MATH IN MOSCOW

Calculus on manifolds

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Contents

Manifolds are topological spaces that locally have the structure of a coordinate space $\mathbb{R}^n$. They appear naturally in almost all branches of modern mathematics. The goal of the course is to develop the language that allows one to translate to the case of abstract manifolds the notions of classical calculus: integration, differentiation etc. We try to emphasize the geometrical meaning of all introduced objects and to present them, if possible, in an invariant coordinate-free form. Even if the studied manifold is a domain in $\mathbb{R}^n$, it is useful do not restrict ourself to the choice of a particular coordinate system. Following this language simplifies and unifies many notions and statements of classical calculus. The introduced notations allow one to fulfil complicated computations without making mistakes. They are so natural that there is no need to keep in mind complicated formulas any more: most of the relations are implied by the naturality of the constructions.

1 Smooth curves in the plane and in the 3-space

We start with the notion of Euclidean space. The simplest Euclidean spaces are the line, the plane, and the 3-space. As about formal definitions, the *line* is the set of real numbers, denoted $\mathbb{R}$ or $\mathbb{R}^1$. The *plane* is the set of pairs of real numbers (x_1, x_2), and it is denoted by $\mathbb{R}^2$. The 3-*space* is the set of triples (x_1, x_2, x_3), denoted by $\mathbb{R}^3$. The point also is an Euclidean space, the 0-dimensional space $\mathbb{R}^0$. This notion can be generalized to that of n-dimensional Euclidean space $\mathbb{R}^n$, the space of n-tuples $(x_1, x_2, \dots, x_n)$.

Each value x_i can be considered as a function on the space $\mathbb{R}^n$. These functions are called the *standard coordinates.* For small dimensions, we will often denote the standard coordinates by x, y, z, not using the subscribes.

In common life, we rarely deal with spaces of dimension greater than three (although the space-time has dimension four, and modern theories of the microworld assume that its dimension is either 10 or 26). However, spaces of various dimensions naturally arise in mathematical research and they proved already their efficiency in the study of complex objects, say our 3-dimensional space with additional structures on it.

Functions on spaces are constructed from functions on $\mathbb{R}^1$ (that is, functions in one variable). We will be interested only in smooth functions. A function is *smooth* if it can be differentiated arbitrarily many times. Here are some examples of smooth functions:

- any constant;
- the function x;
- any polynomial, e.g., $x^3 - x$;
- the exponent e^x;
- the functions $\sin x$, $\cos x$;
- the function $\frac{1}{x^2+1}$.

Some other common functions are not smooth:

- $\frac{1}{x}$ is not defined at the origin, but were it be defined, it would not be continuous, not saying about differentiable, at the origin;
- the same is true for the function $\tan x = \frac{\cos x}{\sin x}$, this time not only at the origin, but also at the points $x = 2\pi n$, $n \in \mathbb{Z}$;
- the function $|x - 2|$ is not smooth at the point $x = 2$, although it is continuous at this point.

However, outside the points mentioned above, these functions are also smooth, and they will be useful for us in what follows.

1.1. Draw the graphs of the functions enlisted above.

Definition 1.1. A *smooth curve* on the plane is a smooth mapping $\gamma : [a, b] \to \mathbb{R}^2$, i.e., a mapping given by two infinitely differentiable functions $(\gamma_1(t), \gamma_2(t))$.

A smooth curve in a space of arbitrary dimension is defined similarly.

1.2. Interpret as a smooth curve a) a circle on the plane; b) the trefoil knot in the space. [Hint: Draw the trefoil knot on a torus in the 3-space.]

1.3. Can the following curves on the plane be given as smooth curves: a) an eight-figure; b) a square; c) the semicubic $x^2 = y^3$?

1.4. Prove that the function

$$y = \begin{cases} 0, & \text{for } x \le 0; \\ e^{-\frac{1}{x}}, & \text{for } x > 0 \end{cases}$$

is infinitely differentiable at the origin.

Recall that the partial derivative $\frac{\partial F}{\partial x}$ is the derivative of F restricted to the straight line $x = \text{const}$, and similarly for other coordinates.

A 1-*form* on the plane is an expression of the form

$$\omega = f(x, y)dx + g(x, y)dy,$$

or, in the multidimensional case,

$$\omega = \omega_1(x_1, \dots, x_n)dx_1 + \dots + \omega_n(x_1, \dots, x_n)dx_n.$$

The *differential* of a function F on the plane is the 1-form

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy,$$

and similarly for the multidimensional case.

Definition 1.2. The *curvilinear integral* of a 1-form ω along a curve $\gamma : [a, b] \to \mathbb{R}^2$ is the number

$$\int_\gamma \omega = \int_a^b \sum \omega_i(\gamma(t)) \frac{d\gamma^i}{dt} dt.$$

Theorem 1.1. *The curvilinear integral*

a) *is invariant with respect to a change of the parameter along the curve;*

b) $\int_\gamma \omega = - \int_{\overline{\gamma}} \omega$, *where $\overline{\gamma}$ is the same curve passed in the opposite direction.*

c) *The following formula (the Newton–Leibnitz formula, or the one-dimensional Stokes formula) is valid for the curvilinear integral: $\int_\gamma dF = F(B) - F(A)$, where γ is an arbitrary curve connecting A and B.*

All these properties are immediate corollaries of the similar properties for the usual one-variable integral.

1.5. Compute the integral $\int_\gamma (x\,dy - y\,dx)$, where the curve γ connects the points $(0, 0)$ and $(1, 2)$ a) along the segment; b) along the curvilinear segment of the parabola $y = 2x^2$; c) along the broken line $(0, 0)$—$(1, 0)$—$(1, 2)$. Deduce from the results that the 1-form $xdy - ydx$ is not a differential of a function.

1.6. The same question for the integral $\int_\gamma (x\,dy + y\,dx)$.

1.7. Compute $\int xy^2\,dy - x^2y\,dx$ over the arc of the circle $x^2 + y^2 = R^2$, $y \ge 0$.

1.8. Compute $\int (x-y)\,dx - (x+y)\,dy$ over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

1.9. Compute $\int x^2\,dx + y^3\,dy + \cos z\,dz$ over the curve $\gamma: \ \ x = t^2 + t^3;\ y = \sqrt{t^2+1};\ z = e^{t^2};\ -1 \le t \le 1$.

1.10. Compute $\int \frac{x\,dy - y\,dx}{x^2+y^2}$ over a closed contour around the origin. [**Hint:** rewrite the integrand in polar coordinates.]

