (b_1, \dots, b_r)$ differ by a permutation, then the corresponding multisingularity varieties $X(\alpha)$ and $X(\beta)$ are isomorphic. However, the corresponding isomorphism $X(\alpha) \simeq X(\beta)$ does not necessary preserve the bundle ζ. This isomorphism does preserve ζ, if the permutation fixes the first element $a_1 = b_1$. Therefore, the index 1 is distinguished in our notation for the multisingularity varieties $X(\alpha)$.

Together with the subvariety $X(1, \alpha) \subset X^{r+1}$ we shall consider also r subvarieties $Z_1, \dots, Z_r$ isomorphic to $X(\alpha)$ via the diagonal embeddings

$$Z_i = \delta_i(X(\alpha)) \subset X^{r+1}, \quad \delta_i : (x_1, \dots, x_r) \mapsto (x_i, x_1, \dots, x_r).$$

The varieties Z_i are isomorphic to each other. However, the class $\theta = \theta_i$ associated with the line bundle $\zeta = \zeta_i$ considered on the ith copy of $X(\alpha)$ corresponds to the distinguished projection to the ith factor and is different for different i. Therefore, a more relevant notation for these varieties would be

$$Z_i = X(a_i, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r).$$

The varieties $X(1, \alpha)$, Z_1, …, Z_r are the components of the fiber product space $X \times_Y X(\alpha)$ defined as the solution scheme of the equation $f(x_0) = q^\alpha(x_1, \dots, x_r)$,

$$\begin{array}{ccccc} & & X(1,\alpha) & & \\ & & \downarrow & \searrow^{f^\alpha} & \\ Z_i & \longrightarrow & X \times_Y X(\alpha) & \longrightarrow & X(\alpha) \\ & & \downarrow & & \downarrow{\scriptstyle q^\alpha} \\ & & X & \xrightarrow{\ f\ } & Y \end{array}$$

Theorem 2.16 *Let $f : X \to Y$ be a proper Morin map. Then the following equalities hold for arbitrary partition $\alpha = (a_1, \dots, a_r)$:*

$$[X(1, \alpha)] = [X] \times_Y [X(\alpha)] - \sum_{i=1}^{r} a_i\, c_\ell(\nu_f + \varkappa - \zeta_i \otimes \varkappa^{a_i}) \frown [Z_i], \tag{17}$$

$$\theta^k\, [X(1, \alpha)] = -\sum_{i=1}^{r} a_i\, c_{\ell+k}(\nu_f + \varkappa - \zeta_i \otimes \varkappa^{a_i}) \frown [Z_i], \quad k \geq 1.$$

Remark that relations between the Chern classes of Morin maps (see Sect. 3.1 below) imply the equalities

$$c_{\ell+k}(\nu_f + \varkappa - \zeta_i \otimes \varkappa^{a_i}) = (\theta_i + a_i\, t)^{k-1} c_{\ell+1}(\nu_f + \varkappa - \zeta_i \otimes \varkappa^{a_i}) = (\theta_i + a_i\, t)^{k-1} \sigma(\theta_i + a_i\, t), \tag{18}$$

where $\sigma(\theta)$ is given by (14).

The two equalities of the theorem can be combined in the following way. Let $P(\theta)$ be a cohomology class on $X(1, \alpha)$ expressed as a polynomial in the variables c_i, t, and θ. Represent it in the form

$$P(\theta) = P(0) + Q\, \theta,$$

where $P(0)$ is free of θ and $Q = (P(\theta) - P(0))/\theta$. Then from (17) and (18) we get

$$P(\theta) \frown [X(1,\alpha)] = (P(0) \frown [X]) \times_Y [X(\alpha)] - \sum_{i=1}^{r} a_i \, \Phi(\theta_i + a_i t) \frown [Z_i],$$

where

$$\Phi(\theta) = c_\ell(\nu_f + \varkappa - \zeta)\, P(0) + c_{\ell+1}(\nu_f + \varkappa - \zeta)\, Q = \frac{\sigma(\theta)\, P(\theta) - \sigma(0)\, P(0)}{\theta}.$$

The term $[X] \times_Y [X(\alpha)]$ is defined as the pull-back of the class dual to the diagonal $\Delta_Y \subset Y \times Y$ under the product map $f \times q^\alpha : X \times X(\alpha) \to Y \times Y$. The meaning of this class is clarified after pushing it forward to $X(\alpha)$, X, or Y, respectively.

Corollary 2.17 *The following relations hold in the cohomology of $X(\alpha)$, X, and Y, respectively:*

$$f_*^\alpha(P(\theta)) = q^{\alpha *} f_*(P(0)) - \sum_{i=1}^{r} a_i \Phi(\theta_i + a_i t), \tag{19}$$

$$p_*^{1,\alpha}(P(\theta)) = P(0) \cdot f^*(n_\alpha) - \sum_{i=1}^{r} a_i \, p_*^{\alpha_i}(\Phi(\theta + a_i t)), \tag{20}$$

$$q_*^{1,\alpha}(P(\theta)) = f_*(P(0)) \cdot n_\alpha - \sum_{i=1}^{r} a_i \, q_*^{\alpha_i}(\Phi(\theta + a_i t)), \tag{21}$$

where $\alpha_i = (a_i, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r)$, the map $p^\alpha : X(\alpha) \to X$ is induced by the projection to the first factor, $q^\alpha = f p^\alpha : X(\alpha) \to Y$, and where

$$\Phi(\theta) = \frac{\sigma(\theta)\, P(\theta) - \sigma(0)\, P(0)}{\theta}, \qquad \sigma(\theta) = c_{\ell+1} + (\theta - t) \sum_{i=0}^{\ell} \theta^i c_{\ell - i}.$$

These relations accompanied with the relations

$$p_*^{a_0,\alpha}(P(\theta)) = p^{1,\alpha}(\widetilde{R}_{a_0}(\theta)\, P(\theta)), \qquad q_*^{a_0,\alpha}(P(\theta)) = q^{1,\alpha}(\widetilde{R}_{a_0}(\theta)\, P(\theta)), \tag{22}$$

following from Corollary 2.13 are sufficient to compute recursively the multisingularity cohomology classes $m_\alpha = p_*^\alpha(1)$ and $n_\alpha = q_*^\alpha(1)$ for any particular multisingularity α. It follows with no further computation that, for example, the resulting expression for $n_\alpha \in H^*(Y)$ is a polynomial combination of the classes of the form $f_*(R)$ for certain polynomials R in the relative Chern classes $c_i(f)$. The detailed study of these recursive relations leads to the general form (7)–(10) of the final expression and to the formula of Theorem 2.7 for residual polynomials.

Theorem 2.16 provides a freedom in the choice of the inductive strategy. For example, we can treat the multisingularity variety $X(\alpha)$ associated with the partition $\alpha = (a_1, \dots, a_r)$ as the multisingularity variety associated with the partition $(a_1, \dots, a_{r-1})$ for the derived map $f^{a_r} : X(1, a_r) \to X(a_r)$, see Sect. 5.4. This strategy (applied to the cycles of multiple points and the derived map $X(1,1) \to X$) is used in the paper [20]. Another possibility is to treat $X(\alpha)$ as the singularity locus $A_{a_1 - 1}$ for the derived map $f^{a_2,\dots,a_r} : X(1, a_2, \dots, a_r) \to X(a_2, \dots, a_r)$, see Sect. 5.5. The last choice is actually used in [21] under the name of the 'method of Hilbert schemes'. Here we consider this method rather as a variation of the iteration principle.

Remark 2.18 In order the push-forward homomorphism $p_* : H^*(X^{r+1}) \to H^*(X)$ induced by the projection to the first factor to be well-defined the manifold X must be compact. In fact, we require a weaker assumption that the map f is proper. This assumption implies that the maps p^α and q^α are also proper, the homomorphisms p^α_* and q^α_* are defined and the equalities of Corollary 2.17 make sense. To derive the corollary we observe that the classes entering Eq. (17) are supported on the variety of the fiber product

$$X \times_Y X(\alpha) = X(1,\alpha) \cup Z_1 \cup \cdots \cup Z_r.$$

The restriction of $p : X^{r+1} \to X$ to this variety is proper and the direct image of the corresponding classes are well defined. Thus Eq. (17) should be considered in the relative cohomology group $H^*(X^{r+1},\ X^{r+1} \setminus X \times_Y X(\alpha))$ (or in $A_*(X \times_Y X(\alpha))$ in the algebraic setting).

3 Characteristic classes of Morin maps

3.1 Relations between relative Chern classes

Let $f : X \to Y$ be a holomorphic map of relative dimension $\ell = \dim Y - \dim X$. Denote by $\nu_f = f^*TY - TX$ the virtual normal bundle of f and by $c_i = c_i(\nu_f) \in H^*(X)$ the corresponding relative Chern classes.

Proposition 3.1 *If f is of corank one then the classes c_i satisfy the relations*

$$c_i\, c_j - c_{\imath'}\, c_{\jmath'} = 0$$

for all $i, j, \imath', \jmath' \geq \ell + 1$ with $i + j = \imath' + \jmath'$.

Proof. Consider the variety F_2 formed by all flags of the form

$$0 = D_{0x} \subset D_{1x} \subset D_{2x} \subset \ker df_x \subset T_x X,$$

$x \in X$, $\dim D_{ix} = i$. Denote by $p_2 : F_2 \to X$ the natural projection. By a formula of [11], we have

$$p_{2*}(t_1^{s_1} t_2^{s_2}) = c_{\ell+s_1+1} c_{\ell+s_2+2} - c_{\ell+s_1+2} c_{\ell+s_2+1}$$

for all $s_1, s_2 \geq 0$, where $t_i = -c_1(D_i/D_{i-1})$. This formula can be applied if certain transversality condition holds implying that F_2 is non-singular of expected codimension. In the case when f is a map of corank one the variety F_2 is empty and the right hand side of the equality vanishes. □

Definition 3.2 The *ring $\mathcal{M}$ of universal characteristic classes of corank one maps* of relative dimension ℓ is the quotient of the ring of polynomials in the variables $c_1, c_2, \ldots$ over the ideal K generated by the relations of Proposition 3.1.

By proposition, for any corank one map the *characteristic homomorphism* $\mathcal{M} \to H^*(X)$ sending c_i to $c_i(\nu_f)$ is well defined.

There is the following convenient description of the ideal K. Namely, introduce an auxiliary variable t with $\deg t = 1$ and consider the homomorphism

$$\rho : \mathcal{C} \to \mathbb{Z}[t, c_1, \dots, c_{\ell+1}], \quad \rho(c_k) = \begin{cases} c_k & \text{if } k \le \ell + 1, \\ t^{k-\ell-1} c_{\ell+1} & \text{if } k > \ell + 1. \end{cases}$$

One can see that K is the kernel of this homomorphism, $K = \ker \rho$. Therefore $\mathcal{M}$ can be identified with the image of ρ. It means that an element of $\mathcal{M}$ can be represented by a polynomial in $t, c_1, \dots, c_{\ell+1}$ provided that any monomial containing t is divisible by $c_{\ell+1}$. To find its representation in terms of the variables c_i one can replace simply any monomial of the form $t^k c_{\ell+1}$ by $c_{\ell+k+1}$. This representation is not unique: two polynomials in c_i represent the same class in $\mathcal{M}$ iff they differ by an element of K.

Let $f : X \to Y$ be a Morin map, and $\eta : \Sigma \to X$ be the embedding of the submanifold $\Sigma = \overline{\Sigma}^1(f)$ of critical points. Over Σ the derivative of f has a non-trivial kernel. Denote by $\varkappa = \ker df$ the corresponding line bundle over Σ and set $t = -c_1(\varkappa) = c_1(\varkappa^\vee)$. Then the induced morphism $T_\Sigma X/\varkappa \to f^*TY|_\Sigma$ is injective and we get that $\nu_f|_\Sigma + \varkappa$ is a vector bundle on Σ of rank $\ell + 1$. Therefore,

$$0 = c_{i+1}(\nu_f|_\Sigma + \varkappa) = c_{i+1} - t\, c_i \quad \text{for} \quad i \ge \ell + 1$$

and $c_{\ell+1+k} = t^k c_{\ell+1}$, where c_i are relative Chern classes $c_i = c_i(\nu_f)$ (restricted to Σ). Therefore, *we may consider $-t$ as the first Chern class of a 'virtual extension' of the line bundle $\varkappa$ to the whole X.* In another interpretation we may consider the multiplication by t as the linear operation on $\mathcal{M}$ that sends any element of the form $c_k a$ with $k \ge \ell + 1$ to $c_{k+1} a$.

In fact, the extension of $\varkappa = \ker df$ to X is not necessary defined and t does not necessary correspond to any globally defined cohomology class on X. But in many important cases such an extension does exist.

Example 3.3 Assume that the given map $X \to Y$ can be decomposed as

$$X \overset{i}{\hookrightarrow} W \overset{\pi}{\to} Y,$$

where the first map is an embedding of codimension $\ell + 1$ with the normal bundle ν and the second one is submersive of relative dimension 1. Denote also by $\varkappa = \ker \pi_*$ the line bundle on W formed by vectors tangent to the fibers of π and set $t = c_1(\varkappa^*) = -c_1(\varkappa)$. Then from the exact sequences

$$0 \to \varkappa \to TW \to \pi^*TY \to 0, \qquad 0 \to TX \to T_X W \to \nu \to 0$$

we get $\nu = \nu_f + \varkappa$ and

$$c(\nu) = (1 + c_1(\nu_f) + \dots)\,(1 - t).$$

Since ν is a vector bundle of rank $\ell + 1$ we get $0 = c_i(\nu) = c_i(\nu_f) - t c_{i-1}(\nu_f)$ for $i > \ell + 1$ that is $c_{\ell+1+k}(\nu_f) = t^k c_{\ell+1}(\nu_f)$. Thus we obtain that in this example the 'virtual' class t is the first Chern class of a line bundle globally defined on X.

One can see that in the example considered above there is no *a priori* relation between the classes $t, c_1(\nu), \ldots, c_{\ell+1}(\nu)$ and so those between the classes $t, c_1(\nu_f), \ldots, c_{\ell+1}(\nu_f)$. In other words, there exists a map having the form of the example above such that the characteristic homomorphism $\mathcal{M} \to H^*(X)$ is injective up to a given degree. Therefore, we obtain the following general principle:

In the computation of characteristic classes of various singularities of Morin maps it is sufficient to assume that the given map f has the form of the example above.

Algebraically this means that in the intermediate steps of computations we are allowed to use polynomials in $t, c_1, \ldots, c_{\ell+1}$ which are not in the image of ρ and thus do not actually belong to $\mathcal{M}$. We shall see that the application of this principle simplifies considerably all computations. After the desired formula is found the arguments for deriving it can be easily adjusted in such way that they can be applied to general Morin maps (however, this is not really necessary due to the general statements about the existence of universal formulas).

We illustrate the application of this principle in the proof of the following statement. Let $f : M \to Y$ be a Morin map with relative Chern classes $c_i = c_i(\nu_f)$. Let $\Sigma = \overline{\Sigma}^1(f)$ be the critical set, $\eta : \Sigma \to X$ be the natural embedding, $\varkappa = \ker df$ be the kernel line bundle over Σ, and $t = -c_1(\varkappa)$. Consider an arbitrary polynomial P in variables $t, c_1, c_2, \ldots$.

Proposition 3.4 *The homomorphism $\eta_* : H^*(\Sigma) \to H^*(X)$ acts on P as the multiplication by $c_{\ell+1}$,*

$$\eta_*(P) = c_{\ell+1}\, P.$$

Here on the left-hand side t is considered as a well defined cohomology class and by c_i we mean the restriction of $c_i(\nu_f)$ to Σ. The polynomial on the right-hand side is considered as a universal characteristic class of corank 1 maps, that is $c_{\ell+1}t^k$ is just the notation for $c_{\ell+1+k}(\nu_f)$.