1.11. Compute $\int (x+y)\,dx + (x-y)\,dy$ over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

From the physical point of view, a *field of forces* (in the plane and in the space) is described by a 1-form. The *work* along a path is nothing but the integral of this 1-form over this path. A field of forces is said to be *potential* if the corresponding 1-form is the differential of a function (which, in this case, is called the *potential* of the field).

1.12. Find the work of the force $F = kr$, proportional to the radius vector, along the arc of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ x \ge 0,\ y \ge 0$.

1.13. Find the work of the Newton force $F = kr/|r|^3$ along an arbitrary path connecting two given points in $\mathbb{R}^3$.

2 Submanifolds

Definition 2.1. A *k-dimensional submanifold* in $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$ such that each point $x^* \in M$ has a neighborhood $U(x^*) \subset \mathbb{R}^n$ such that in $U(x^*)$ the set M is given by a system of equations $f_1(x) = \cdots = f_{n-k}(x) = 0$, where all the functions f_i are smooth and the rank of the matrix of partial derivatives $(\partial f_i/\partial x_j)$ is $n-k$ (i.e., the rank is the maximal possible).

2.1. Prove that the following subsets are submanifolds: a) the space $\mathbb{R}^n$ itself; b) the vector subspace $x_1 = 0$ in $\mathbb{R}^n$; c) the circle $x^2 + y^2 = 1$ in the plane; d) the sphere $x^2 + y^2 + z^2 = 1$ in the 3-space; e) the graph of a smooth function $z = f(x,y)$ in $\mathbb{R}^3$.

2.2. For which values of the parameter a the surface in the 3-space given by the equation $x^2 + y^2 - z^2 = a$ is a submanifold?

2.3. For which values of the parameter a the curve on the plane given by the equation $y^2 = x^3 - x + a$ is a submanifold?

2.4. Are the following subsets submanifolds: a) the eight-figure in the plane; b) the boundary of a square in the plane; c) the interior of a square in the plane; d) the trefoil knot in the 3-space; e) the semicubic curve $x^2 - y^3 = 0$ in the plane; f) the set in the 4-space given by the system of equations

$$\begin{cases} 3x + y - z + u^2 = 0 \\ x + y + 2z + u = 0 \\ 2x + 2y - 3z + 2u = 0? \end{cases}$$

Sometimes (but rarely) it is possible to introduce on a whole submanifold a system of coordinates. For example, the functions $x_2, \ldots, x_n$ form a system of coordinates on the hyperplane $x_1 = 0$ (and, more generally on the graph of an arbitrary function $x_1 = f(x_2, \ldots, x_n)$). However, much more often this is not the case: a set of functions producing a set of coordinates on the whole subvariety does not exist and it is possible to introduce a system of coordinates only locally, in a neighborhood of each point $x^* \in M$.

Definition 2.2. A set of functions $y_1, \dots, y_k : \mathbb{R}^n \to \mathbb{R}$ is a set of *local coordinates* in a neighborhood of $x^* \in M \subset \mathbb{R}^n$ of a k-dimensional submanifold M given by a system of equations $f_1 = \dots = f_{n-k} = 0$ if the functions $y_1, \dots, y_k, f_1, \dots, f_{n-k}$ together give a system of local coordinates in this neighborhood of $x^* \in \mathbb{R}^n$. (This means that the matrix of the first partial derivatives of the set $y_1, \dots, y_k, f_1, \dots, f_{n-k}$ is nondegenerate at x^*.)

In other words, the local structure of a submanifold of dimension k coincides with that of a k-space.

2.5. Choose local coordinates in a) the circle; b) the sphere; c) the trefoil. Suggest several versions.

Definition 2.3. A subset $M \subset \mathbb{R}^n$ is *open* if for any $x \in M$ there is an open ball $U(x) \subset M$. A subset is *closed* if its complement is open.

2.6. Give examples of open and closed subsets in a) the line; b) the plane; c) the space.

2.7. Prove that an open subset in the line is a union of at most countable set of intervals.

2.8. Prove that the following sets are submanifolds in the spaces of matrices: a) the set of $n \times n$-matrices having nonzero trace; b) the set of $n \times n$-matrices having zero trace; c) the set of $n \times n$-matrices with determinant 1; d) the set of orthogonal $n \times n$-matrices, i.e., matrices A such that $AA^t = I$; e) the set of unitary $n \times n$-matrices, i.e., complex matrices A such that $A\bar{A}^t = I$.

(The set of real $n \times n$-matrices is the vector space $\mathbb{R}^{n^2}$; the entries of the matrices can serve as coordinates in this space. The set of complex $n \times n$-matrices is the vector space $\mathbb{R}^{2n^2}$; for coordinates, the real and the imaginary part of the entries can be chosen.)

2.9. What is the dimension of the submanifolds from the previous problem?

2.10. Prove that any open subset in $\mathbb{R}^n$ is a submanifold.

Definition 2.4. A set $M \subset \mathbb{R}^n$ is said to be *compact* if it is bounded and closed.

2.11. Prove that M is compact iff a finite subcovering can be chosen from any covering of M by open sets.

2.12. What submanifolds from the above examples are compact?

3 Smooth mappings of submanifolds, diffeomorphisms, and manifolds

Definition 3.1. A *smooth function* on a submanifold $M \subset \mathbb{R}^n$ is the restriction to M of a smooth function in $\mathbb{R}^n$.

Two smooth functions $F_1, F_2 : \mathbb{R}^n \to \mathbb{R}$ determine the same function on M iff the restriction of their difference $F_1 - F_2$ to M is zero.

Let $y_1, \dots, y_k$ be local coordinates on M. Each smooth function $F : M \to \mathbb{R}$ can be written in the local coordinates; i.e., it can be represented as $F(y_1, \dots, y_k)$. Indeed, suppose M admits local equations $f_1 = \dots = f_{n-k} = 0$. Then in a neighborhood of a point $x^* \in M \subset \mathbb{R}^n$ the function F can be represented as $F(y_1, \dots, y_k, f_1, \dots, f_{n-k})$. Setting $f_i = 0$, we express the restriction of F to M in terms of $y_1, \dots, y_k$.