Proof. In the case of the map of the example above Σ is defined as the locus of points where the fibers of π are tangent to X that is by vanishing of a section of the bundle $\mathrm{Hom}(\varkappa, \nu) = \varkappa^* \otimes \nu = \varkappa^* \otimes (\nu_f + \varkappa) = \varkappa^* \otimes \nu_f + \mathbb{C}$. Therefore, $\eta_*(1) = c_{\ell+1}(\varkappa^* \otimes \nu_f + \mathbb{C}) = c_{\ell+1}$. Since the class t is globally defined on X, we get by the projection formula,

$$\eta_*(P) = \eta_*(1)\ P = c_{\ell+1}P.$$

By the formulated principle the relation we just obtained is valid for arbitrary Morin map. □

It is worthwhile to compare very simple arguments above with the computations that can be applied in the general case (see, e.g. [11]). Let $f : X^m \to Y^{m+\ell}$ be a Morin map. Then Σ is smooth. The kernel line bundle $\varkappa = \ker df$ is defined now on Σ only. Set $t = -c_1(\varkappa)$. The formula of the proposition can be reformulated as the equality

$$\eta_*(t^k) = t^k c_{\ell+1} = c_{\ell+1+k}.$$

To prove this equality an extra construction is needed. Denote by $\pi : PTX \to X$ the projective bundle associated with the tangent bundle of X. The restriction of this bundle to Σ admits a natural section $s : \Sigma \to PTX$ which assigns to a point $x \in \Sigma \subset X$

the line $\ker df(x) \subset T_xX$. By construction, the bundle $\varkappa$ on Σ coincides with $s^*(O(-1))$ where $O(-1)$ is the tautological line bundle over PTX. Respectively, the class $t = -c_1(\varkappa)$ is induced form the class $c_1(O(1)) = -c_1(O(-1))$ on PTX that we denote again by t.

Furthermore, the differential df defines a section of the bundle $\mathrm{Hom}(O(-1), f^*TY)$ and the submanifold $s(\Sigma) \subset PTX$ is the zero locus of this section. Therefore,

$$[s(\Sigma)] = c_{m+l}(\mathrm{Hom}(O(-1), f^*TY)) = \sum_{i+j=m+\ell} t^i c_j(f^*TY).$$

Using the well known equality $\pi_*(t^i) = c_{i-m+1}(-TX)$ we get

$$\begin{aligned}\eta_*(t^k) &= \pi_*(t^k[s(\Sigma)]) = \pi_*\Big(t^k \sum_{i+j=m+\ell} t^i c_j(f^*TY)\Big) = \pi_* \sum_{i+j=m+\ell+k} t^i c_j(f^*TY) \\ &= \sum_{i+j=\ell+1+k} c_i(-TX) c_j(f^*TY) = c_{\ell+1+k}(f^*TY - TX)\end{aligned}$$

as required. □

3.2 Thom polynomials of monosingularities

According to Thom's principle, the cohomology class in X dual to the locus of any singularity of the map f, say, of the singularity $\overline{\Sigma}^{1_p}$, can be represented as a universal polynomial P in the relative classes $c_i(\nu_f)$. If the map is of corank at most 1 then this polynomial P is defined up to an element of the relation ideal K only. Of course, among the polynomials representing the same element in $\mathcal{M}$ there is a preferable choice, namely the polynomial representing the class dual to the locus of the singularity $\overline{\Sigma}^{1_p}$ for *any* sufficiently 'generic' map. The problem of finding a universal formula for such a representative seems to be hopeless since it depends on the complete classification of map germ singularities which is known to be highly irregular. Nevertheless, a universal formula can be given for this class up to an element in K. The following theorem is a reformulation of Theorem 2.4.

Theorem 3.5 *If $f : X^m \to Y^{m+\ell}$, $\ell \geq 0$, is a Morin map then the cohomology class dual to the closure of the locus of the singularity $\overline{\Sigma}^{1_p}$ is given by the polynomial*

$$[\overline{\Sigma}^{1_p}] = \prod_{k=1}^{p} \sigma_p \in \mathcal{M}, \qquad \sigma_k = c_{\ell+1} + (k-1) \sum_{j=0}^{\ell} k^j t^{j+1} c_{\ell-j},$$

evaluated on the characteristic homomorphism $\mathcal{M} \to H^*(X)$ associated with the map f.

Note that the smallest degree of a non-trivial element in K is $2(\ell+2)$ which coincides with the codimension of the singularity $\overline{\Sigma}^2$. Therefore, the formula of the theorem can be applied in the presents of $\overline{\Sigma}^2$-singularities if $p(\ell+1) < 2(\ell+2)$. In particular, for $p = 2$ this formula holds for any map and results to the following expression known since [32],

$$[\overline{\Sigma}^{1,1}] = c_{\ell+1}^2 + \sum_{j=0}^{\ell} 2^j\, c_{\ell+j+2}\, c_{\ell-j}.$$

For $\ell = 0$ the formula reduces to

$$[\overline{\Sigma}^{1_p}] = c_1\,(c_1 + t)\,(c_1 + 2\,t)\dots(c_1 + (p{-}1)t)$$

In this form it has been proved by Vic. Kulikov in [24]. It is equivalent to the Porteous' [28] recursive formula

$$[\overline{\Sigma}^{1_p}] = [\overline{\Sigma}^{1_{p-1}}]\,c_1 + (p{-}1)\,[\overline{\Sigma}^{1_{p-2}}]\,c_2 + (p{-}1)(p{-}2)\,[\overline{\Sigma}^{1_{p-3}}]\,c_3 + \dots + (p{-}1)!\,c_{\ell+1}.$$

Similarly, for $\ell = 1$ we get

$$[\overline{\Sigma}^{1_p}] = c_2\,(c_2 + t\,c_1 + 2\,t^2)\,(c_2 + 2\,t\,c_1 + 6\,t^2)\dots(c_2 + (p{-}1)\,t\,c_1 + (p{-}1)p\,t^2)$$

P r o o f. Let $\Sigma = \overline{\Sigma}^1$ be the critical manifold of f and $\eta : \Sigma \to X$ the natural embedding. Then, by proposition 3.4 we get for $p = 1$,

$$[\overline{\Sigma}^1] = \eta_*(1) = c_{\ell+1}.$$

Now, denote by $E_1 = \nu_\Sigma X$ the normal bundle and by $\varkappa = \ker df$ the corresponding kernel line bundle over Σ. Then, $\overline{\Sigma}^{1,1} \subset \Sigma$ is defined as the locus of points where the lines of $\varkappa$ are tangent to Σ that is by vanishing a section of the bundle $E_2 = \mathrm{Hom}(\varkappa, E_1) = \varkappa^* \otimes E_1$. Therefore, we have $[\overline{\Sigma}^{1,1}] = \eta_*(c_{\ell+1}(E_2))$ and the normal bundle of $\overline{\Sigma}^{1,1}$ in Σ coincides with the restriction of E_2. $\overline{\Sigma}^{1,1,1} \subset \overline{\Sigma}^{1,1}$ is defined as the locus of points where the lines of $\varkappa$ are tangent to $\overline{\Sigma}^{1,1}$ that is by vanishing a section of the bundle $E_3 = \mathrm{Hom}(\varkappa, E_2) = \varkappa^{*2} \otimes E_1$. Therefore, we have $[\overline{\Sigma}^{1,1,1}] = \eta_*(c_{\ell+1}(E_2)c_{\ell+1}(E_3))$ and the normal bundle of $\overline{\Sigma}^{1,1,1}$ in $\overline{\Sigma}^{1,1}$ coincides with the restriction of E_3.

Continuing this way we obtain by induction

$$[\overline{\Sigma}^{1_p}] = \eta_*(\,c_{\ell+1}(E_2)\dots c_{\ell+1}(E_p)), \qquad E_k = \varkappa^{*(k-1)} \otimes E_1.$$

The homomorphism η_* is known by Proposition 3.4. It remains to compute the Chern class of the bundle $E_1 = \nu_\Sigma X$. According to the principle formulated in Sect. 3.1, we may assume that f is as in the example 3.3. Then in the notations of this example Σ is the zero locus of a section of $\mathrm{Hom}(\varkappa, \nu) = \varkappa^* \otimes \nu_f + \mathbb{C}$ where the bundle $\varkappa$ is defined on the whole X. Therefore, E_1 is the restriction to Σ of this bundle, $E_1 = \varkappa^* \otimes \nu_f + \mathbb{C}$, and

$$\begin{aligned} c(E_k) &= c(\varkappa^{*k} \otimes \nu) = c(\varkappa^{*k} \otimes \nu_f + \varkappa^{*(k-1)})) \\ &= (1 + (k{-}1)t)\,\left((1 + kt)^\ell + (1 + kt)^{\ell-1}c_1(\nu_f) + \dots + c_\ell(\nu_f) + \frac{c_{\ell+1}(\nu_f)}{1 + kt} + \dots\right) \end{aligned}$$

which gives $c_{\ell+1}(E_k) = c_{\ell+1} + (k-1)t\sum_{j=0}^{\ell}(kt)^j c_{\ell-j} = \sigma_k$. Therefore,

$$[\overline{\Sigma}^{1_p}] = \eta_*(\sigma_2 \dots \sigma_p) = c_{\ell+1}\,\sigma_2 \dots \sigma_p. \qquad \square$$

For the purity, let us show how to compute $c(E_1) = c(\nu_\Sigma X)$ in the general case. We use the notations from the proof of Proposition 3.4. The subvariety $\Sigma \subset X$ can be identified with its natural lifting $s(\Sigma) \subset PTX$ so that $\varkappa = s^*(O(-1))$, where $\pi : PTX \to X$ is the projectivization of the tangent bundle and $s(\Sigma)$ is the zero locus of a section of the bundle

$\mathrm{Hom}(O(-1), f^*TY) = O(1) \otimes f^*TY$. Since f is a Morin map the fibers of π are never tangent to $s(\Sigma)$ and we get the exact sequence

$$0 \to V|_{s(\Sigma)} \to O(1) \otimes f^*TY|_{s(\Sigma)} \to E_1 \to 0$$

where $V = \ker \pi_*$, the bundle of vectors tangent to the fibers of π. The total Chern class of V can be computed using the adjunction exact sequence

$$0 \to \mathbb{C} \to O(1) \otimes \pi^*TX \to V \to 0.$$

Combining all this together we arrive at the same conclusion as above,

$$E_1 = \varkappa^* \otimes f^*TY - \varkappa^* \otimes TX + \mathbb{C} = \varkappa^* \otimes \nu_f + \mathbb{C},$$

where $\varkappa$ is not necessary extendable to the whole X. □

3.3 Restriction equations for residual polynomials of multisingularities

The computation of residual polynomials for multisingularities done in this section is based on the existence of these polynomials provided by the equalities (8)–(11) proved by topological arguments in [16]. The idea of this method is to apply the equality (8) to a number of certain test maps for which both sides of the equality can be computed independently. The relations obtained from these examples are often sufficient to recover the polynomials uniquely. This method motivated by the Szücs' cobordism theory for maps with prescribed singularities was first applied with a great success by Rimányi [30] to the computation of Thom polynomials for local singularities. At present this is the most efficient known way to compute the class of any particular singularity or multisingularity in many setups [16], [17].

Proof of Theorem 2.7. In the case of Morin maps it is sufficient to consider just one test map constructed in the following way. Let $\Lambda \simeq \mathbb{C}^{\ell+1}$ and $\varkappa \simeq \mathbb{C}$ be vector spaces. Set

$$V = (\Lambda \otimes \varkappa^*) \oplus (\Lambda \otimes \varkappa^{*2}) \oplus \cdots \oplus (\Lambda \otimes \varkappa^{*n})$$

for some big n. For any $x \in \varkappa$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in V$ we set

$$F(x, \lambda) = \lambda_1 x + \cdots + \lambda_n x^n \in \Lambda.$$

Set $X = \varkappa \oplus V$, $Y = \Lambda \oplus V$, and define the polynomial mapping given by

$$f : X \to Y, \qquad (x, \lambda) \mapsto (\lambda_0 = F(x, \lambda), \lambda).$$

This mapping is of corank one. It is infinitely degenerated on the subspace given by the equation $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, and on the complement to this subspace it acquires only Morin singularities. The subspace of infinite degeneracy has a very high codimension, therefore, it does not affect any relation in the cohomology of any fixed degree and will be ignored below.

Since the definition of the mapping f is invariant, we can treat V and $\varkappa$ as vector bundles over some base B. Then X and Y are also total spaces of the corresponding vector bundles over B. The projection to the base induces an isomorphism

$$H^*(X) = H^*(Y) = H^*(B)$$

that we always imply below. The total relative Chern class of f is given by

$$c(f)=\frac{c(\Lambda)}{c(\varkappa)}=1+(c_1(\Lambda)+t)+(c_2(\Lambda)+t\,c_1(\Lambda))+\dots,$$

where $t=-c_1(\varkappa)$. Moreover, the bundle $\varkappa$ restricted to the critical set f coincides with the kernel bundle $\ker df$ which justifies our notation for the bundle $\varkappa$. We shall use also the explicit form of the top Chern class of the bundle $\Lambda\otimes\varkappa^{*k}$:

$$c_{\ell+1}(\Lambda\otimes\varkappa^{*k})=\sigma_k,$$

where σ_k is expressed in the relative Chern classes $c_i(f)$ by (3).

The base B can be chosen arbitrary. We require only that the Chern classes $c_i(E)$, $i=1,\dots,\ell+1$, and $t=-c_1(\varkappa)$ are multiplicatively independent up to some high degree. This implies that the characteristic homomorphism $\mathcal{M}\to H^*(X)$ of Definition 3.2 is injective. It follows that our test map is representative enough to identify the residual polynomial of any multisingularity.

Let us compute the push-forward homomorphism f_*. Since f maps the subvariety given by the equation $x=0$ isomorphically to the subvariety given by the equation $\lambda_0=0$, we obtain that $f_*(c_1(\varkappa))=c_{\ell+1}(\Lambda)$ or $f_*(-t)=\sigma_0$. It follows from the projection formula that f_* is given by

$$f_*(a)=-\frac{\sigma_0}{t}a.$$

The right hand side of this expression is not always a polynomial. The visible contradiction is explained by the fact that f is *not* proper and the homomorphism f_* is not defined. However, the restriction of f to the hypersurface $x=0$ *is* proper so that f_* is defined on the polynomials divisible by t and is given on those by the formula above.

The multisingularity locus $X(\alpha)$ associated with the map f and the partition $\alpha=(a_1,\dots,a_r)$ is parameterized by the tuples

$$(u_1,\dots,u_r,\mu_{|\alpha|},\dots,\mu_n),\qquad u_i\in\varkappa,\quad \mu_i\in\Lambda\otimes\varkappa^{*i}.$$

The map $q^\alpha:X(\alpha)\to Y$ sends this tuple to the coefficients $(\lambda_0,\lambda_1,\dots\lambda_n)$ of the polynomial column

$$g(x)=(x-u_1)^{a_1}\dots(x-u_r)^{a_r}(\mu_{|\alpha|}+\dots+\mu_n x^{n-|\alpha|})=\lambda_0+\lambda_1x+\dots+\lambda_nx^n.$$

Similarly to the computation of the homomorphism f_* above we obtain by comparing the top Chern classes of the bundles $X(\alpha)\to B$ and $Y\to B$ the equality

$$(-t)^r n_\alpha=\sigma_0\sigma_1\dots\sigma_{|\alpha|-1}=\sigma_0\,R_{|\alpha|}.$$

Substituting this to (8) we obtain the following:

$$\exp\left(\sum_{i_1,i_2,\dots} f_*(R_{1_{i_1},2_{i_2},\dots})\frac{\tau_1^{i_1}}{i_1!}\frac{\tau_2^{i_2}}{i_2!}\dots\right)$$

$$=1+\sum_{i_1,i_2,\dots}\sigma_0(-t)^{-i_1-i_2-\dots}R_{i_1+2\,i_2+\dots}\frac{\tau_1^{i_1}}{i_1!}\frac{\tau_2^{i_2}}{i_2!}\dots.$$

Taking logarithm of both sides and applying the formula for f_* we obtain the required equality (11) of Theorem 2.7.

The technical difficulty appearing from the fact that f is not proper can be avoided by different means. For example, instead of the test map considered above one could construct in a similar way the test map produced from the local normal form of the singularity A_k given in Sect. 2.1. The map constructed in this way is proper and the homomorphism f_* for it is well defined. This map has less symmetries and the characteristic homomorphism for it is not injective. However, considering the series of such maps for all integer k is sufficient to restore the residual polynomial completely. We leave to the reader the details of this computation. □

4 Geometry of Morin maps

4.1 Proof of the iteration principle

In this section we prove Theorem 2.11 and some of its refinements. Since all statements of this theorem are local, it is sufficient to verify them for a germ of Morin map given by the normal form of some A_k-singularity. It will be convenient for us to use a local model for corank 1 maps which is slightly different from (1)–(2). Denote by $P_k \simeq \mathbb{C}^{(\ell+1)(k+1)}$ the space of polynomial maps $\mathbb{C} \to \mathbb{C}^{\ell+1}$ of the form

$$x \mapsto g(x) = \begin{pmatrix} x^{k+1} + \lambda_{0,k}x^p + \cdots + \lambda_{0,1}x + \lambda_{0,0} \\ \lambda_{1,k}x^k + \cdots + \lambda_{1,1}x + \lambda_{1,0} \\ \cdots \\ \lambda_{\ell,k}x^p + \cdots + \lambda_{k,1}x + \lambda_{\ell,0} \end{pmatrix}.$$

The coefficients $\lambda_{i,j}$ are the coordinates in P_k. Consider the mapping

$$\psi_k : \mathbb{C} \times P_{k-1} \to P_k$$

that maps the value $u \in \mathbb{C}$ and the column $g \in P_{k-1}$ of polynomials to the following column of polynomials:

$$\psi_k(u, g) = (x - u)g(x) \in P_k. \tag{23}$$

The components of ψ_k are the coefficients of the polynomial map obtained by expanding the product $(x - u)g(x)$.

Lemma 4.1 *The mapping ψ_k has a stable singularity A_k at the origin. Moreover, it is left-right equivalent to a one-dimensional extension of the mapping given by Eq.* (1)–(2)

Indeed, the additive group $\mathbb{C}$ acts by shifts of the argument x on the source and the target of ψ_k and ψ_k is equivariant with respect to this action. The induced mapping of the orbit spaces differs from the mapping (1)–(2) by an evident polynomial change of variables in the source. □

P r o o f o f T h e o r e m 2.11. By the lemma, we may assume without loss of generality that we have isomorphisms $X = \mathbb{C} \times P_{k-1}$, $Y = P_k$, and that the map $f = \psi_k : X \to Y$ is given by (23)

A column $h \in P_k$ of polynomials belongs to the image to the map ψ_k if it is divisible by a linear factor. If it is a common image of several points then it is divisible by several linear factors. More general, the column h is the image point of the multisingularity $\alpha = (a_1, \dots, a_r)$ if this column can be represented in the form

$$h(x) = (x - u_1)^{a_1} \dots (x - u_r)^{a_r} g(x), \qquad g \in P_{k-|\alpha|},$$

where all u_i are pairwise different. Passing to the closure, we see that the numbers $u_1, \dots, u_r$ and the coefficients of the polynomial column g can be taken as coordinates in the smooth variety $X(\alpha) \simeq \mathbb{C}^r \times P_{k-|\alpha|}$. This proves the first statement of Theorem 2.11. A more direct proof is given in [27].

The second statement is proved by similar arguments. If $f = \psi_k$ is a model map (23), then the source manifold $X(1, \alpha)$ of the derived map f^α parameterizes polynomial columns represented in the form

$$(x - u_0)(x - u_1)^{a_1} \dots (x - u_r)^{a_r} g(x), \qquad g \in P_{k-|\alpha|-1}.$$

The derived map f^α is given by expanding the coefficients in the product $(x - u_0)\, g(x)$ leaving the other linear factors unchanged. Comparing with (23) we see that f^α coincides with the model map $\psi_{k-|\alpha|}$ trivially extended with help of parameters $u_1, \dots, u_r$. The statement follows.

The other statements of Theorem 2.11 follow trivially from similar arguments. $\square$

Corollary 4.2 (of the proof). *Let $f : X \to Y$ be a proper Morin map. Then all varieties $X(\alpha)$ with fixed $|\alpha| = k$ can be identified as submanifolds in the manifold $X(1_k) = X(1, \dots, 1)$ (k units). Local singularities of the partition of $X(1_k)$ into submanifolds of the form $X(\alpha)$ are isomorphic to the singularities of the standard hyperplane arrangement A_k (cut out by the hyperplanes $u_i = u_j$ on the hyperplane $u_0 + \dots + u_k = 0$ in $\mathbb{C}^{k+1}$) multiplied by an affine space of appropriate dimension.*

4.2 Refined iteration principle

Theorem 2.11 admits the following refinement. Assume that the numbers a_i of the given partition $\alpha = (a_1, \dots, a_r)$ are divided into s subsets containing $k_1, \dots, k_s$ elements with $\sum k_i = r$, such that the elements of ith subset are equal to each other and are equal to b_i:

$$(a_1, \dots, a_r) = (\underbrace{b_1, \dots, b_1}_{k_1}, \dots, \underbrace{b_s, \dots, b_s}_{k_s}).$$

In this case the group $G = S(k_1) \times \dots \times S(k_s)$ of permutations inside each subsets consists of automorphisms of the partition α so it acts on the manifold $X(\alpha)$. Denote by $V(\boldsymbol{\beta})$ the quotient space with respect to this action where by

$$\boldsymbol{\beta} = (b_1, \dots, b_1; \dots; b_s, \dots, b_s) = ((b_1)_{k_1}; \dots; (b_s)_{k_s})$$

we denote the corresponding subdivided partition. The variety $V(\boldsymbol{\beta})$ is called the *refined multisingularity variety.* We do not require that all b_i are distinct. In particular, the usual multisingularity varieties $X(\alpha)$ correspond to the case when each subset contains exactly one element and the group G is trivial. All the statements of Theorem 2.11 are valid for the varieties $V(\boldsymbol{\beta})$ as well.

Theorem 4.3 *Assume that $f : X \to Y$ is a proper Morin map of holomorphic manifolds of relative dimension $\ell = \dim Y - \dim X$. Then the following statements hold:*

1. The refined multisingularity variety $V(\boldsymbol{\beta})$ is non-singular for any subdivided partition $\boldsymbol{\beta}$.

2. The natural projection $f^{\boldsymbol{\beta}} : V(1;\boldsymbol{\beta}) \to V(\boldsymbol{\beta})$ is a proper Morin map of the same relative dimension ℓ.

3. For any other subdivided partition $\boldsymbol{\beta}' = ((b'_1)_{k'_1};\dots;(b'_{s'})_{k'_{s'}})$ the multisingularity variety associated with the subdivided partition $\boldsymbol{\beta}'$ and the derived map $f^{\boldsymbol{\beta}}$ is naturally identified with the variety $V(\boldsymbol{\beta}',\boldsymbol{\beta})$ associated with the original map f, where $(\boldsymbol{\beta}',\boldsymbol{\beta}) = ((b'_1)_{k'_1};\dots;(b'_{s'})_{k'_{s'}};(b_1)_{k_1};\dots;(b_s)_{k_s})$.

P r o o f. The first statement is a manifestation of the well known fact that the orbit space of the reflection group A_k is smooth, or, in other formulation, that the symmetric powers of one-dimensional non-singular varieties are smooth. If $f = \psi_k$ is the model map (23), then the variety $V((b_1)_{k_1};\dots;(b_s)_{k_s})$ parameterizes polynomial maps represented in the form

$$x \mapsto \prod_{j=1}^{s}(x^{k_j} + \mu_{j,k_j-1}x^{k_j-1} + \dots + \mu_{j,0})^{b_j}\ g(x), g \in P_{k-|\boldsymbol{\beta}|},$$

where $|\boldsymbol{\beta}| = \sum k_j b_j$. For the coordinates on the smooth variety $V(\boldsymbol{\beta})$ we can take the coefficients $\mu_{j,i}$ and the coefficients of the polynomials forming the column g. This explicit presentation of the variety $V(\boldsymbol{\beta})$ implies all statements of Theorem 4.3 similarly to the proof of Theorem 2.11. $\square$

5 Residual intersections (after S.Kleiman)

The version of the residue intersection formula discussed in this section is basically due to S.Kleiman. We present it in a form more suitable for our applications. To avoid repetitions, we do not present the complete proof. Instead, we outline some arguments serving *to derive* this formula without rigorous justification. We hope that these arguments would help the reader to understand the details of the proof given in [20].

5.1 Residue intersection formula

Let $g : V \to Y$ be a holomorphic map of nonsingular complex varieties and $X \subset Y$ be a nonsingular subvariety. If g is transversal with respect to X then by the implicit function theorem $U = g^{-1}(X)$ is also nonsingular of the same codimension, $\operatorname{codim}_V U = \operatorname{codim}_Y X$.

$$\begin{array}{ccc} U & \longrightarrow & V \\ {\scriptstyle g|_U}\downarrow & & \downarrow{\scriptstyle g} \\ X & \longrightarrow & Y \end{array}$$

Moreover, the normal bundle ν_U of the submanifold $U \subset V$ and the fundamental class $[U]$ (in the Chow group A_*V or in the cohomology group $H^*(V)$) are the pull-backs of the corresponding normal bundle ν_X and the fundamental class $[X]$ for the submanifold $X \subset Y$,

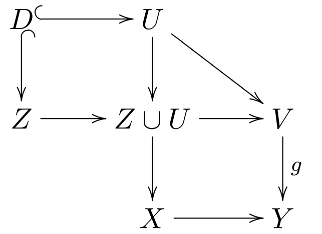
$$\nu_U = g^*\nu_X, \qquad [U] = g^*[X]. \tag{24}$$

The residue intersection formulas deal with the case when the transversality condition for the map g fails. Denote by $W = g^{-1}(X)$ the scheme-theoretical inverse image of X. Since W might be singular and even non-reduced it is more natural to use in our setup the Chow group of algebraic cycles modulo rational equivalence instead of cohomology. However, all relations of this section hold in the cohomology as well.

Assume that $W = g^{-1}(X)$ consists two components, $W = U \cup Z$, where U is smooth and has the expected dimension, $\operatorname{codim}_V U = \operatorname{codim}_Y X$. Then the relations (24) should be corrected by some 'residual' terms supported on Z. The structure of these terms depends on the structure of singularities of Z and $D = U \cap Z$. The assumptions used in the theorems of this section are as follows.

— *Z is nonsingular.* We denote by $\ell = \dim Z - \dim U$ the exceed of the dimension of Z with respect to the expected one.

— *the intersection $D = U \cap Z$ is a smooth divisor in U.* Denote by ζ the line bundle of the divisor $D \subset U$ and set $\theta = -[D] = c_1(\zeta^*) = -c_1(\zeta)$. The restriction of ζ to D is isomorphic to the normal bundle $\zeta|_D = \nu_D U$.



Our assumptions mean that near any point of D the variety W is given in appropriate coordinates by the equations

$$x_1 = \cdots = x_k = z\, y_1 = \cdots = z\, y_{\ell+1} = 0. \tag{25}$$

In these coordinates U and Z are given by $x = y = 0$ and $x = z = 0$, respectively.

Theorem 5.1 *Under the assumptions above, the Chern classes of the normal bundles ν_U and $g^*\nu_X$ on U differ by terms supported on D:*

$$c(\nu_U - g^*\nu_X) = c(\eta - \eta \otimes \zeta),$$

where η is the rank $\ell + 1$ bundle over $D = U \cap Z$ given by

$$\eta = \nu_D Z = \nu_D - \nu_Z|_D = (\nu_U - \nu_Z + \zeta)|_D.$$

The bundle η is defined on D only and does not extend necessary to the whole U. However, by Remark 2.14 the equality of the theorem makes sense. Substituting the expression for η to this equality and expanding the brackets we obtain the following equivalent relation:

$$c(\nu_U - \nu_Z + \zeta) = c((g^*\nu_X - \nu_Z + \zeta) \otimes \zeta^*) \tag{26}$$

(again, viewed in A_*U with a proper treatment).

Sketch of the proof. Our assumption on singularities of W implies that locally X admits an embedding $X \subset M \subset Y$ such that M is nonsingular, g is transversal with respect to M, and such that $Z \subset g^{-1}(M)$ is a divisor. In the local coordinates (25) such M can be given by the equation $x = 0$.

Assume that $M \subset Y$ is defined globally. In this case the bundle $\eta = \nu_D Z$ extends to U as the normal bundle of the embedding $U \subset g^{-1}(M)$, i.e. $\eta = \nu_U g^{-1}(M)$. The differential dg induces morphisms of vector bundles over U:

$$\nu_U/\eta \to g^*\nu_W, \qquad \eta \to g^*\nu_X W.$$

The first morphism is an isomorphism. The second one is an isomorphism outside D and it vanishes on D, which implies an isomorphism $\eta \otimes \zeta \simeq g^*\nu_X W$. Combining this together we obtain the following equality in the Grothendieck ring $K(U)$:

$$g^*\nu_X = g^*\nu_M + g^*\nu_X M = \nu_U - \eta + \eta \otimes \zeta.$$

The equality of the theorem follows.

The assumption on the global existence of M can be dropped. This can be justified by the arguments similar to those used in Section 3.1: we expect that the answer is a polynomial expression involving Chern classes of the normal bundles of the manifolds involved into the consideration, and the computation above determines this expression uniquely. An alternative direct approach is provided by the standard methods of intersection theory (blow up along Z then use Grothendieck's key formula etc.). The computations are more involved but they lead actually to the same formula, cf. [20]. □

Theorem 5.2 ([20]) *Under the assumptions above, the following relations hold in A_*W:*

$$[U] = h^*[X] - c_\ell(g^*\nu_X - \nu_Z) \frown [Z], \tag{27}$$

$$\theta^k \frown [U] = -c_{\ell+k}(g^*\nu_X - \nu_Z) \frown [Z], \qquad k \geq 1. \tag{28}$$

Before giving the proof, we make some general comments about residual intersections. Let $g : V \to Y$ be a holomorphic map, $X \subset Y$ be a submanifold, and $W = g^{-1}(X)$. Then the basic intersection formula gives

$$\{c(\nu_X) \frown s(W, V)\}_d = g^*[X], \tag{29}$$

$$\{c(\nu_X) \frown s(W, V)\}_k = 0, \qquad k < d, \tag{30}$$

where $d = \dim V - \operatorname{codim}_Y X$ is the 'expected' dimension of $W = g^{-1}(X)$. The key point of the left hand side expressions is that they involve the Segre classes $s(W, V)_k$ which are defined completely by the subscheme $W \subset V$ and do not depend explicitly on the map g.