A set $F = (F_1, \dots, F_r)$ of smooth functions determines a *smooth mapping* of M to $\mathbb{R}^r$. The image of this mapping can prove to be a submanifold in $\mathbb{R}^r$ (as in the case of the trefoil parameterized by the circle), or to be not a submanifold (the square also can be parameterized smoothly by the circle; another example is given by the projection of the unit sphere in the 3-space to a plane). If the image $F(M)$ of a smooth mapping belongs to a submanifold $N \subset \mathbb{R}^r$, then we also say that F defines a smooth mapping from M to N.

Definition 3.2. Let $F : M \to \mathbb{R}^r$ be a smooth mapping of a submanifold $M \subset \mathbb{R}^n$ whose image $N = F(M) \subset \mathbb{R}^r$ also is a submanifold and suppose F admits a smooth inverse mapping $F^{-1} : N \to M$. Then F is called a *diffeomorphism.* Two submanifolds M and N are called *diffeomorphic* (denoted $M \sim N$) if there is a diffeomorphism of one of them onto another.

Submanifolds in spaces of distinct dimension may well be diffeomorphic.

3.1. Prove that the dimensions of diffeomorphic manifolds coincide.

3.2. Prove that the relation of being diffeomorphic is an equivalence relation, that is, a) $M \sim M$; b) $M \sim N \Rightarrow N \sim M$; c) $M \sim N, N \sim L \Rightarrow M \sim L$.

This equivalence relation splits all submanifolds into disjoint classes. Any two submanifolds belonging to the same class are diffeomorphic, while two submanifolds from distinct classes are not diffeomorphic to each other. Diffeomorphic submanifolds, in a very broad sense, behave similarly, and it is natural to treat them as a same object.

Definition 3.3. A *manifold* is an equivalence class of pairwise diffeomorphic submanifolds.

Definition 3.4. A submanifold is said to be *connected* if it cannot be represented as a union of two disjoint nonempty open sets. If a submanifold M is represented as a disjoint union $M = M_1 \cup M_2 \cup \dots$ of nonempty open submanifolds, then these submanifolds are called the *connected components* of M.

3.3. Prove that a diffeomorphism of submanifolds preserves a) the property of being connected; b) the number of connected components; c) the property of being compact.

Dimension, compactness and connectedness are the simplest (but extremely important) examples of *invariants of manifolds*, that is, these properties are preserved under diffeomorphisms. Since each manifold is a disjoint union of connected ones, we will be interested mainly in calculus on connected manifolds.

3.4. Find the numbers of connected components in the examples of submanifolds from the previous lecture.

3.5. Prove that a submanifold $M \subset \mathbb{R}^n$ is connected iff it is *path connected*, that is iff any two points $x, y \in M$ can be connected by a smooth path $\gamma : [0,1] \to M$, $\gamma(x) = 0$, $\gamma(x) = 1$.

Now we are able to classify all one-dimensional manifolds. We know two examples of one-dimensional manifolds: the circle and the line.

3.6. Construct a diffeomorphism of an open interval onto the line.

These two manifolds are not diffeomorphic to each other: the circle is compact, while the line is not. Here is another proof of this fact: if we puncture the circle it remains connected, while after puncturing the line we obtain a disconnected submanifold, consisting of two connected components.

Theorem 3.1 (Classification of one-dimensional manifolds). *Each one-dimensional manifold is a disjoint union of circles and lines.*

3.7. Prove the classification theorem.

3.8. Prove that the torus in $\mathbb{R}^3$ is diffeomorphic to the submanifold in $\mathbb{C}^2 \cong \mathbb{R}^4$ given by the equations $|z_1| = |z_2| = 1$ (here z_1 and z_2 are the standard complex coordinates in $\mathbb{C}^2$).

3.9. Prove that the following subsets are subvarieties in the matrix spaces: a) $\mathrm{SO}(n) \subset O(n) \subset \mathbb{R}^{n^2}$, orthogonal matrices with determinant 1; b) $\mathrm{SU}(n) \subset U(n) \subset \mathbb{C}^{n^2} \cong \mathbb{R}^{n^2}$, unitary matrices with determinant 1.

3.10. Which of the following submanifolds are diffeomorphic: a) S^3; b) $\mathrm{SO}(3)$; c) $O(3)$; d) $U(2)$; e) $\mathrm{SU}(2)$; f) $\mathrm{SL}(2)$?

4 Abstract manifolds, tangent spaces

Definition 4.1. An *abstract manifold* is a set M represented as a union $M = \bigcup_\alpha U_\alpha$, together with bijections $\varphi_\alpha : U_\alpha \to V_\alpha$, where $V_\alpha \subset \mathbb{R}^n$ is an open domain. The maps φ_α are referred to as *charts*. An *atlas* is a collection of all charts. We require that the charts must be *compatible*: if $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$ then the maps $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_a(U_{\alpha\beta}) \to \varphi_b(U_{\alpha\beta})$ are diffeomorphisms of the corresponding domains in the Euclidean space.

4.1. Define the notions of admissible charts, equivalent atlases, smooth maps, diffeomorphic manifolds.

The atlas determines the structure of a topological space on M. We always assume that the following two additional conditions are satisfied: the topological space M is *Hausdorf* and *separable*. The first condition means that for any two distinct points x_1, x_2 there exist admissible charts $U_1 \ni x_1$, $U_2 \ni x_2$, whose intersection is empty. The second condition means that there exist an atlas containing at most countable set of charts.

4.2. Define a structure of an abstract manifold on a submanifold of a Euclidean space.

One can think that the notion of an abstract manifold is more general than that of a submanifold of a Euclidean space. In fact, both notions are equivalent:

Theorem 4.1. *Every manifold is diffeomorphic to a submanifold of a Euclidean space.*

Equivalently, one can say that every manifold admits an *embedding* to a Euclidean space.

4.3. Construct an embedding $M \hookrightarrow \mathbb{R}^N$ in the case when M is compact.

There is, however, a number of important examples of manifolds that appear naturally as abstract manifolds.

4.4. Define the structure of a smooth manifolds on the following spaces. Determine the dimension in each case. How many charts do you need to define an atlas?

(a) projective space $\mathbb{R}P^n$ formed by all lines in the Euclidean space $\mathbb{R}^{n+1}$;

(b) Grassmann manifold $G_{k,n}$ of k-planes in the Euclidean space $\mathbb{R}^n$;

(c) complex projective space $\mathbb{C}P^n$ and complex Grassmann manifold $G_{k,n}^{\mathbb{C}}$, formed by complex subspaces in the corresponding complex vector space.

4.5. Which compact manifolds of dimension 2 are known to you?

4.6. A generic homogeneous polynomial of degree d in three variables determines a *complex projective curve* $C \subset \mathbb{C}P^2$. As a real manifold, it is an oriented surface of some genus g, i.e. a sphere with g handles. What is the genus g (say, in the simplest cases $d = 1, 2, 3$)?

4.7. What is the dimension of the following manifolds? Which of them are diffeomorphic?

(a) S^3;
(b) $\mathbb{R}P^3$;
(c) $S^2 \times S^1$;
(c) $SO(3)$;
(d) $SU(2)$;
(e) the space of unit tangent vectors to the sphere S^2;
(f) the space of (non-oriented) tangent directions to the sphere S^2;
(g) the subset in $\mathbb{C}^3$ given by the equations $z_1^2 + z_2^2 + z_3^2 = 0$, $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$.

To each point x of a smooth manifold x one associates the *tangent space* $T_x M$. It is a vector space of dimension $n = \dim M$. Elements of this vector space are called *tangent vectors*. For the actual definition of a tangent vector one can use either of three following equivalent ones.

Definition 4.2. A *tangent vector* at a point $x \in M$ is a class of equivalent smooth curves $\gamma : \mathbb{R} \to M$, $\gamma(0) = x$. Two curves γ_1, γ_2 are *equivalent*, if $|\gamma_1(t) - \gamma_2(t)| = o(t)$ for some (and hence, for any) coordinate system around x.

Definition 4.3. A *tangent vector* at a point $x \in M$ is a collection of its coordinates $v = (v_1, \dots, v_n)$ assigned to each coordinate system. With a change of coordinate system the column of coordinates is multiplied by the Jacobi matrix of the coordinate change.

A tangent vector can also be identified with the differential operator given by the derivative in the direction of this vector.

Definition 4.4. A *tangent vector* at a point $x \in M$ is a *derivation*, i.e. it is an operation $f \mapsto D(f)$ defined on the space of smooth functions on M with values in $\mathbb{R}$, which is $\mathbb{R}$-linear and satisfies the *Leibnitz rule*:

$$D(f\,g) = D(f)\,g(x) + f(x)\,D(g).$$

4.8. Find the correspondence between these definitions. Prove the equivalence of them.

The last definition suggests that the most invariant way to write down the tangent vector with the coordinates $v_1, \dots, v_n$ is

$$v = v_1 \frac{\partial}{\partial x_1} + \dots v_n \frac{\partial}{\partial x_n},$$

so that $\partial/\partial x_i$ is the ith basic vector directed along the ith coordinate line.

4.9. Find the expression for the basic vectors $\partial/\partial x$, $\partial/\partial y$ on the plane $\mathbb{R}^2$ in polar coordinates.

4.10. Find the basic tangent vectors $\partial/\partial \varphi$, $\partial/\partial r$ of the polar coordinate system on the plane $\mathbb{R}^2$ in the Euclidean coordinates.

Consider a smooth map $f : M \to N$ of manifolds of arbitrary dimensions. Set $y = f(x)$. The *derivative* is a linear map of tangent spaces $f_* : T_x M \to T_y N$.

4.11. Give the definition of the derivative map in terms of every of the three definitions of the tangent vectors.

If M, N are two charts on the same manifold and f is the transition function then the formula for f_* expresses the transformation rule under a change of coordinates.

4.12. Make more precise the transformation rule for tangent vectors under a change of coordinates. (Which Jacobi matrix should be applied?)

4.13. Prove the *chain rule*: for a composition of smooth maps $M \xrightarrow{f} N \xrightarrow{g} L$ we have

$$(g \circ f)_* = g_* \circ f_* : T_x M \to T_{g(f(x))} L.$$

5 Vector fields

Definition 5.1. A *vector field* on a smooth manifold M is a smooth family of tangent vectors $v(x) \in T_x M$ parameterized by the points x of the manifold.

Locally a vector field is written in the form

$$v = v_1(x)\,\partial_{x_1} + \cdots + v_n(x)\,\partial_{x_n}, \qquad \partial_{x_i} = \frac{\partial}{\partial x_i},$$

where $v_1, \ldots, v_n$ are smooth functions.

The derivative $f \mapsto v\,f = \sum v_i \frac{\partial f}{\partial x_i}$ along the vector field determines an operation on the ring of smooth functions which is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$v\,(fg) = f\;(vg) + g\;(vf).$$

Any such operation (i.e. linear and satisfying the Leibniz rule) is called *derivation*.

5.1. Prove that every derivation on the ring of smooth functions is a derivative along some vector field.

5.2. Prove that the commutator $[u, v] = uv - vu$ of two derivations is again a derivation (while the composites uv and vu are not derivations).

Let w be the vector field corresponding to the derivation $[u, v]$. This field is called the *commutator of vector fields* u, v.

5.3. Find the expression for the commutator in coordinates.

5.4. Prove that the commutator of vector fields is skew-symmetric and satisfies the *Jacobi identity*

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0.$$

(One says that the vector fields constitute a *Lie algebra*)

The vector field is the natural language for the ODE theory.

Definition 5.2. The *phase curve* of a vector field v is a curve $\gamma : I \to M$ ($I \subset \mathbb{R}$ is an open interval) which is tangent to the field at any point. Finding phase curves is equivalent to solving ODE

$$\dot\gamma = v(\gamma(t)), \quad \gamma = (\gamma_1(t), \ldots, \gamma_n(t)), \quad v = (v_1(x), \ldots, v_n(x)), \quad \dot{} = \frac{d}{dt}.$$

According to the main theorem of ODE, for every initial point $x \in M$ there exists a unique phase curve passing through this point (and defined for small values of t). The phase curve through the point x is denoted by $t \mapsto g_v^t x$.

If M is compact then the phase curve extends to the whole range $\mathbb{R}$ of values of t. In general, this is not always possible (give examples). If every phase curve is defined for all values of t then these curves define the *phase flow*, a one-parameter family of diffeomorphisms

$$g_v^t : M \to M, \qquad t \in \mathbb{R}.$$

5.5. Prove that this family of diffeomorphisms forms a *group*: $g_v^0 = \mathrm{id}$, $g_v^t \circ g_v^s = g_v^{t+s}$.

The notions of phase curves, phase flows extend to the case when the field v depends smoothly in time t (this case corresponds to a non-autonomous ODE). In this case the family of diffeomorphisms g_v^t is not a group any more. Conversely, any one-parameter family of diffeomorphisms is a flow of some vector field. This field is autonomous iff the family of diffeomorphisms forms a group.