The first formula (29) is actually taken as the definition of the 'intersection product' class $V \cdot X = g^*[X]$ in [8]. The second one (30) follows immediately from the basic construction of the intersection product. Since is not formulated explicitly in [8] we provide a proof here. By the 'deformation to the normal cone' principle it is sufficient to assume that Y is the space of projective bundle of the form $\pi : P(E \oplus \mathbf{1}) \to X$ over X where E is a vector bundle of rank n and $\mathbf{1} = \mathcal{O}_X$ is the trivial line bundle of rank one; $X \to Y$ is the embedding of the section $P(0 \oplus \mathbf{1})$ and $g : V \to Y$ is an embedding. The bundle E can be identified with the normal bundle ν_X of X. We shall denote by $c_i = c_i(E)$ the Chern classes themselves as well as the pull-backs of these classes to Y via π. Then the class dual to the subvariety $X \subset Y$ is given by

$$[X] = c_n(\mathrm{Hom}(O(-1), E)) = t^n + c_1\, t^{n-1} + \cdots + c_n, \qquad t = -c_1(O(1)).$$

Therefore,

$$V \cdot X = \pi_*((t^n + c_1\, t^{n-1} + \cdots + c_n) \frown [V]).$$

The right hand side is the degree $d = (\dim V - n)$ term in the class

$$\pi_*(\sum_{i,j} c_i t^j \frown [V]) = c(E) \frown \pi_*(\sum_j t^j \frown [V]) = c(E) \frown s(W, V).$$

The terms of degree higher than d vanish since $\sum_{i+j=k} c_i t^j$ is the kth Chern class of the rank n quotient vector bundle $\pi^*(E \oplus \mathbf{1})/\mathcal{O}(-1)$ which is trivial for $k > n$.

In practice, the total Segre class $s(W, V) = \sum_k s(W, V)_k$ is usually computed not from the definition but from the following two properties:

1) If W is nonsingular, then

$$s(W, V) = c(-\nu_W V) \frown [W].$$

In other words, the Segre class of the pair (W, V) is reduced in this case to the Segre class of the normal bundle $\nu_W V$. The same is applied if the possibly singular subscheme W is *regularly embedded* (that is W has pure codimension and admits only complete intersection singularities). In this case the normal bundle $\nu_W V$ is still defined and the formula above makes sense. In particular, if $W = H$ is a divisor then we have

$$s(H, V) = \frac{H}{1 + H} = \sum_{k=0}^{\infty} (-1)^k H^{k+1}.$$

2) If $\rho : \widetilde{V} \to V$ is a proper morphism of degree 1 of equally dimensional schemes and $\widetilde{W} = \rho^{-1} W \subset \widetilde{V}$ then

$$s(W, V) = \rho_* s(\widetilde{W}, \widetilde{V}).$$

These two properties determine the Segre classes completely. Indeed, blowing V along W one can reduce the problem of computation of Segre classes for any subsceme to the case of a divisor.

Sketch of the proof of Theorem 5.2. The direct computation provided by the methods of intersection theory involves a sequence of blowups reducing the problem to the case of a divisor. As a result of these computations we obtain that the classes $g^*[X] - [U]$ and $\theta^k \frown [U]$ have support on Z and are expressed as certain polynomials in the Chern classes of the normal bundles of manifolds involved into the consideration (this conclusion is quite clear from the topological point of view).

Instead of passing through these tedious computations we try to find the final expression by studying a slightly simplified situation. Namely, similarly to the proof of

Theorem 5.1 we assume that there exists a non-singular closed submanifold $M \subset Y$ containing X such that g is transversal with respect to M and such that $Z \subset g^{-1}(M)$ is a divisor. Observe that the classes $g^*[X]$, $\theta^k \frown [U]$, and $c_k(g^*\nu_X - \nu_Z)$ are determined by the restriction of g to $g^{-1}(M)$. Therefore, *without loss of generality, we can assume that $M = Y$, and $Z \subset V$ is a divisor.*

If $Z \subset V$ is a divisor, then the bundle ζ (defined originally on U) extends to V as the bundle of the divisor Z. The Segre class $s(W, V)$ is given in this case by

$$s(W,V) = s(Z,V) + c(-(\nu_U + \mathbf{1}) \otimes \zeta) \frown [U], \tag{31}$$

where $\mathbf{1} = \mathcal{O}_U$ is the trivial bundle of rank 1. Indeed, setting $n = \operatorname{codim}_V U$, we have

$$\begin{aligned} c(-(\nu_U + \mathbf{1}) \otimes \zeta) \frown [U] &= \sum_{j \geq 0} (1-\theta)^{-n-1-j} c_j(-\nu_\varphi) \frown [U] \\ &= \sum_{k \geq j \geq 0} \binom{n+k}{k-j} \theta^{k-j} \frown s(U,V)_{\dim U - j}. \end{aligned}$$

This is exactly the expression for the residue class $s(W,V) - s(Z,V)$ given in [8, Proposition 9.2].

Since $\nu_Z = \zeta$, Eq. (26) can be rewritten as $\nu_U \otimes \zeta = g^*\nu_X$. Therefore, Eq. (31) takes the form

$$c(-g^*\nu_X - \zeta) \frown [U] = s(W,V) - s(Z,V).$$

Multiplying both sides by $c(g^*\nu_X)$ we get

$$c(-\zeta) \frown [U] = c(\nu_X) \frown s(W,V) - c(g^*\nu_X - \nu_Z) \frown [Z].$$

Taking homogeneous terms of this equality we see that equations (27) and (28) of the theorem follow from (29) and (30), respectively. □

Remark 5.3 The assertions of Theorems 5.1 and 5.2 remain true if smooth submanifolds are replaced by regular embeddings. The standard technique of intersection theory necessary for the proof of this generalization is developed in [8], see also [20].

5.2 Characteristic classes for the derived map $f^1 : X(1,1) \to X$

Let $f : X \to Y$ be a Morin map. In this section we apply the theorems of the previous section to the study of the derived map $f^1 : X(1,1) \to X$ induced by the projection $X \times X \to X$ to the second factor.

Let us apply the residue intersection formula to the product map $g = f \times f : X \times X \to Y \times Y$ and the diagonal embedding $\Delta_Y \subset Y \times Y$. The inverse image of the diagonal $W = g^{-1}(\Delta_Y)$ consists of two components, the 'expected' one $U = X(1,1)$ and the residual one Z which is just the diagonal $Z = \Delta_X \subset X \times X$. The assumptions of the previous section are satisfied and we may apply Theorems 5.1 and 5.2. In order to identify the terms entering these theorems we need the following lemma.

The intersection $D = U \cap Z$ is the singularity variety $X(2)$. By Corollary 4.2, $X(2)$ can be viewed as a smooth hypersurface in $X(1,1)$. Denote by ζ the normal line bundle of the embedding $X(2) \to X(1,1)$. Since $X(2)$ is the critical set of f, it carries also the kernel line bundle $\varkappa = \ker df$.

Lemma 5.4 *The line bundles ζ and $\varkappa$ over $D = X(2)$ are naturally isomorphic.*

This lemma proves the second statement of Theorem 2.12.

P r o o f (D. Mond). The exact sequence

$$0 \to TX(2) \to T_{X(2)}X(1,1) \to \nu \to 0$$

admits a natural splitting. The construction of this splitting is as follows. Consider the involution on $X(1,1)$ induced by the permutation of the factors in $X \times X$. The subvariety $X(2) \subset X(1,1)$ is the locus of fixed points for this involution. Hence, the normal bundle ζ can be identified with the subbundle $\xi_{-1} \subset T_{X(2)}X(1,1)$ formed by eigenspaces with the eigenvalue -1.

Denote by $p : X(1,1) \to X$ the restriction of the projection $X^2 \to X$ to the first factor. It remains to verify that the derivative dp maps ξ_{-1} isomorphically to $\varkappa$. The last statement is local. It is sufficient to verify it for the sample map (23) which is quite easy.□

In view of this lemma all terms participating in Theorems 5.1 and 5.2 can be easily identified and we get (see details in [20] and in the next section)

$$\begin{aligned} c(\nu_{f^1}) &= c(\nu_f \otimes \varkappa^* + \mathbf{1} - \varkappa), \\ [X(1,1)] &= [X] \times_Y [X] - c_\ell \frown [X] \\ t^k \frown [X(1,1)] &= -c_{\ell+k} \frown [X] \qquad (k \geq 1), \end{aligned} \tag{32}$$

where $c_i = c_i(\nu_f)$.

Since $t\,[X(1,1)] = -[X(2)] = -c_{\ell+1}$ and $c_{\ell+1+k} = t^k\, c_{\ell+1}$, the last equality follows also from Proposition 3.4.

5.3 Characteristic classes for the derived map $f^\alpha : X(1,\alpha) \to X(\alpha)$

If the partition α consists of one element, $\alpha = (k)$, then geometrically the space of the fiber product $X \times_Y X(k)$ consists of two components, the 'expected' one $X(1,k)$, and the 'residual' one $X(k)$. Both components are nonsingular and intersect in the way satisfying the assumptions of Sect. 5.1. However, Theorems 5.1 and 5.2 are *not* applied directly. The reason is that as a scheme, the residual component is *not reduced.* One can verify by the local computations (see below) that it enters to $X \times_Y X(k)$ with the multiplicity k. In the case of a general partition $\alpha = (a_1, \dots, a_r)$ the situation is even worse: the residual scheme $Z = Z_1 \cup \dots \cup Z_r$ has several components entering with some multiplicities. Fortunately, in all these cases the residual component Z is *regularly embedded* to $X \times X(\alpha)$. According to Remark 5.3, this observation permits us to apply Theorems 5.1 and 5.2 and to complete the computation of the multisingularity classes.

In the lemma below we use the notations of Sections 2.4 and 2.5. In particular, we denote by $\theta = -c_1(\zeta)$ the negative of the first Chern class of the line bundle of the divisor $D = Z \cap X(1,\alpha) \subset X(1,\alpha)$, and by $\theta_i = -c_1(\zeta_i)$ a similar class on $Z_i \simeq X(a_i, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r)$ viewed as a subvariety of $X(1, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r)$.

Lemma 5.5 1. *The residual component $Z = Z_1 \cup \dots \cup Z_r$ of the fiber product scheme $X \times_Y X(\alpha)$ is regularly embedded to $X \times X(\alpha)$.*

2. *The component $Z_i \subset Z$ enters to the fundamental class of Z with the multiplicity a_i:*

$$[Z] = a_1\,[Z_1] + \cdots + a_r\,[Z_r].$$

3. *The restriction of the bundles ζ_i and ζ to $D_i = Z_i \cap X(1,\alpha)$ are related by*

$$\zeta|_{D_i} \simeq \zeta_i \otimes \varkappa^{a_i}, \qquad \theta = \theta_i + a_i t. \tag{33}$$

4. *The restriction of the normal bundle ν_Z to the component Z_i is related to the normal bundle ν_{Z_i} of this component by*

$$c(\nu_Z)|_{Z_i} = c(\nu_{Z_i} + \zeta - \varkappa) = c(\nu_{Z_i} + \zeta_i \otimes \varkappa^{a_i} - \varkappa). \tag{34}$$

Remark that the bundle $\varkappa$ is well defined on Z_i if $a_i > 1$. In the case $a_i = 1$ this bundle is defined only on a smaller its subvariety. However, one can verify that all classes participating in the relations of the lemma are correctly defined either by Remark 2.14 or by Proposition 3.4.

Proof. Since the assertions on the singularities of the scheme $W = X \times_Y V$ are local, it is sufficient to prove them for the model map $f = \psi_k$ given by (23). For this map the variety $V = X(\alpha)$ is parameterized by polynomial columns of the form

$$x \mapsto (x - u_1)^{a_1} \ldots (x - u_r)^{a_r}\, g(x), \quad g \in P_{p-|\alpha|},$$

and the scheme W admits an embedding to the space $M = \mathbb{C} \times \mathbb{C}^r \times P_{p-|\alpha|}$ with coordinates $(u, u_1, \ldots, u_r)$ on $\mathbb{C} \times \mathbb{C}^r$ as the subscheme given by the equations

$$(u - u_1)^{a_1} \ldots (u - u_r)^{a_r} g(u) = 0 \in \mathbb{C}^{\ell+1}.$$

This subscheme is reducible. It has components U given by the equation $g(u) = 0$ and Z which is a divisor in M with the equation $(u - u_1)^{a_1} \ldots (u - u_r)^{a_r} = 0$. This implies all assertions on the singularities of the schemes W, Z, and D.

The computation above implies that locally Z can be embedded as a divisor of the form $Z = a_1\,Z_1 + \cdots + a_r\,Z_r$ in a non-singular subvariety M in $X \times X(\alpha)$. Assume that M is defined globally. For example, if X, Y, and the map f are fibered over some base B with $\dim X - \dim B = 1$, then we can set $M = X \times_B X(\alpha)$.

If such M does exist then the line bundle $\varkappa$ can be extended from D_i to Z_i as the normal bundle of the divisor $Z_i \subset M$. Besides, identifying line bundles and the corresponding divisors on M we get

$$\zeta = [Z] = \sum_{j=1}^{r} a_j\,[Z_j], \quad \zeta_i = \sum_{j \neq i} a_j\,[Z_j] = [Z] - a_i[Z_i],$$

which implies (33).

Now, since the restriction to Z_i of the line bundles ζ and $\varkappa$ are the normal bundles of the embeddings $Z \subset M$ and $Z_i \subset M$, respectively, we have

$$\nu_M|_{Z_i} = (\nu_Z - \zeta)|_{Z_i} = (\nu_{Z_i} - \varkappa)|_{Z_i}$$

which leads to (34).

The proof above is carried out under the assumption on the existence of M. This assumption can be dropped by the arguments similar to those discussed in the proofs of Theorems 5.1 and 5.2. □

Proof of Theorems 2.12 and 2.16. Using the lemma we can identify the terms entering equalities of Theorems 5.1 and 5.2. In particular, we have

$$\begin{aligned}
\nu_U &= TX + TX(\alpha) - TX(1,\alpha) = TX + \nu_{f^\alpha}, \\
\nu_Z\big|_{Z_i} &= \nu_{Z_i} + \zeta - \varkappa = TX + TX(\alpha) - TZ_i + \zeta - \varkappa = TX + \zeta - \varkappa, \\
\nu_U - \nu_Z + \zeta &= \nu_{f^\alpha} + \varkappa, \\
(\nu_{\Delta_Y} - \nu_Z + \zeta)\big|_{Z_i} &= TY - TX + \varkappa = \nu_f + \varkappa.
\end{aligned}$$

Therefore, Eq. (26), (27), and (28) give, respectively,

$$\begin{aligned}
c(\nu_{f^\alpha}) &= c((\nu_f + \varkappa)\otimes\zeta^* - \varkappa), \\
[X(1,\alpha)] &= [X]\times_Y [X(\alpha)] - c_\ell(\nu_f + \varkappa - \zeta)\frown[Z] \\
\theta^k \frown [X(1,\alpha)] &= -c_{\ell+k}(\nu_f + \varkappa - \zeta)\frown[Z] \qquad (k \geq 1),
\end{aligned} \tag{35}$$

The first equation is actually the formula of the first assertion of Theorem 2.12. The second assertion of Theorem 2.12 is proved by Lemma 5.4. Thus, Theorem 2.12 is proved completely.

Now, by Lemma 5.5, we have

$$c_k(\nu_f + \varkappa - \zeta)\frown[Z] = \sum_{i=1}^{r} a_i\, c_k(\nu_f + \varkappa - \zeta_i\otimes\varkappa^{a_i})\frown[Z_i])$$

which proves the equalities of Theorem 2.16. □

For the last equation in (35) an independent proof can be given. Remark that the relations of Sect. 3.1 between Chern classes of Morin maps imply the equality

$$c_{\ell+k}(\nu_f + \varkappa - \zeta) = \theta^{k-1}\, c_{\ell+1}(\nu_f + \varkappa - \zeta),$$

where θ extends form $D \subset U$ to the ith component Z_i of Z as $\theta = \theta_i + a_i t$. Therefore, the required equation for arbitrary k is implied by this equation for $k = 1$.

In the case $k = 1$ we observe that

$$-\theta\frown[X(1,\alpha)] = [D] = a_1[D_1] + \cdots + a_r[D_r].$$

The inclusion $D_i \subset Z_i$ can be identified with the inclusion of the singularity loci $A_{a_{i-1}} \subset A_{a_i}$ for the derived map $f^{a_1,\dots,a_{i-1},a_{i+1},\dots,a_r}$. Therefore, by Corollary 2.13 the cohomology class dual to the submanifold $D_i \subset Z_i$ equals

$$\sigma(\theta_i + a_i t) = \sigma(\theta) = c_{\ell+1}((\nu_f + \varkappa)\otimes\zeta^*) = c_{\ell+1}(\nu_f + \varkappa - \zeta),$$

as required.

5.4 Solving recursion equations for the map $f^k : X(1,k) \to X(k)$.

In this section we apply the derived map f^k, $k \geq 1$, associated with a one part partition (k) to give a direct proof of Theorems 2.6 and 2.7. The map $q^{k,\alpha} : X(k,\alpha) \to Y$ associated to any partition $(k,\alpha) = (k, a_1, \dots, a_r)$ factors through the map $p^{k,\alpha} : X(k,\alpha) \to X$ induced by the projection $X^{r+1} \to X$ to the first factor. Let us set

$$m_{k,\alpha} = p_*^{k,\alpha}[X(k,\alpha)] \in H^*(X)$$

then we obtain evidently

$$n_{k,\alpha} = f_* m_{k,\alpha}. \tag{36}$$

The class $m_{k,\alpha}$ is supported on $X(k) \subset X$, and by the iteration principle, *$m_{k,\alpha}$ is equal to the target multisingularity class n_α associated with the derived map f^k*,

$$\begin{array}{ccccc} X(1,k) & \xrightarrow{f^k} & X(k) & \hookrightarrow & X \\ \downarrow{\scriptstyle p^{1,k}} & & \downarrow{\scriptstyle q^k} & \swarrow{\scriptstyle f} & \\ X & \xrightarrow{f} & Y & & \end{array}$$

The derived map f^k associated with a one part partition has a remarkable property: expressions for its Chern classes do not involve the extra class θ. By Theorem 2.12 the relative Chern classes of f^k are given by

$$c(\nu_{f^k}) = c((\nu_f + \varkappa) \otimes \varkappa^{*k} - \varkappa) = \frac{(1+k\,t)^{\ell+1}}{1-t} \left(1 + \frac{c_1 - t}{1+k\,t} + \frac{c_2 - t\,c_1}{(1+k\,t)^2} + \dots\right), \tag{37}$$

where $c_i = p^{1,k*}c_i(f)$ are the relative Chern classes of f lifted to $X(1,k)$ from X (not from $X(k)$). Denote by

$$\Psi_k : \mathbb{Z}[t, c_1, \dots, c_{\ell+1}] \to \mathbb{Z}[t, c_1, \dots, c_{\ell+1}]$$

the homomorphism that sends c_i to the ith homogeneous component of (37). Then, Theorem 2.16 (or, more precisely, equality (19)) leads to the following

Inductive rule for the computation of the class $m_{k,\alpha}$: *take the known expression for n_α, replace in this expression any class of the form $f_*(P)$ by*

$$f_*(P) \rightsquigarrow f^* f_*(P) + \Phi_k(P), \quad \textit{where} \quad \Phi_k(P) = \frac{\sigma_0 P - \Psi_k(\sigma_0 P)}{t},$$

and multiply the result by R_k.

This rule together with (36) is sufficient to compute by induction the class of any particular multisingularity. Before passing to explicit computations, recall that the Thom polynomial R_k of the singularity A_{k-1} is given by

$$R_k = \sigma_1 \dots \sigma_{k-1},$$

where $\sigma_i = c_{\ell+1}((\nu_f + \varkappa) \otimes \varkappa^{*i})$. The homomorphism Ψ_k acts on these classes by

$$\Psi_k(\sigma_i) = c_{\ell+1}((\nu_{f^k} + \varkappa) \otimes \varkappa^{*i}) = c_{\ell+1}((\nu_f + \varkappa) \otimes \varkappa^{*(k+i)}) = \sigma_{k+i}$$

which implies the equality

$$R_k\Psi_k(\sigma_0 R_i) = \sigma_1 \dots \sigma_{k-1}\sigma_k \dots \sigma_{k+i-1} = R_{k+i} \tag{38}$$

valid for any k and i.

Therefore, for partitions with small number of parts the inductive rule gives:

$$\begin{aligned}
m_r &= R_r; \qquad n_r = f_*(m_r) = f_*(R_r);\\
m_{q,r} &= R_q\Big(f^*f_*(R_r) + \Phi_q(R_r)\Big) = R_q f^*f_*(R_r) + R_{q,r}, \quad \text{where}\\
R_{q,r} &= R_q\Phi_q(R_r) = R_q\frac{1}{t}(\sigma_0 R_r - \Psi_q(\sigma_0 R_r)) = \frac{1}{t}(\sigma_0 R_q R_r - R_{q+r});\\
n_{q,r} &= f_*(m_{q,r}) = f_*(R_q)f_*(R_r) + f_*(R_{q,r});\\
m_{p,q,r} &= R_p\Big(\Big(f^*f_*(R_q) + \Phi_p(R_q)\Big)\Big(f^*f_*(R_r) + \Phi_p(R_r)\Big) + f^*f_*(R_{q,r}) + \Phi_p(R_{q,r})\Big)\\
&= R_p f^*f_*(R_q)f^*f_*(R_r) + R_{p,q}f^*f_*(R_r) + R_{p,r}f^*f_*(R_q) + R_p f^*f_*(R_{q,r}) + R_{p,q,r},\\
n_{p,q,r} &= f_*(m_{p,q,r})\\
&= f_*(R_p)f_*(R_q)f_*(R_r) + f_*(R_{p,q})f_*(R_r) + f_*(R_{p,r})f_*(R_q) + f_*(R_p)f_*(R_{q,r})\\
&\quad + f_*(R_{p,q,r}), \qquad \text{where}\\
R_{p,q,r} &= R_p\Phi_p(R_q)\Phi_p(R_r) + R_p\Phi_p(R_{q,r});\\
&\quad \text{etc.}
\end{aligned}$$

Substituting the known expressions for Φ_p, $R_{q,r}$, expanding brackets and applying (38) we obtain the expression for $R_{p,q,r}$ as in Theorem 2.7. Continuing this way, we get a uniquely determined formula for any classes n_α as a polynomial expression in the classes of the form $f_*(P)$ where P is a polynomial in $t, c_1, \dots, c_{\ell+1}$. It remains to identify these polynomials. In the computations below we admit formally rational functions with some power of t in the denominator. Since the final answer is unique, it will automatically contain only polynomials.

Proof of Theorems 2.6 and 2.7. To organize the computation it is convenient to introduce the following generating functions considered as formal series in the supplementary variables $\tau_1, \tau_2, \dots$:

$$\begin{aligned}
\mathcal{Q} &= 1 + \sigma_0 \sum_{i_1,i_2,\dots} (-t)^{-i_1-i_2-\dots} R_{i_1+i_2+\dots}\frac{\tau^{i_1}}{i_1!}\frac{\tau^{i_2}}{i_2!}\dots\\
&= 1 - \sigma_0\frac{\tau_1}{t} + \sigma_0\sigma_1\Big(\frac{\tau_1^2}{2\,t^2} - \frac{\tau_2}{t}\Big) + \sigma_0\sigma_1\sigma_2\Big(-\frac{\tau_1^3}{3!\,t^3} + \frac{\tau_1\tau_2}{t^2} - \frac{\tau_3}{t}\Big) + \dots;\\
\mathcal{R} &= -\frac{t}{\sigma_0}\log(\mathcal{Q}),\\
\mathcal{N} &= \exp(f_*(\mathcal{R})),\\
\mathcal{M}_k &= f^*(\mathcal{N})\,\frac{\partial\mathcal{R}}{\partial\tau_k} = \exp\Big(f^*f_*(\mathcal{R})\Big)\,\frac{\partial\mathcal{R}}{\partial\tau_k}.
\end{aligned}$$

By the equations (38) the partial derivatives of $\mathcal{Q}$ are given by

$$\frac{\partial\mathcal{Q}}{\partial\tau_k} = -\frac{\sigma_0}{t}\,R_k\,\Psi_k(\mathcal{Q}).$$

This yields
$$\frac{\partial \mathcal{R}}{\partial \tau_k} = -\frac{t}{\sigma_0}\,\mathcal{Q}^{-1}\,\frac{\partial \mathcal{Q}}{\partial \tau_k} = R_k \exp\Bigl(\frac{\sigma_0\mathcal{R} - \Psi_k(\sigma_0\mathcal{R})}{t}\Bigr).$$
This can be viewed as a recursive equation for the coefficients of $\mathcal{R}$. It provides a formal proof of the fact that these coefficients are indeed polynomials in c_i and t. Substituting this partial derivative to the definition of $\mathcal{M}_k$ we get

$$\mathcal{M}_k = R_k\,\exp\Bigl(f^*f_*(\mathcal{R}) + \frac{\sigma_0\mathcal{R} - \Psi_k(\sigma_0\mathcal{R})}{t}\Bigr).$$

We see that the coefficients of the series $\mathcal{M}_k$ are obtained from the coefficients of the series $\mathcal{N} = \exp(f_*(\mathcal{R}))$ by applying the inductive rule of this section. In addition, by the projection formula we get

$$f_*(\mathcal{M}_k) = \exp\Bigl(f_*(\mathcal{R})\Bigr)\,\frac{\partial f_*(\mathcal{R})}{\partial \tau_k} = \frac{\partial}{\partial \tau_k}\exp\Bigl(f_*(\mathcal{R})\Bigr) = \frac{\partial \mathcal{N}}{\partial \tau_k}$$

which is equivalent to (36). It follows by induction that $\mathcal{M}_k$ and $\mathcal{N}$ are exactly the generating functions for multisingularity classes in the source and target manifolds, respectively, introduced in Sect. 2.3.

The equality $\mathcal{R} = -\frac{t}{\sigma_0}\log(\mathcal{Q})$ is equivalent to the formula of Theorem 2.7. Equalities of Theorem 2.6 are obtained as byproducts of our computations. □

5.5 Solving recursion equations for the map $f^\alpha : X(1,\alpha) \to X(\alpha)$.

Let $(a_0,\alpha) = (a_0,a_1,\dots,a_r)$ be a partition, $p^{a_0,\alpha} : X(a_0,\alpha) \to X$ be the map induced by the projection $X^{r+1} \to X$ to the first factor and $q^{a_0,\alpha} = f\,p^{a_0,\alpha} : X(a_0,\alpha)$ be induced by f. The multisingularity classes

$$m_{a_0,\alpha} = p_*^{a_0,\alpha}(1) \in H^*(X) \quad \text{and} \quad n_{a_0,\alpha} = f_*(m_{a_0,\alpha}) = q_*^{a_0,\alpha}(1) \in H^*(Y)$$

can also be computed directly from the recursive formulas (20) and (21). The advantage of this approach is that we do not need to use explicitly the relative Chern classes of the map f^α but only those for the map f pulled back to $X(a_0,\alpha)$. Instead, we have to add to the consideration the additional class θ. Classes of the form

$$p_*^{a_0,\alpha}(\theta^k) \in H^*(X)$$

are called *derived* or *higher* multisingularity classes. By the inductive formula (20) every class of this form is represented as a polynomial combination of the classes of the form $f^*f_*(P)$, where P is a polynomial in the relative Chern classes c_i of f, and also in the classes c_i themselves.

Definition 5.6 The *residual polynomial associated with the higher multisingularity class* $p_*^{a_0,\alpha}(\theta^k)$ is the sum of terms not containing classes of the form $f_*f^*(P)$ in the universal formula for $p_*^{a_0,\alpha}(\theta^k)$. In other words, the residual polynomial is obtained from the universal formula for $p_*^{a_0,\alpha}(\theta^k)$ by setting $f^*f_*(P) = 0$ for every term of this form.

In terms of the multisingularity class $q_*^{a_0,\alpha}(\theta^k)$ on the target, the residual polynomial corresponds to the terms which are *linear* in the classes of the form $f_*(P)$ in the universal formula of $q_*^{a_0,\alpha}(\theta^k)$.

In particular, the residual polynomials participating in Theorem 2.7 are those associated with the usual multisingularity classes. It follows that only the second term of (20) gives the contribution to the residual polynomial. This results to the following procedure for its computation. Let $\sigma(\theta)$ and $\widetilde{R}_k(\theta)$ be as in Corollary 2.13. Define inductively polynomials $\widetilde{R}_\alpha(\theta)$ for arbitrary $\alpha = (a_1, \ldots, a_r)$ with $r > 1$ by the recursive formula

$$\widetilde{R}_{a_1,a_2,\ldots,a_r}(\theta) = -\sum_{i=2}^{r} a_i\, \widetilde{R}_{a_i}(\theta)\, \frac{\sigma(\theta + a_i t)\, \widetilde{R}_{\alpha_i}(\theta + a_i t) - \sigma(0)\, \widetilde{R}_{\alpha_i}(0)}{\theta + a_i t} \quad (r > 1), \qquad (39)$$

where $\alpha_i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r)$.

Theorem 5.7 *The residue polynomials for multisingularities of corank one maps are given by*

$$R_\alpha = \widetilde{R}_\alpha(0).$$

The fact that the terms involving classes of the form $f_*(P)$ in the resulting expression for $m_{k,\alpha} = p_*^{k,\alpha}(1)$ are organized in the form prescribed by the formula (10) can also be easily proved by induction in this approach.

Remark 5.8 It is possible to rewrite the equation (39) in terms of generating functions. For that we introduce the series

$$\widetilde{\mathcal{R}}_k(\theta) = \sum_{i_1,i_2,\ldots} \widetilde{R}_{k,1_{i_1},2_{i_2},\ldots}(\theta) \frac{\tau_1^{i_1}}{i_1!} \frac{\tau_2^{i_2}}{i_2!} \cdots.$$

The residue polynomials of Theorem 2.6 can be recovered from this series by setting $\theta = 0$,

$$\widetilde{\mathcal{R}}_k(0) = \frac{\partial \mathcal{R}}{\partial \tau_k},$$

and by Theorem 5.7 the series itself is determined by the relation

$$\widetilde{\mathcal{R}}_k(\theta) = \widetilde{R}_k(\theta) + \sum_{a=1}^{\infty} a\, \widetilde{R}_a(\theta)\, \frac{\sigma(\theta + a\,t)\, \widetilde{\mathcal{R}}_k(\theta + a\,t) - \sigma(0)\, \widetilde{\mathcal{R}}_k(0)}{\theta + a\,t}\, \tau_a.$$

It follows, in particular, that all polynomials entering the partial derivative $\frac{\partial \mathcal{R}}{\partial \tau_k}$ are defined over integers for every $k \geq 1$ (in opposite to the series $\mathcal{R}$ itself whose coefficients are in general rational).