5.6. Write the differential equations corresponding to the following fields:

$$x\,\partial_x + y\,\partial_y, \quad x\,\partial_x - y\,\partial_y, \quad x\,\partial_y - y\,\partial_x.$$

Solve these equations and find the corresponding phase flows. Draw the phase curves in each case.

5.7. The same questions for the field $(x^2 - y^2)\,\partial_x + 2\,x\,y\,\partial_y$ (Hint: apply the inversion transformation for the plane $\mathbb{R}^2$.)

In general, phase flows of different vector fields u, v do not commute: $g_u^t \circ g_v^s \neq g_v^s \circ g_u^t$. The non-commutativity of these flows is measured by the commutator of the fields:

Theorem 5.1. 1. *The flows g_u^t, g_v^s commute for all t, s iff the commutator $[u, v]$ of the fields u, v vanishes identically on M.*

2. *For any fields u, v, initial point $x \in M$, and small values t, s one has*

$$g_v^{-s} \circ g_u^{-t} \circ g_v^s \circ g_u^t x = x - t\,s\,[u, v] + \ldots,$$

where the dots denote the terms of higher order in t, s.

Corollary. *Let $\xi_1, \ldots, \xi_n$ be a collection of vector fields which are linearly independent in a neighborhood of some point on an n-dimensional manifold. The following properties are equivalent:*

(i) *there exists a coordinate system $(x_1, \ldots, x_n)$ such that $\xi_i = \frac{\partial}{\partial x_i}$;*

(ii) *the fields ξ_i commute pairwise, $[\xi_i, \xi_j] \equiv 0$.*

5.8. Prove the corollary.

5.9. Prove that there exists a coordinate system s, t in a neighborhood of the point $x = y = 0$ such that the fields

$$u = (1+y)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \quad \text{and} \quad v = x\frac{\partial}{\partial x} + (1+y)\frac{\partial}{\partial y}$$

take the form $\dfrac{\partial}{\partial s}$ and $\dfrac{\partial}{\partial t}$ respectively. Find this coordinate system.

5.10. Consider a unit sphere as a model for the globe. Denote by N, E the fields of length one vectors directed to the North and East respectively.

a) Express the fields N, E in the spherical coordinates.

b) Find the flows g_N^t, g_E^s of these fields.

c) Compute the commutator of the flows $g_E^{-s} \circ g_N^{-t} \circ g_E^s \circ g_N^t$. Expand this transformation as a series in t, s and find the leading term of the expansion.

d) Compute the commutator $[N, E]$ of the fields. Compare the result with the computation of the previous question.

Let v be any differential geometry structure (function, vector field, covector field, differential form, etc.) on a given manifold. The phase flow g_u^t of the vector field u translates this structure and produces a family v_t of such structures. In particular, if v is a vector field, we set $v_t(x) = (g_u^t)_*^{-1} v(g_u^t)$. The *Lie derivative* $L_u v$ of the structure v along the field u is defined as

$$L_u v = \frac{d}{dt} v_t|_{t=0}.$$

5.11. Prove that if f is a function then $L_u f = u\, f$ is the usual directional derivative.

5.12. Let u, v be two vector fields. Prove the formula

$$L_u v = [u, v] = -L_v u.$$

(Hint: apply the Leibnitz rule: $L_u(v\, f) = (L_u v) f + v\,(L_u f)$.) Verify this formula for the fields N, E of Problem 5.10 above.

6 Frobenius Theorem

The assertion of Problem 5.12 can be reformulated as follows.

Lemma 6.1. *Let v_t be the family of vector fields obtained from the initial field $v = v_0$ by the action of the one-parameter family of diffeomorphisms g^t. Then the propagation of this family is described by the equation*

$$\frac{dv_t}{dt} = [u, v_t],$$

where u is the vector field (perhaps, non-autonomous) that generates the flow g^t.

Proof of Theorem 5.1. The assertion that the flows of the fields u, v commute can be reformulated by saying that the flow of the field u preserves the flow of the field v, or, equivalently, that the flow of the field u preserves the field v itself, i.e. that $\frac{dv_t}{dt} \equiv 0$ where v_t is obtained from v by applying the flow g_u^t of the field u. By the lemma this is equivalent to the equality $[u, v] \equiv 0$.

Another application of the lemma above is the proof of the following **Frobenius theorem**.

Definition 6.1. A 2-*distribution* in the space $\mathbb{R}^3$ is any field of tangent 2-planes. The distribution is called *integrable* if in a neighborhood of any point there exists a smooth function without critical points such that the planes of the distribution are formed by the planes tangent to the level surfaces of the function.

6.1. Prove that the distribution is integrable iff there exists a coordinate system x, y, z for which the planes of the distribution are *horizontal* (i.e. spanned by the vectors ∂_x and ∂_y).

6.2. Prove that the distribution is integrable iff for each point $x \in \mathbb{R}^3$ there exists a smooth integral surface passing through this point (a submanifold is called integral if its tangent spaces are contained in the planes of the distribution; in other words, if the planes of the distribution are tangent to that submanifold at any point).

Theorem 6.2. *Suppose that the planes of the distribution are spanned by two vector fields ξ, η. The distribution is integrable iff the commutator $[\xi, \eta]$ also belongs to the distribution, i.e. if*

$$[\xi, \eta] = a\,\xi + b\,\eta$$

for some smooth functions a, b.

6.3. Prove the necessity of the condition.

Proof. For a given point $x \in \mathbb{R}^3$ we need to find an integral surface through this point. It s very easy to construct a candidate for such a surface: first, starting from the point x we draw the phase curve $\gamma(t) = g_u^t x$ of the field u, and then we draw the phase curves $g_v^s \gamma(t)$ of the field v through each point of γ. We need to show that the surface S swept by these curves is an integral one.

Lemma 6.3. *If the criterium of Theorem is satisfied then the flow of any field tangent to the distribution preserves the distribution. In particular, this flow translates integral submanifolds to integral ones.*

By construction, the phase curves of the flow g_v^s through γ are tangent to the distribution. Besides, by the lemma, the image of the curve γ under the flow g_v^s is also tangent to the distribution. Since the tangent lines to these curves span the tangent space to S, we obtain, that S is an integral surface, which proves Theorem.

Proof of the lemma. Let w be some field tangent to the distribution, let $\gamma(t)$ be its phase curve. Let $\xi_0 \in T_x\mathbb{R}^3$ be some tangent vector contained in the plane of the distribution. Applying the flow of the field w we obtain the family $\xi_t \in T_{\gamma(t)}\mathbb{R}^3$. We need to show that ξ_t also belongs to the distribution. We look for ξ_t in the form $\xi_t = a(t)\,u(\gamma(t)) + b(t)\,v(\gamma(t))$. From the previous lemma we find that the coefficients must satisfy certain linear ODE. Solving this ODE we find the field ξ_t, and from the uniqueness of solutions of ODE we find that this field is indeed obtained form ξ_0 by applying the flow of w. By construction, this field belongs to the distribution. The lemma is proved.

6.4. Prove that the distributions spanned by the following vectors are integrable; find any function f whose level surfaces are fibers of the distribution:

a) $u = (1 + x\,y)\,\partial_x + y\,z\,\partial_z$, $v = (1 + x\,y)\,\partial_y + x\,z\,\partial_z$;

b) $u = \cos x\,\cos z\,\partial_x + \sin x\,\sin z\,\partial_z$, $v = \cos y\,\cos z\,\partial_y - \sin y\,\sin z\partial_z$.

7 Differential forms

Definition 7.1. A *cotangent vector* is an element of the dual space $T_x^* M$, i.e. a linear function on the space of tangent vectors at the given point. A *differential* 1*-form* is a family of cotangent vectors depending smoothly on a point of the manifold.

Definition 7.2. If $f : M \to \mathbb{R}$ is a smooth function, then its *differential* df is the 1-form whose value on a tangent vector v is given by the directional derivative along this vector,

$$df(v) = v\,f.$$

7.1. Show that $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$, where dx_i are differentials of coordinate functions.

7.2. Show that the covectors $dx_1, \ldots, dx_n$ form a *basis* in the cotangent space dual to the basis $\partial_{x_1}, \ldots, \partial_{x_x}$.

Definition 7.3. An *exterior k-form* is a multilinear skew-symmetric form in k tangent vectors at some point. A *differential k-form* is a family of exterior k-forms depending smoothly on the point of the manifold.

7.3. Exterior forms form a vector space. Find its dimension. [Answer: $\binom{n}{k}$.]

7.4. Prove that the forms $\omega_{i,j}$, $i < j$, such that $\omega_{ij}(\partial_{x_i}, \partial_{x_j}) = -\omega_{ij}(\partial_{x_j}, \partial_{x_i}) = 1$, and ω_{ij} vanishes on other pais of basic vectors, form a basis in the space of exterior 2-forms.

Definition 7.4. *Exterior product* $\alpha \wedge \beta$ of two 1-forms α and β is the 2-form given by $\alpha \wedge \beta(u, v) = \alpha(u)\,\beta(v) - \alpha(v)\,\beta(u)$.

7.5. In notation of Problem 7.4, show that $\omega_{ij} = dx_i \wedge dx_j$.

7.6. Find formulas for the transformation of a) 1-forms, b) 2-forms under the coordinate change.

7.7. Is that true that any 1-form is a differential of some function on a) $\mathbb{R}^1$, b) S^1, c) $\mathbb{R}^2$, d) S^2?

7.8. Compute $df \wedge df$.

7.9. Describe all 2-forms on a) $\mathbb{R}^1$, b) $\mathbb{R}^2$.

7.10. Give an example of a 2-form on S^2 that does not vanish at any point.

An *area form* on an oriented surface $S \subset \mathbb{R}^3$ is the 2-form whose value on the tangent vectors (u, v) is equal to the (oriented) area of the parallelogram spanned by these vectors.

7.11. Write the area form on the torus given parametrically by

$$\begin{cases} x = (a + R\cos\theta)\,\cos\psi \\ y = (a + R\cos\theta)\,\sin\psi \\ z = R\sin\theta \end{cases}$$

8 Exterior (wedge) product

The space of differential k-forms on the manifold M is denoted by $\Omega^k(M)$. For example, $\Omega^0(M) = \mathcal{F}(M)$ is the space of smooth functions. The operation of *exterior product* $\Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ takes the pair of a k-form α and an l-form β to the $(k+l)$-form $\alpha \wedge \beta$ given by

$$\alpha \wedge \beta(\xi_1, \ldots, \xi_{k+l}) = \sum_{\substack{\sigma_1 < \cdots < \sigma_k \\ \sigma_{k+1} < \cdots < \sigma_{k+l})}} (-1)^{\operatorname{sign}\sigma}\, \alpha(\xi_{\sigma_1}, \ldots, \xi_{\sigma_k})\, \beta(\xi_{\sigma_{k+1}}, \ldots, \xi_{\sigma_{k+l}}),$$

where $(-1)^{\operatorname{sign}\sigma}$ is the sign of the permutation $(1, \ldots, k+l) \mapsto (\sigma_1, \ldots, \sigma_{k+l})$.

8.1. Show that the exterior product is associative and (weighted) skew-commutative:

$$\alpha^k \wedge \beta^l = (-1)^{kl}\, \beta^l \wedge \alpha^k.$$

8.2. Prove that every differential k-form can be presented in local coordinates uniquely in the form

$$\omega = \sum_{i_1 < \cdots < i_k} a_{i_1,\dots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

The $\binom{n}{k}$ smooth functions $a_{i_1,\dots,i_k}$ are called the *coordinates* of the form ω.

Let $F : M^m \to N^n$ be a smooth map. The tangent map (or the derivative) $F_* : T_xM \to T_{F(x)}N$ takes tangent vectors on M to the tangent vectors on N. Nevertheless, for vector fields the operation F_* is in general not defined (unless F is a diffeomorphism, for example, a change of coordinates). However, for differential forms there is a well-defined operation of taking the *pull-back* $F^* : \Omega^k(N) \to \Omega^k(M)$ acting in the direction *opposite* to the direction of the map. By definition,

$$F^*\omega(\xi_1, \dots, \xi_k) = \omega(F_*\xi_1, \dots, F_*\xi_k).$$

A philosophical conclusion of this definition is that the differential forms have better functorial properties comparing with vector fields. This becomes apparent in the behavior of vector fields and differential forms under the coordinate changes.

8.3. Let $p : \mathbb{R}^2 \setminus \{0\} \to S^1$ be the central projection to the unit circle. Compute $p^*d\varphi$ where $d\varphi$ is the *arc length* 1-*form* on the unit circle. [Answer: $\frac{y\,dx - x\,dy}{x^2+y^2}$.]