6 Refinements

6.1 Refined integer relations for the source multisingularity classes

The universal formula for a particular multisingularity class n_α followed by Eq. (8) and (11) involves integer coefficients and is valid over integers in spite of rational coefficients in the expansion of the exponent and the logarithm. This follows from the recursive procedures for obtaining this formula outlined in Sect. 5.4 and 5.5. However, the profit of using this integer relation is not so high since the classes involved in it have usually large integer factors. In this section we explain how to get rid of these redundant integer factors. We borrowed the idea of this refinement from [21], where it is outlined for the case of classes of multiple points.

Let $f : X \to Y$ be a proper Morin map, $(k,\alpha) = (k, a_1, \dots, a_r)$ be a partition and $X(k,\alpha)$ be the corresponding multisingularity variety. Denote by

$$\overline{m}_{k,\alpha} \in H^*(X), \qquad \overline{n}_{k,\alpha} \in H^*(Y)$$

the classes dual to the target varieties $p^{k,\alpha}(X(k,\alpha)) \subset X$ and $q^{k,\alpha}(X(k,\alpha)$, respectively, considered with their *reduced* structures. From the known multiplicities of the mappings $p^{k,\alpha}$ and $q^{k,\alpha}$ to their images we get

$$m_{k,\alpha} = |\mathrm{Aut}(\alpha)|\,\overline{m}_{k,\alpha}, \qquad n_{k,\alpha} = |\mathrm{Aut}(k,\alpha)|\,\overline{n}_{k,\alpha}.$$

The class $\overline{m}_{k,\alpha}$ determines $\overline{n}_{k,\alpha}$ not uniquely but up to a torsion of order s,

$$f_*(\overline{m}_{k,\alpha}) = s\,\overline{n}_{k,\alpha},$$

where $s = |\mathrm{Aut}(k,\alpha)|/|\mathrm{Aut}(\alpha)|$ is the number of occurrences of k in the multi-index $(k, a_1, \dots, a_r)$.

The coefficients of the series $\mathcal{N}$ and $\mathcal{M}_k$ introduced in Sect. 2.3 are defined over integers,

$$\mathcal{N} = \sum_{i_1,i_2,\dots} \overline{n}_{1_{i_1},2_{i_2},\dots} \tau_1^{i_1}\tau_2^{i_2}\dots, \qquad \mathcal{M}_k = \sum_{i_1,i_2,\dots} \overline{m}_{k,1_{i_1},2_{i_2},\dots} \tau_1^{i_1}\tau_2^{i_2}\dots.$$

Besides, we shall consider the series $\partial\mathcal{R}/\partial\tau_k$ whose coefficients are polynomials in $c_1, \dots, c_{\ell+1}, t$ of the form

$$\overline{R}_{k,\alpha} = \frac{1}{|\mathrm{Aut}(\alpha)|}\,R_{k,\alpha}.$$

Remark that if β is obtained from α by a permutation of entries, then we have $R_\alpha = R_\beta$ while the corresponding reduced polynomials $\overline{R}_\alpha$ and $\overline{R}_\beta$ differ by a rational factor.

Theorem 6.1 *The reduced residual polynomials $\overline{R}_{k,\alpha}$ have only integer coefficients and the equality*

$$\mathcal{M}_k = \frac{\partial\mathcal{R}}{\partial\tau_k}\,f^*(\mathcal{N})$$

holds over integers.

Over rationals this equality is proved in Sect. 5.4. To prove it over integers, we present an algorithm to express the reduced multisingularity class $\overline{m}_{k,\alpha}$ as linear combination of the classes of the form $f^*(\overline{n}_\beta)$ for various β whose coefficients are polynomials in the relative Chern classes $c_i = c_i(f)$ of the map f. Since the desired representation is unique (up to the ideal K of relations between Chern classes c_i of Morin maps), we get that the obtained expression for $\overline{m}_{k,\alpha}$ must be equal to that of Theorem 6.1. It is not difficult also to prove the theorem directly from the recursive relations presented below.

Denote by $U(k,\alpha) = X(k,\alpha)/\mathrm{Aut}(\alpha)$ and $V(\alpha) = X(\alpha)/\mathrm{Aut}(\alpha)$ the refined multisingularity loci considered in Sect. 4.2. Then we have

$$\overline{m}_{k,\alpha} = \overline{p}^{k,\alpha}_*(1), \qquad \overline{n}_\alpha = \overline{q}^{\alpha}_*(1),$$

where $\overline{p}^{k,\alpha} : U(1,\alpha) \to X$ and $\overline{q}^\alpha : V(\alpha) \to Y$ are induced by $p^{k,\alpha}$ and q^α, respectively. The variety $U(1,\alpha)$ can be identified as the 'principal part' of the fiber product space $X \times_Y V(\alpha)$. Local computations similar to those carried out in Sect. 5.3 show that the residual component of this fiber product is regularly embedded and Theorem 5.2 is applied. As a result we arrive at the following recursive relations similar to those of Theorem 2.16 and its corollary:

$$P(\theta) \frown U(1,\alpha) = (P(0) \frown [X]) \times_Y [V(\alpha)] - \sum_i a_i\, \Phi(\theta + a_i t) \frown [Z_i],$$

$$\overline{p}^{1,\alpha}_*(P(\theta)) = P(0) \cdot f^*(\overline{n}_\alpha) - \sum_i a_i\, \overline{p}^{\alpha_i}_*(\Phi(\theta + a_i t)),$$

where $\alpha_i = (a_i, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r)$, $Z_i = U(\alpha_i)$,

$$\Phi(\theta) = \frac{\sigma(\theta)P(\theta) - \sigma(0)P(0)}{\theta},$$

and where the sum runs over the set of *pairwise different* entries a_i in the partition α. Theorem 6.1 follows from these relations by induction. The equation for the (reduced) residual polynomials obtained in this way is equivalent to that of Remark 5.8 □

Remark 6.2 In the case when the partition α has the form $\alpha = (1, \ldots, 1) = (1_r)$ the corresponding varieties $U(1_{r+1}) = X(1_{r+1})/S(r)$ and $V(1_r) = X(1_r)/S(r)$ have been used in [21] by S.Kleiman in his 'method of Hilbert schemes'.

6.2 Maps possessing higher singularities

The residue polynomials R_α in the relative Chern classes $c_i = c_i(\nu_f)$ are introduced in this paper in such a way that they are defined modulo the ideal K introduced in Sect. 3.1. In the equivalence class of polynomials representing the same element $R_\alpha \in \mathcal{M}$ there is a preferable representative: according to [16], *with an appropriate choice of polynomials* R_α*, the formulas* (7)–(11) *hold true for any generic map, possessing possibly the singularities of corank greater than one.*

Denote by $R^{\mathrm{actual}}_\alpha$ the representative for R_α suggested by the statement above and by $R^{\mathrm{naive}}_\alpha$ the polynomial obtained form (11) by expanding all brackets, and then by applying the equality $t^k c_{\ell+1} = c_{\ell+1+k}$ once to each monomial: if

$$R_\alpha = c_{\ell+1}(R^0_\alpha + R^1_\alpha\, t + R^2_\alpha\, t^2 + \ldots)$$

with R^i_α free of t then we set

$$R^{\mathrm{naive}}_\alpha = R^0_\alpha\, c_{\ell+1} + R^1_\alpha\, c_{\ell+2} + R^2_\alpha\, c_{\ell+3} + \ldots.$$

Then obviously we have $R^{\mathrm{actual}}_\alpha = R^{\mathrm{naive}}_\alpha$ if the degree of the polynomials is less then $2(\ell+2)$ for K is trivial in those degrees. For greater degree the difference $R^{\mathrm{actual}}_\alpha - R^{\mathrm{naive}}_\alpha \in K$ is not trivial. The method for finding the residue polynomials $R_\alpha^{\rm actual}$ is described in [16]. For the classes of small codimension and $\ell = 0, 1$ the corresponding 'correction term' is presented below. In the table we use the notation $d_{i,j} = c_i c_j - c_{i-1}c_{j+1}$.

$\ell = 0$		$\ell = 1$	
α	$R_\alpha^{\rm actual} - R_\alpha^{\rm naive}$	α	$R_\alpha^{\rm actual} - R_\alpha^{\rm naive}$
(5)	$2\,d_{2,2}$	(4)	$d_{3,3}$
$(3,3)$	$-12\,d_{2,2}$	$(1,1,3)$	$18\,d_{3,3}$
$(2,4)$	$-8\,d_{2,2}$	$(1,2,2)$	$24\,d_{3,3}$
$(2,2,3)$	$48\,d_{2,2}$	$(1_4,2)$	$408\,d_{3,3}$
(2_4)	$-288\,d_{2,2}$	(1_7)	$12240\,d_{3,3}$
(6)	$10\,c_1d_{2,2} + 12\,d_{2,3}$	$(1,4)$	$-12\,c_1d_{3,3} - 28\,d_{3,4}$
$(3,4)$	$-72\,c_1d_{2,2} - 72\,d_{2,3}$	$(2,3)$	$-12\,c_1d_{3,3} - 30\,d_{3,4}$
$(2,5)$	$-60\,c_1d_{2,2} - 60\,d_{2,3}$	$(1_3,3)$	$-366\,c_1d_{3,3} - 576\,d_{3,4}$
$(2,3,3)$	$504\,c_1d_{2,2} + 432\,d_{2,3}$	$(1,1,2,2)$	$-440\,c_1d_{3,3} - 656\,d_{3,4}$
$(2,2,4)$	$432\,c_1d_{2,2} + 384\,d_{2,3}$	$(1_5,2)$	$-14640\,c_1d_{3,3} - 18480\,d_{3,4}$
$(2_3,3)$	$-3648\,c_1d_{2,2} - 2880\,d_{2,3}$	(1_8)	$-700560\,c_1d_{3,3} - 776160\,d_{3,4}$
(2_5)	$31488\,c_1d_{2,2} + 23040\,d_{2,3}$		

6.3 Morin maps of negative relative dimension

Let $f : X \to Y$ be a generic holomorphic map of non-positive relative dimension $\ell = \dim Y - \dim X \le 0$. We say that f is a *Morin map* if the following two conditions are satisfied:

1) The rank of $\operatorname{coker} df$ does not exceed 1 for any point in X. The genericity condition implies that the critical locus Σ of f is smooth and has the codimension $1-\ell$ in X.

2) The restriction of f to Σ is a corank one map.

The list of possible local singularities of Morin maps is exhausted by the stable singularities of the type A_k, $k \ge 1$ (or $\Sigma^{1-\ell,1,\dots,1}$ in the Boardman notation) with the following normal form:

$$(x, y_1, \dots, y_{-\ell}, \lambda) \mapsto (x^{k-1} + y_1^2 + \dots + y_{-\ell}^2 + \lambda_{k-1}x^{k-1} + \dots + \lambda_1 x, \lambda).$$

Denote by the tuple $\alpha = (a_1, \dots, a_r)$ the type of the multisingularity $(A_{a_1-1}, \dots, A_{a_r-1})$. Since the singularity A_k is determined for $k \ge 1$ only, we assume that $a_i \ge 2$ for all i. By the same reason, we do not include τ_1 in the generating functions $\mathcal{N}^-$ and $\mathcal{M}_k^-$ for the multisingularity classes of Morin maps with $\ell \le 0$:

$$\mathcal{N}^- = 1 + \sum_{i_2,i_3,\dots} n_{2_{i_2},3_{i_3},\dots} \frac{\tau_2^{i_2}}{i_2!}\frac{\tau_3^{i_3}}{i_3!}\dots,$$

$$\mathcal{M}_k^- = m_k + \sum_{i_2,i_3,\dots} m_{k,2_{i_2},3_{i_3},\dots} \frac{\tau_2^{i_2}}{i_2!}\frac{\tau_3^{i_3}}{i_3!}\dots.$$

The induced map

$$f^\Sigma : \Sigma \to Y$$

is a corank one map of positive relative dimension $\dim Y - \dim\Sigma = 1$ for any $\ell \le 0$. However, it is *not* a Morin map since it is highly degenerate considered as a usual holomorphic map. This is due to the fact that this map is *Legendrian*: it factors through Legendrian embedding $\Sigma \hookrightarrow PT^*Y$, see [3], [17]. Therefore, Theorems 2.6 and 2.7 cannot be applied to f^Σ directly.

Consider first the case $\ell = 0$ of maps of equidimensional manifolds. In this case f is finite and the results of this paper are applied. The residual polynomials of multisingularities are supported on Σ and expressed in terms of two variables, c_1 and t. Let us make a change of coordinates and replace c_1 by $u = \sigma_0 = c_1 - t$. With this notation our formulas for the multisingularity classes can be written as follows:

$$\begin{aligned} \mathcal{N}^- &= \exp(f^\Sigma_*(\mathcal{R}^-)), \\ \mathcal{M}^-_k &= f^{\Sigma *}(\mathcal{N}^-)\,\frac{\partial \mathcal{R}^-}{\partial \tau_k} = \exp(f^{\Sigma *} f^\Sigma_*(\mathcal{R}^-))\,\frac{\partial \mathcal{R}^-}{\partial \tau_k}, \\ \frac{\partial \mathcal{N}^-}{\partial \tau_k} &= f^\Sigma_*(\mathcal{M}^-_k), \end{aligned} \tag{40}$$

where

$$\begin{aligned} \mathcal{R}^- &= \frac{1}{\sigma_0\sigma_1}\log\left(1 + \sigma_0\sigma_1 \sum_{i_2,i_3,\dots} \sigma_2\sigma_3\dots\sigma_{(2i_2+3\,i_3+\dots)-1}(-t)^{1-(i_2+i_3+\dots)}\frac{\tau_2^{i_2}}{i_2!}\frac{\tau_3^{i_3}}{i_3!}\dots\right), \\ \sigma_k &= u + k\,t. \end{aligned}$$

The homomorphism f^Σ_* decomposes as $f^\Sigma_* = f_* j_*$, where $j : \Sigma \to X$ is the embedding. By Proposition 3.4, the homomorphism j_* is given by $j_* t^k = c_1\,t^k = c_{k+1}$. Remark that u is the first Chern class of the *cokernel line bundle* $\operatorname{coker} df = (\ker df^*)^*$ defined over Σ. It follows that j_* is determined also equivalently by

$$j_*(u^k) = c_{k+1}(-\nu^*_f), \qquad c(-\nu^*_f) = \frac{1}{1 - c_1 + c_2 - \dots}.$$

Theorem 6.3 *Let $f : X \to Y$ be a Morin map of arbitrary non-positive relative dimension $\ell = \dim Y - \dim X \le 0$ such that the restriction of f to the critical variety $\Sigma \subset X$ is proper. Then all relations* (40) *hold true, if we set $u = c_1(\operatorname{coker} df)$ and*

$$t = \frac{1}{2}c_1(f^\Sigma) - u = (\tfrac{\ell}{2} + 1)\,c_1(f) - u.$$

The only difference in the case $\ell < 0$ is the following: if f itself is proper then f^Σ_* also decomposes as $f^\Sigma_* = f_* j_*$, but in this case the homomorphism j_* acts as

$$j_*(u^k) = c_{-\ell+1+k}(-\nu^*_f).$$

S k e t c h o f t h e p r o o f. As we mentioned above, the corank one map f^Σ is not generic. However, considered as a Legendrian mapping it is generic and acquires only standard Legendrian singularities A_k for different k. Thus, the theory of Morin maps of non-positive relative dimension is essentially the theory of Legendrian Morin maps.