8.4. Let $p : \mathbb{R}^3 \setminus \{0\} \to S^2$ be the central projection to the unit sphere. Compute $\omega_0 = p^*\sigma$ where σ is the *area* 2-*form* on the unit sphere.
[Answer: $\omega_0 = \frac{1}{r^3}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)$, where $r = \sqrt{x^2 + y^2 + z^2}$.]

8.5. Prove that the wedge product commutes with the operation of taking the pull-back,

$$F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta.$$

8.6. Prove that differentiating a function $f \mapsto df$ commutes with the operation of taking the pull-back.

8.7. Find the transformation rule under a change of coordinates for the *volume form* $dx_1 \wedge \cdots \wedge dx_n$ in $\mathbb{R}^n$ of the top degree n.

8.8. Compute the value of the wedge product of k arbitrary 1-forms $\alpha_1 \wedge \cdots \wedge \alpha_k$ on arbitrary collection of vectors $v_1, \dots, v_k$.

8.9. Show that $\alpha \wedge \alpha = 0$ for any 1-form α. Is that true that $\omega \wedge \omega = 0$ for any 2-form? Give an example of a 2-form ω such that $\omega \wedge \omega \neq 0$. Give an example of a 2-form in $\mathbb{R}^{2n}$ such that $\omega^n = \omega \wedge \cdots \wedge \omega \neq 0$.

9 Gelfand-Leray and area forms

9.1. Let $\alpha_1, \dots, \alpha_k$ be a collection of linearly independent 1-forms and β be another 1-form such that $\beta \wedge \alpha_1 \wedge \cdots \wedge \alpha_k = 0$. Prove:

(a) β is a linear combination of $\alpha_1, \dots, \alpha_k$;

(b) There exists a $(k-1)$-form η such that $\beta \wedge \eta = \alpha_1 \wedge \cdots \wedge \alpha_k$.

9.2. Let $\beta \neq 0$ be a 1-form. Prove that for any k-form ω one has $\beta \wedge \omega = 0$ if and only if there exists a $(k-1)$-form η such that $\omega = \beta \wedge \eta$.

9.3. Let $f = x_1^{a_1} + \cdots + x_n^{a_n}$. Find any form η such that $df \wedge \eta = f\,dx_1 \wedge \cdots \wedge dx_n$.

Consider the Euclidean space $\mathbb{R}^n$ with the standard *volume form* $\Omega = dx_1 \wedge \cdots \wedge dx_n$. Let $H \subset \mathbb{R}^n$ be a smooth hypersurface given implicitly by the equation $F = 0$ where F is a smooth function on $\mathbb{R}^n$ such that $dF \neq 0$.

Definition 9.1. The *Gelfand-Leray* $(n-1)$*-form* on H is the restriction to H of any form η such that $dF \wedge \eta = \Omega$. The Gelfand-Leray form is denoted by $\frac{\Omega}{dF}$.

9.4. There is a freedom in the choice of the form η such that $dF \wedge \eta = \Omega$. Prove that the restriction of η to H is independent of this choice.

9.5. Prove that the Gelfand-Leray form $\frac{\Omega}{dF}$ does not vanish at any point of H. Moreover,

$$dF(\vec{n})\,\frac{\Omega}{dF} = \sigma,$$

where σ is the *area form* on H, and $\vec{n}$ is the *unit normal vector* to H.

9.6. Find a form ω such that $(x\,dx + y\,dy + z\,dz) \wedge \omega = (x^2 + y^2 + z^2)\,dx \wedge dy \wedge dz$. Prove that the restriction of ω to the unit sphere is the area form.

The form $\omega_0 = \frac{1}{r^3}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)$ from the answer to Problem 8.4 is induced from the form ω of the last problem by the map $(x, y, z) \mapsto (\frac{x}{r}, \frac{y}{r}, \frac{z}{r})$, $r = \sqrt{x^2 + y^2 + z^2}$. Remark that this fact implies that the forms ω_0 and $r^3\omega_0 = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$ are invariant under the orthogonal transformation of the Euclidean space $\mathbb{R}^3$. This is not evident from the coordinate representation of these forms. For example, the form $dy\wedge dz+dz\wedge dx+dx\wedge dy$ is not invariant at all.

10 Differential

One of the reason to consider skew-symmetric forms instead of arbitrary tensor fields is that they admit a natural operation $d : \Omega^k(M) \to \Omega^{k+1}(M)$ called the *exterior derivative* or the *differential*. In coordinates it is given by differentiating the coefficients of the form.

Definition 10.1. If a k-form ω is given in coordinates as $\omega = \sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, then its *differential* is the $(k+1)$-form given by

$$d\omega = \sum_{i_1,\ldots,i_k} da_{i_1,\ldots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_{l,i_1,\ldots,i_k} \frac{\partial a_{i_1,\ldots,i_k}}{\partial x_l}\, dx_l \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

It is possible to verify by direct computations that $d\omega$ does not depend on the choice of coordinates. Alternatively, the invariance of the definition can be derived from equivalent definitions given below.

10.1. Show that the differential satisfies the following properties:
(a) Linearity.
(b) If $f \in \Omega^0(M)$ is a smooth function, then df is the usual differential.
(c) $dd = 0$.
(d) $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$.
Show that these properties determine the differential completely. They provide an axiomatic definition of the differential which is equivalent to 10.1.

The following explicit formula gives yet another even more invariant definition of the differential:

10.2. Prove that the value of df on the collection $v_0, \dots, v_k$ of *pairwise commuting* vector fields is given by

$$d\omega(v_0, \dots, v_k) = \sum_{i=0}^{k} (-1)^i v_i \omega(v_0, \dots, \hat{v}_i, \dots, v_k).$$

10.3. Find the value of $d\omega$ on arbitrary collection of $k+1$ vector fields. [Answer:

$$d\omega(v_0, \dots, v_k) = \sum_{i=0}^{k} (-1)^i v_i \omega(v_0, \dots, \hat{v}_i, \dots, v_k) + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([v_i, v_j], v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_k).]$$

For small k the last formula gives:

$$n = 2: \qquad d\omega(\xi, \eta) = \xi\omega(\eta) - \eta\omega(\xi) - \omega([\xi, \eta]);$$

$$n = 3: \qquad d\omega(\xi, \eta, \zeta) = \xi\omega(\eta, \zeta) - \omega([\xi, \eta], \zeta) + \circlearrowleft,$$

where $\circlearrowleft$ denotes the terms obtained by cyclic permutation of the fields ξ, η, ζ.