The theory of Legendrian mappings and related to them characteristic classes is parallel to the general theory of Thom polynomials, see [15], [16]. The iteration principle extends to the case of Legendrian corank one singularities and leads to the family of derived

Legendrian maps $X(2,\alpha) \to X(\alpha)$. In the case of *real* corank one Legendrian mappings a version of this principle is formulated in [33]. The whole theory of this paper can be repeated in the Legendrian case. As a result, we obtain an algorithm for computing classes of multisingularities for Legendrian Morin maps. The residue polynomials in this theory are expressed as universal polynomials in the classes $u = c_1(\mathcal{O}_{PT^*Y}(1))$ and $c_1(f^\Sigma)$. The formula obtained in this way can be applied as well to the case of the Legendrian mapping f^Σ associated with a usual Morin map of equidimensional manifolds. Due to the universality, the formula which is valid for this particular case is valid also for any generic Legendrian mapping, in particular, for that arising from a Morin map f of any non-positive relative dimension ℓ. □

Remark 6.4 The equalities $t = (\frac{\ell}{2}+1)c_1(f) - u$ and $j_*(u^k) = c_{-\ell+1+k}(-\nu_f^*)$ determine the residual classes R_α as polynomials in the relative Chern classes $c_i(f)$. In fact, these polynomials are determined not uniquely but up to the ideal of relations satisfied for any Morin map of relative dimension $\ell \leq 0$. This ideal is generated by the classes 'supported on the loci of singularities more complicated than the Morin ones'. For this ideal an explicit description can be given, however, it is not so nice and we do not describe it here.

Remark also that the appearance of $1/2$ in the case when ℓ is odd is due to our choice for the representative of R_α modulo relations. In fact, all classes R_α are defined over integers.

7 Applications

Corank one maps between equally dimensional manifolds ($\ell = 0$) appearing in applications are usually decomposed as

$$X \stackrel{j}{\to} W \stackrel{\pi}{\to} Y, \tag{41}$$

where j is an embedding of codimension 1 and π is a locally trivial fibration with one-dimensional fibers. Since the number of regular preimages over any multisingularity stratum in Y is determined by the degree of f, we exclude parts of length 1 from the notation of multisingularity classes. Respectively, we exclude the variable τ_1 from the considered generating series. All residual classes are expressed in terms of two generators. For these generators we take $t = -c_1(\varkappa)$ where $\varkappa = TW/Y$ is the relative tangent line bundle of the fibration, and $u = i_*(1) = \sigma_0 = c_1(f) - t$, the class of the divisor $X \subset W$. Set

$$\begin{aligned}\mathcal{S}(u,t) &= j_*(\mathcal{R}) = u\,\mathcal{R} \\ &= -t\,\log\Big(1 + \sum_{i_2,i_3,\dots} (-t)^{-i_2-i_3-\dots}\Big(\prod_{j=0}^{(2\,i_2+3\,i_3+\dots)-1}(u+j\,t)\Big)\frac{\tau_2^{i_2}}{i_2!}\frac{\tau_3^{i_3}}{i_3!}\dots\Big) \\ &= u(u+t)\tau_2 + u(u+t)(u+2t)\tau_3 - u(u+t)(2u+3t)\tau_2^2 \dots.\end{aligned} \tag{42}$$

The coefficients of this series are homogeneous polynomials in u and t. The principle formula (8) gives

$$\mathcal{N} = \exp(f_*\mathcal{R}) = \exp(\pi_*\mathcal{S}).$$

Thus, to complete the computation of the multisingularity classes it remains to describe the action of the homomorphism π_* on every monomial in u and t. This action depends on the particular geometric problem.

7.1 Hilbert's and de Jonquieres' formulas

Denote by $Y_d \simeq \mathbb{C}P^d$ the projectivization of the space of degree n binary forms (homogeneous polynomials in two variables). Every form Φ that can be factorized as

$$\Phi = L_1^{a_1} \dots L_r^{a_r},$$

where L_i is linear for all i. The partition $(a_1, \dots, a_r)$ of the number d is called *factorization type* of Φ. The degree of the variety $Y_d(a_1, \dots, a_r)$ of forms with the given factorization type is given by the Hilbert's formula

$$\deg Y_d(a_1, \dots, a_r) = \frac{r!}{|\mathrm{Aut}(a_1, \dots, a_r)|} \Big(\prod_i a_i\Big).$$

It will be convenient to pack Hilbert's numbers into the generating series

$$\mathcal{N}(d) = \sum \deg Y_d(1_{d-\sum a_i}, a_1, \dots, a_r)\, h^{\sum(a_i-1)}\, \tau_{a_1} \dots \tau_{a_r}$$

where $h = c_1(\mathcal{O}_{Y_d}(1))$ and the sum is taken over the set of all partitions $(a_1, \dots, a_r)$ with parts of length $a_i \geq 2$. We claim that the strata $Y_d(\alpha)$ are the multisingularity loci of certain mapping and $\mathcal{N}(d)$ is the generating series of the target multisingularity classes. Namely, the mapping is

$$Y_1 \times Y_{d-1} \to Y_d$$

that sends a couple consisting of a 1-form and $(d-1)$-form to their product. This mappings factors as in the diagram (41) with the trivial fibration $\pi : Y_1 \times Y_d \to Y_d$. It is easy to compute $u = h - \frac{d}{2}\, t$ and the homomorphism π_* is determined by

$$\pi_*(t) = -\chi(\mathbb{C}P^1) = -2, \qquad \pi_*(t^k) = 0 \quad (k > 1).$$

In other words, the homomorphism π_* selects the coefficient of t in the given polynomial multiplied by -2.

Corollary 7.1 *The generating function of Hilbert's numbers is given by*

$$\mathcal{N}(d) = \exp\Big(-2\, \frac{\partial \mathcal{S}(h - \frac{d}{2}\, t, t)}{\partial t}\, \Big|_{t=0}\Big),$$

where $\mathcal{S}$ is given by (42), *and $h = c_1(\mathcal{O}_{Y_d}(1)) \in H^2(Y_d)$.*

The fact that two presented expressions for $\mathcal{N}(d)$ agree is very easy to verify using computer but I was not able to derive a formal proof.

Generalizing the mapping studied above consider the diagram (41) with $\pi : C \times Y \to Y$ a trivial bundle over arbitrary nonsingular base Y with a fixed genus g curve C as the fiber. Then we have necessarily that $u = \pi^*(h) - \frac{d}{2-2g}\, t \in H^2(C \times Y)$ where d is the degree of f and h is some class defined on Y. The homomorphism π_* is given by

$$\pi_*(t) = -\chi(C) = 2g - 2, \qquad \pi_*(t^k) = 0 \quad (k > 1).$$

Corollary 7.2 *If the bundle π of the diagram* (41) *is trivial, then the generating function for the multisingularity classes is given by*

$$\exp\Big((2g-2)\,\frac{\partial\mathcal{S}(\frac{d}{2g-2}t+h,t)}{\partial t}\;\Big|_{t=0}\Big)=\big(\mathcal{N}(\tfrac{d}{1-g})\big)^{1-g},$$

where $\mathcal{S}$ is given by (42)*, d is the degree of f, g is the genus of C, and $\mathcal{N}(d)$ is the generating function of Corollary* 7.1*.*

As a consequence we obtain that the class $\overline{n}_\alpha$ of any multisingularity $\alpha=(a_1,\dots,a_r)$ is proportional to some power of h. The argument above is not applied in the case $g=1$. However, the coefficients in the series of the corollary are polynomial with respect to g and when evaluated at $g=1$ provide the correct answer. This claim is left to the reader as an exercise.

In a particular case, let $Y=P\mathcal{O}(L)$ be the parameter space of a generic linear system of divisors on C. Then the corollary above describes the degrees of the varieties of divisors consisting of tuples of points with prescribed multiplicities. In more geometric terms, we consider a projective curve $C\subset\mathbb{C}P^n$ of degree d, and take for $Y=\check{\mathbb{C}}P^n$ the dual space parameterizing hyperplanes in $\mathbb{C}P^n$. Then the formula of Corollary 7.2 describes the degrees of varieties of hyperplanes with prescribed tangency types with respect to the curve. The source manifold X of the underlying corank one mapping is the *incidence variety* formed by the couples (a point of C, a hyperplane passing through this point).

On the other hand, for the degrees in question an explicit classical de Jonqieres' formula is known. If the degeneracy variety is determined by the partition $(a_1,\dots,a_r)$ of the number $d=\deg C$, then its degree is given by

$$\frac{\prod_i a_i}{|\mathrm{Aut}(a_1,\dots,a_r)|}\sum_{j=0}^{r}(r-j)!j!\binom{g}{j}S_j(a_1-1,\dots,a_r-1),$$

where S_j is the jth elementary symmetric function of its arguments. Since the generating series of Corollary 7.2 is expressed in terms of that of Corollary 7.1, we obtain as a consequence that de Jonqieres' formula (which is valid for arbitrary g) is a formal consequence of the Hilbert's formula (which is derived in a particular case $g=0$).

Moreover, both formulas are valid for arbitrary base variety Y instead of the projective space. For example, we get without any computation the following result. Let $Y=S^{(d)}C$ be the dth symmetric power of a given genus g smooth curve C. The points of Y are represented by tuples of d unordered points on C. This variety is stratified by partitions of the number d according to the multiplicities of repeating points in the tuple. Denote by h the divisor given by the tuples containing a fixed point $x_0\in C$.

Corollary 7.3 *The classes of all strata of the same codimension m are proportional to the mth power of the divisor h in the cohomology of the dth symmetric power $Y=S^{(d)}C$ of the given curve C. The proportionality coefficient is given by the de Jonqieres' formula.*

A standard approach to the proof of Hilbert's and de Jonquieres' formulas is based on the study of a resolution of the corresponding degeneracy locus $Y(\alpha)\subset Y=\mathbb{C}P^n$. The standard resolution used in these proofs is nothing but the corresponding multisingularity variety $X(\alpha)$ or its refined version of Sect. 4.2.

7.2 Enumeration of contact of lines with hypersurfaces and Thom polynomials for coincidence root varieties

A slightly more general situation is when f decomposes as

$$X \longrightarrow P(E) \xrightarrow{\pi} Y ,$$

where the first map is an embedding of a smooth hypersurface and $\pi : P(E) \to Y$ is the projective bundle associated with a rank 2 vector bundle E over Y. In this case we have $t = -2c_1(\mathcal{O}(1)) - c_1(E)$ and $u = [X] = -\frac{d}{2}t + \pi^*(h)$, where d is the degree of f and h is certain class defined on Y. The push-forward homomorphism π_* is given by

$$\begin{cases} \pi_*(t^{2k}) = 0, \\ \pi_*(t^{2k+1}) = -2\,w^k, \quad \text{where } w = c_1^2(E) - 4\,c_2(E). \end{cases}$$

As a corollary, we get that the multisingularity classes $\overline{n}_\alpha$ are expressed as universal polynomials in $h \in H^2(Y)$ and $w \in H^4(Y)$. It is remarkable that, in general, h is not related to E, and the characteristic classes of E enter with the combination $w = c_1^2 - 4\,c_2$. This is explained by the fact that the structure group of $\mathbb{C}P^1$-bundles is $PGL(2)$ and the ring of $\mathbb{Q}$-characteristic classes for this group is $\mathbb{Q}[w]$. In more details: if L is a line bundle over Y then the projective bundles $P(E)$ and $P(E \otimes L)$ are naturally isomorphic. Therefore, the class $\pi_* t^k$ should not change if we replace E by $E \otimes L$. But w is just the combination of characteristic classes that is invariant with respect to such a twist of E.

The formula for π_* can be represented in the following form. Over $P(E)$ the bundle E splits and we have $c(E) = (1 - v_1)(1 - v_2)$, where $-v_1$ and $-v_2$ are the first Chern classes of the canonical subbundle $\mathcal{O}(-1)$ and the quotient line bundle $E/\mathcal{O}(-1)$. It follows that t is expressed as $t = v_2 - v_1$ and the homomorphism π_* is given by the 'divided difference' operation

$$\pi_* = \partial : P(v_1, v_2) \mapsto \frac{P(v_1, v_2) - P(v_2, v_1)}{v_1 - v_2}.$$

Corollary 7.4 *If π is the projective line bundle $P(E) \to Y$ associated with a rank 2 vector bundle E, then the generating function for the multisingularity classes is given by*

$$\mathcal{N} = \exp\Big(\partial\, \mathcal{S}(-\tfrac{d}{2}(v_2 - v_1) + h, v_2 - v_1)\Big),$$

where $\mathcal{S}$ is given by (42)*, d is the degree of f, and $v_1 + v_2 = -c_1(E)$, $v_1\, v_2 = c_2(E)$.*

A particular case of this formula is that for the 'coincidence root Thom polynomials' from [7] generalizing Hilbert's formula. Let $E \to Y$ be a rank two vector bundle and s be a generic section of the bundle $S^d(E^*)$ viewed as the family of degree d binary forms defined on the fibers of E. The factorization type of the forms $s(y)$, $y \in Y$, determines a stratification on Y whose strata are labelled by partitions α of d. The cohomology class $[Y(\alpha)] \in H^*(Y)$ dual to the closure of the stratum $Y(a) \subset Y$ can be expressed as a universal polynomial $\mathrm{Tp}(\alpha)$ called the *coincidence root polynomial* in the Chern classes $c_1(E)$ and $c_2(E)$. Some formulas for $\mathrm{Tp}(\alpha)$ are given in [7].

We claim that the coincidence root locus $Y(\alpha)$ can be identified as the multisingularity variety for certain Morin map $f : X \to Y$. Namely, X is the hypersurface in $P(E)$ formed by all lines $l \subset E_y$ such the form $s(y)$ vanishes on l. The genericity of s implies that X is nonsingular and the natural projection $X \to Y$ is a Morin map. Therefore, Corollary 7.4 is applied. Remark that X can be identified as the zero locus of a section of the bundle $\mathcal{O}_{P(E)}(d)$ with the first Chern class $u = d\,v_2 = -\frac{d}{2}\,t - \frac{d}{2}\,c_1(E)$.

Corollary 7.5 *The generating function for the Thom polynomials of coincidence root varieties is given by that of Corollary 7.4 with $h = \frac{d}{2}(v_1 + v_2)$, $v_1 + v_2 = -c_1(E)$, and $v_1\,v_2 = c_2(E)$:*

$$\mathcal{N} = \exp\Big(\partial\,\mathcal{S}(d\,v_1, v_2 - v_1)\Big),$$

Expanding the exponent we obtain explicit expressions in terms of divided differences for Thom polynomials corresponding to partitions with small number of parts. They are equivalent to those derived in [7].

Here is a situation where this kind of Thom polynomials can be applied. Let $V \subset \mathbb{C}P^n$ be a degree d nonsingular hypersurface and $Y \simeq G_{2,n+1}$ be the variety of projective lines in $\mathbb{C}P^n$. This variety is stratified according to the contact type of lines with V. The strata are the multisingularity varieties for the Morin map $X \to Y$, where X is parameterized by the couples of the form (a point of V, a line through this point). The cohomology of $Y = G_{2,n+1}$ is generated by the Chern classes $c_1(E)$ and $c_2(E)$ of the tautological rank 2 bundle E. The equation of V can be viewed as a section of the bundle $\mathcal{O}_{P(E)}(d)$ over $P(E)$. The source manifold X of f can be identified as the zero locus of this section.

Corollary 7.6 *The Poincaré dual of a degenerate contact locus in $G_{2,n+1}$ is given by the corresponding coincidence root Thom polynomial in the tautological Chern classes.*

If the codimension of the multisingularity α is equal to $\dim G_{2,n+1} = 2\,(n-1)$, then the corollary provides the number of the corresponding degenerate lines. For example, in the case $n = 2$ we obtain the Plücker formulas for the numbers $|n_3|$ of flexes and $|n_{2,2}|$ of bitangent lines to a degree d generic plane curve:

$$\begin{aligned} |n_3| &= 3\,d\,(d-2) \\ |n_{2,2}| &= \frac{1}{2}\,d\,(d-3)(d-2)(d+3) \end{aligned}$$

For a degree d generic surface in $\mathbb{C}P^3$ the corresponding enumerative invariants are

$$\begin{aligned} |n_5| &= 5\,d\,(7\,d-12)(d-4) \\ |n_{2,4}| &= 2\,d\,(d-5)(3\,d-5)(d-4)(d+6) \\ |n_{3,3}| &= \frac{1}{2}\,d\,(d-5)(d-4)(d^3+3\,d^2+29\,d-60) \\ |n_{2,2,3}| &= \frac{1}{2}\,d\,(d-6)(d-5)(d-4)(d^3+9\,d^2+20\,d-60) \\ |n_{2,2,2,2}| &= \frac{1}{12}\,d\,(d-7)(d-6)(d-5)(d-4)(d^3+6\,d^2+7\,d-30) \end{aligned}$$

Respectively, for a degree d generic threefold in $\mathbb{C}P^4$ we have

$$\begin{aligned}
|n_7| &= 35\,d\,(7\,d-12)(3\,d-10)(d-6)\\
|n_{2,6}| &= 5\,d\,(d-7)(d-6)(17\,d^3+69\,d^2-662\,d+840)\\
|n_{3,5}| &= 5\,d\,(d-7)(d-6)(2\,d^4+6\,d^3+91\,d^2-678\,d+840)\\
|n_{4,4}| &= \frac{1}{2}\,d\,(d-7)(d-6)(d^5+d^4+65\,d^3+395\,d^2-3350\,d+4200)\\
|n_{2,2,5}| &= \frac{5}{2}\,d\,(d-8)(d-7)(d-6)(2\,d^4+23\,d^3+13\,d^2-598\,d+840)\\
|n_{2,3,4}| &= d\,(d-8)(d-7)(d-6)(d^5+11\,d^4+95\,d^3+115\,d^2-3030\,d+4200)\\
|n_{3,3,3}| &= \frac{1}{6}\,d\,(d-8)(d-7)(d-6)(d^5+21\,d^4+40\,d^3+225\,d^2-3110\,d+4200)\\
|n_{2,2,2,4}| &= \frac{1}{6}\,d\,(d-9)(d-8)(d-7)(d-6)(d^5+21\,d^4+125\,d^3-165\,d^2\\
&\quad -2710\,d+4200)\\
|n_{2,2,3,3}| &= \frac{1}{4}\,d\,(d-9)(d-8)(d-7)(d-6)(2\,d^5+22\,d^4+105\,d^3\\
&\quad -115\,d^2-2750\,d+4200)\\
|n_{2^4,3}| &= \frac{1}{24}\,d\,(d-10)(d-9)(d-8)(d-7)(d-6)(3\,d^5+33\,d^4+115\,d^3\\
&\quad -345\,d^2-2470\,d+4200)\\
|n_{2^6}| &= \frac{1}{144}\,d\,(d-11)(d-10)(d-9)(d-8)(d-7)(d-6)(d^5+9\,d^4+21\,d^3\\
&\quad -105\,d^2-446\,d+840)
\end{aligned}$$

The transversality condition necessary to apply these formulas is satisfied for $d \geq 4$ if $n = 3$ and for $d \geq 6$ if $n = 4$ since for smaller values of d the hypersurface contain entire lines which is infinitely degenerate from the singularity theory viewpoint. The numbers $|n_\alpha|$ vanish for $d < \sum a_i$ since for this range of d the multisingularity α cannot be realized. Remark that with our approach the class n_α is expressed as a sum of several quite complicated terms and the fact that the final answer has this nice factorization is not seen until the last step of computation.

7.3 Enumeration of contact of lines with varieties of higher codimension

The enumeration of contact of lines with a generic nonsingular subvariety $V \subset \mathbb{C}P^n$ of arbitrary dimension $\dim V = m$ is also covered by Theorems 2.6 and 2.7. The underlying Morin map

$$f : X \xrightarrow{j} P(E) \xrightarrow{\pi} Y = G_{2,n+1}$$

is defined in the same way as in the previous section. This map has relative dimension $\ell = n - m - 1$. All enumerative invariants of V are expressed in terms of characteristic classes $c_i(\nu)$ of the normal bundle $\nu = \nu_V$ of the embedding $V \subset \mathbb{C}P^n$. There is no *a priori* restriction to these classes except that the top Chern class is

$$c_{n-m}(\nu) = [V] = d\,u^{n-m},$$

where d is the degree of V and $u = c_1(\mathcal{O}_{\mathbb{C}P^n}(1))$. For any monomial P in $c_i(\nu)$ we denote by $\chi(P) = \int_V \frac{P}{1-u}$ the corresponding characteristic number equal to the degree of the class $P \frown [V]$ in $H^*(\mathbb{C}P^n)$. In particular, $d = \chi(1)$.

The virtual normal bundle of f is $\nu_f = \nu_j - \nu_\pi = \nu - \varkappa$, where $\varkappa = \ker d\pi$. In particular, we have

$$\sigma_k = c_{\ell+1}(\nu \otimes \varkappa^{*k}) = \sum_{i+j=\ell+1} (k\,t)^i\, c_j(\nu),$$

where $t = -c_1(\varkappa) = K(\pi) = -2\,c_1(\mathcal{O}_{P(E)}(1)) - c_1(E) = v - u$ as in the previous section.

The homomorphism π_* of the decomposition $f_* = \pi_* j_*$ is given by the divided difference operator as in the previous section. Finally, j_* commutes with u and v and for any monomial in $c_i(\nu)$ the class $j_*(P)$ is proportional to the appropriate power of u with $\chi(P)$ as the proportionality coefficient.

All ingredients of Theorem 2.7 are identified. This allows us to compute easily the classes n_α explicitly. For example, if V is a space curve of genus g then setting $x = \chi(c_1(\nu)) = 4\,d - 2 + 2g$ we obtain the following numbers $|n_{1_4}|$ of 4-secants and $|n_{2,1}|$ of tangent lines having an additional crossing with V, respectively:

$$\begin{aligned}
|n_{1_4}| &= \frac{1}{8}\,x^2 - \frac{1}{4}\,(d^2 - 3d + 11)\,x + \frac{1}{12}\,d\,(d^3 - 13\,d + 72),\\
|n_{2,1}| &= (d-6)\,x - 2\,d\,(d-8).
\end{aligned}$$

If $V \subset \mathbb{C}P^4$ is a curve, then, setting $x = \chi(c_1(\nu)) = 5\,d - 2 + 2g$, we get the following number of 3-secants:

$$|n_{1,1,1}| = \frac{1}{6}\,d\,(d-4)(d+10) - \frac{1}{2}\,(d-4)\,x.$$

Respectively, for a surface $V \subset \mathbb{C}P^4$ with characteristic numbers $x = \chi(\nu_1)$ and $y = \chi(\nu_1^2)$ the corresponding enumerative invariants are

$$\begin{aligned}
|n_{2,2}| &= x^2 - (d^2 + 3\,d - 165)\,x + \frac{1}{2}\,d\,(d^3 + 2\,d^2 - 43\,d - 540) - 18\,y,\\
|n_{3,1}| &= 5\,d\,(d^2 - 5\,d - 108) - 6\,(3\,d - 55)\,x + 2\,(d - 18)\,y,\\
|n_{2,1_3}| &= \frac{2}{3}\,(x - d^2 + 26\,d - 210)\,y - 4\,x^2 - \frac{1}{3}(d^3 - 11\,d^2 + 267\,d - 3000)\,x\\
&\quad + \frac{1}{6}\,d\,(d^4 - 10\,d^3 + 129\,d^2 - 20\,d - 8640),\\
|n_{1_6}| &= \frac{1}{18}\,y^2 - \frac{1}{12}(2\,d - 5)\,x\,y + \frac{1}{36}\,(2\,d^3 - 17\,d^2 + 233\,d - 1350)\,y\\
&\quad - \frac{1}{48}\,x^3 + \frac{1}{8}\,(d^2 - 14)\,x^2 - \frac{1}{48}\,(3\,d^4 + 6\,d^3 - 75\,d^2 + 1230\,d - 10960)\,x\\
&\quad + \frac{1}{144}\,d\,(d^5 + 11\,d^4 - 97\,d^3 + 809\,d^2 - 524\,d - 43200).
\end{aligned}$$

The enumerative results formulated in this section agree with those of [5].

Remark 7.7 The enumeration of contact multisingularities of projective varieties with projective subspaces of dimension greater that 1 is also covered by the general multisingularity theory of mappings, see [16]. Similarly to the case of corank one mappings the computation of multisingularity classes is reduced to the computation of the corresponding residual polynomials. A new difficulty appearing in this case is that one has to take into consideration singularities of corank greater than one classification of which is known only up to some codimension (about 10). An algorithm for the computation of residual polynomials and tables of resulting enumerative formulas can be found in [16], [17].

References

[1] Y. Ando, On the higher Thom polynomials of Morin singularities, *Publ. Res. Inst. Math. Sci.* **23** (1987), 195-207.

[2] V. I. Arnold, Braids of algebraic functions and cohomologies of swallowtails. (Russian) *Uspehi Mat. Nauk*, **23** (1968), no. 4(142), 247–248.

[3] V. I. Arnold, V. V. Goryunov, 0. V. Lyashko, and V. A. Vassiliev, *Singularities.* I. *Local and global theory*, Progress in Science and Technology, Current Problems in Mathematics, Fundamental Directions, vol. 6, VINITI, Moscow 1988; English transl., *Dynamical Systems* VI. Encyclopaedia Math. Sci., vol. 6, Springer-Verlag, Heidelberg 1993.

[4] D. S. Chess, A note on classes $[\Sigma_1^k(f)]$, Singularities, Part 1 (Arcata, CA, 1981), Amer. Mat. Soc., Providence, RI 1983, pp. 221-224.

[5] S. J. Colley, Lines having specified contact with projective varieties, Proc. 1984 Vancouver Conf. on Algebraic Geometry (J. Carrell, A. V. Geramita, and P. Russell, eds.), Amer. Math. Soc., Providence, RI 1986, pp. 47-70.

[6] S. J. Colley, Multiple-point formulas and line complexes, *Lecture Notes in Math.* **1311** (1988), 9-22.

[7] L. M. Fehér, A. Némethi, R. Rimányi, Coincident root loci of binary forms, 2004 preprint.

[8] W. Fulton, *Intersection theory*, Springer-Verlag, Berlin 1984.

[9] T. Gaffney, The Thom polynomial of $\Sigma^{1,1,1,1}$, Singularities, Part 1 (Arcata, CA, 1981), Amer. Mat. Soc., Providence, RI 1983, pp. 399-408.

[10] V. Goryunov, Semi-simplicial resolutions and homology of images and discriminants of mappings. *Proc. London Math. Soc.* (3) **70** (1995), no. 2, 363–385.

[11] J. Harris and L. Tu, Chern numbers of kernel and cokernel bundles, *Invent. Math.* **75** (1984), 467-475.

[12] S. Katz, Iteration of multiple point formulas and applications to conics, *Lecture Notes in Math.* **1311** (1988), 147-155.

[13] M. Kazarian, Characteristic classes of singularity theory, The Arnold-Gelfand Mathematical Seminars, Birkhauser, Boston 1997, pp. 325-340.

[14] M. Kazarian, Classifying spaces of singularities and Thom polynomials, New Developments in Singularity Theory (Cambridge, 2000), Kluwer, Dordrecht 2001, pp. 117-134.

[15] M. Kazarian, Thom polynomials for Lagrange, Legendre, and critical point function singularities, *Proc. London Math. Soc.* (3) **86** (2003), Part 3, 707-734.

[16] M. Kazarian, Multisingularities, cobordisms, and enumerative geometry, Russian Math. Surveys 58:4 (2003) 665-724; Uspekhi Mat. Nauk (2003) 58:4 29-88.

[17] M. Kazarian, Thom polynomials, Proc. Conf. on Singularity Theory in Sapporo, 2003.

[18] M. Kazaryan, S. Lando, On intersection theory on Hurwitz spaces. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **68** (2004), no. 5, 91–122; translation in Izv. Math. **68**

[19] S. L. Kleiman, The enumerative theory of singularities, Real and Complex Singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Multisingularities, cobordisms, and enumerative geometry 723 Noordhoff, Alphen aan den Rijn 1977, pp. 297-396.

[20] S. L. Kleiman, Multiple-point formulas. I: Iteration, *Acta Math.* 147:2 (1981), 13-49.

[21] S. L. Kleiman, Multiple-point formulas. II: The Hilbert scheme, *Lecture Notes in Math.* **1436** (1990), 101-138.

[22] S. L. Kleiman, Intersection theory and enumerative geometry: A decade in review (with the collaboration of Anders Thorup on Sect. 3), Algebraic Geometry (Brunswick/Maine, 1985), Amer. Math. Soc., Providence, RI 1987, pp. 321-370.

[23] S. L. Kleiman, J. Lipman, and B. Ulrich, The multiple-point schemes of a finite curvilinear map of codimension one, *Ark. Mat.* **34** (1996), 285-326.

[24] V. S. Kulikov, *Calculus of singularities of embeddings of a general algebraic surface into projective space* P^3, Func. Anal. Appl., **17** 15–27 (1983)

[25] P. Le Barz, Géométrie énumérative pour les multisécantes, *Lecture Notes in Math.* **683** (1978), 116-167.

[26] J. Mather, Stability of C^∞-mappings. I-VI, *Ann. of Math.* (2) **87** (1968), 89-104; **89** (1969), 254-291; *Inst. Hautes Études Sci. Publ.* **35** (1968), 279-308; **37** (1969), 223-248; *Adv. Math.* **4** (1970), 301-336; *Lecture Notes in Math.* **192** (1971), 207-253.

[27] W.L. Marar, D.Mond, Multiple point schemes for corank 1 maps, J. London Math. Soc. (2) 39 (1989), 553–567.

[28] I. R. Porteous, Simple singularities of maps, *Lecture Notes in Math.* **192** (1971), 286-307.

[29] Z. Ran, Curvilinear enumerative geometry. Acta Math. **155** (1985), no. 1–2, 81–101.

[30] R. Rimányi, Thom polynomials, symmetries and incidences of singularities, *Invent. Math.* 143 (2001), 499-521.

[31] R. Rimányi, A. Szücs, Pontrjagin-Thom-type construction for maps with singularities. Topology **37** (1998), no. 6, 1177–1191.

[32] Ronga, F. Le calcul des classes duales aux singularites de Boardman d'ordre deux. (French) Comment. Math. Helv. 47 (1972), 15–35.

[33] V. D. Sedykh, Resolution of singularities of corank 1 of a front in general position. (Russian) *Funktsional. Anal. i Prilozhen.* **37** (2003), no. 2, 52–64; English translation in *Funct. Anal. Appl.* **37** (2003), no. 2, 123–133.

[34] V. D. Sedykh, The topology of corank 1 multi-singularities of stable smooth mappings of equidimensional manifolds. *C. R. Math. Acad. Sci. Paris* **340** (2005), no. 6, 441–444.

[35] Szücs, A. On the cobordism group of Morin maps. Acta Math. Hungar. 80 (1998), no. 3, 191–209.

[36] R. Thom, Les singularités des applications différentiables, *Ann. Inst. Fourier* (*Grenoble*) **6** (1955/1956), 43-87.

[37] V.A. Vassiliev. Complements of Discriminants of Smooth Maps: Topology and Applications. Transl. Math. Monographs. V. 98, Providence, RI: Amer. Math. Soc., 1994. (Extended Russian translation: Moscow, Phasis, 1997).