The invariance of the differential has a more general meaning:

10.4. Prove that the differential commutes with the operation of taking the pull-back,

$$F^*(d\,\omega) = d\,F^*(\omega).$$

The form ω is called *closed* if $d\omega = 0$. It is called *exact* if $\omega = d\eta$ for some form η. Exact forms are closed ($d^2 = 0$). The inverse is not true in general.

10.5. For which value of α the form $\frac{1}{r^\alpha}\omega$ is closed, where $r = \sqrt{\sum x_i^2}$, and

$$\omega = \sum_{i=1}^{n} (-1)^i x_i \, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n?$$

Prove that this form is not exact (at least for $n = 2$).

The language of differential forms gives the following nice reformulation of the Frobenius theorem on integrable distributions. A 2-distribution in $\mathbb{R}^3$ can be given by a 1-form ω: the planes of the distribution are the kernels of ω. This form is defined up to a multiplication by a non-vanishing function. The distribution is integrable iff $\omega = f\,dg$ for some smooth functions f, g.

10.6. Prove that the Frobenius criterium of the integrability of the distribution (see Theorem 6.2 on page 12) is equivalent to the equality

$$\omega \wedge d\omega \equiv 0.$$

10.7. For $\omega = \frac{dx}{yz} + \frac{dy}{xz} + \frac{dz}{xy}$ compute $\omega \wedge d\omega$. Find a function f such that the form $f\omega$ is closed.

10.8. Find the explicit formula for the differential of a 1-form $A\,dx + B\,dy + C\,dz$; for a 2-form $P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$ in $\mathbb{R}^3$.

In $\mathbb{R}^3$ both 1-forms and 2-forms are given by three smooth functions. Therefore, in classical calculus they are identified with vector fields. Similarly, a 3-form can be treated as function. With this identification, the exterior differential $d\omega^k$ for a $k = 0, 1, 2$ is called the *gradient*, *curl*, and *divergence* respectively:

$$\begin{array}{ccccccc} \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 \\ \| & & \| & & \| & & \| \\ \mathcal{F} & \xrightarrow{\text{grad}} & \mathcal{F}^3 & \xrightarrow{\text{curl}} & \mathcal{F}^3 & \xrightarrow{\text{div}} & \mathcal{F} \end{array}$$

Thus the language of differential forms unifies and simplifies various notions and statements of classical calculus. For example, the identities $\operatorname{curl}\operatorname{grad} = 0$, $\operatorname{div}\operatorname{curl} = 0$ are partial cases of the equality $d^2 = 0$, etc.

11 Integration of differential forms

The possibility to differentiate is a very nice property of the differential forms. But the main destination of differential n-forms is that they provide an invariant meaning of the *integration* over smooth n-dimensional manifolds. For example, the rule for a change of the variable in the definite integral $\int f(x)\,dx$ shows that what we actually integrate is *not* the function f but the *differential* 1-*form* $f\,dx$.

The *support* of a differential form is the closure of the locus where the form is different from zero. Assume that the support of an n form $\omega = f(x)\,dx_1 \wedge \cdots \wedge dx_n$ in $\mathbb{R}^n$ is bounded. Without loss of generality assume that the support is contained in the unit cube $I^n = [0,1]^n$. Then we define the integral of such a form as the usual multiple integral

$$\int_{\mathbb{R}^n} \omega = \int_{I^n} f(x)\,dx_1 \wedge \cdots \wedge dx_n = \int_0^1 \!\!\ldots\! \int_0^1 f(x)\,dx_1 \ldots dx_n. \qquad (*)$$

According to the well-known rule, in the case of a change of coordinates in the multiple integrals the function f under integration must be multiplied by the absolute value of the Jacobian of the coordinate change. If the Jacobian is positive, the same rule is used for the transformation of the differential form $f(x)\,dx_1 \wedge \cdots \wedge dx_n$. It follows that *the integral* $(*)$ *is independent of the choice of coordinates provided the new coordinates preserve the orientation of* $\mathbb{R}^n$.

As a consequence we get that a differential n-forms ω over a smooth manifold M can be defined when M is *n-dimensional, compact* and *oriented*. If ω is supported in some coordinate chart $F : U \to M$, $U \subset \mathbb{R}^n$, then we set, by definition,

$$\int_M \omega \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} F^*\omega.$$

In the general case we represent ω as a finite sum $\omega = \sum \omega_i$ such that each ω_i is supported in some chart, then we set

$$\int_M \omega = \int_M \sum \omega_i = \sum \int_M \omega_i.$$

The splitting $\omega = \sum \omega_i$ can be obtained using a partition of unity.

Definition 11.1. A *partition of unity* is a finite collection of smooth functions ρ_i such that (i) each ρ_i is non-negative and supported in some coordinate chart; and (ii) $\sum \rho_i \equiv 1$.

It is clear that every partition of unity provides the necessary splitting of ω: it is sufficient to set $\omega_i = \rho_i \, \omega$.

Theorem 11.1. *Every compact manifold admits a finite partition of unity.*

First note that a function ρ_x satisfying (i) and such that $\rho(x) \neq 0$ exists for any point $x \in M$: choose a *cut-off* (a hat shaped) function in some local coordinate system and extend it by zero to the whole manifold. Since M is compact we can choose a finite number $\rho_{x_1}, \dots, \rho_{x_m}$ of such functions such that at any point y on M at least one of ρ_{x_i}'s is non-vanishing. Then the collection of the functions $\rho_i = \rho_{x_i}/(\sum \rho_{x_j})$ is a partition of unity.

11.1. How the integral changes if we integrate the same form ω over the same manifold M with the opposite choice of the orientation?

11.2. Compute the integral over the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ of the form a) $dx \wedge dy$; b) $z \, dx \wedge dy$; c) $z^2 dx \wedge dy$; d) $\frac{dy \wedge dz}{x} + \frac{dz \wedge dx}{y} + \frac{dx \wedge dy}{z}$.

11.3. Compute the integral over the top half ($z \geq 0$) of the same ellipsoid of the form a) $x^3 dy \wedge dz$; b) $yz \, dz \wedge dx$.

12 Stokes formula

The notion of a smooth manifold extends to the notion of a *manifold with boundary*[1] $(M, \partial M)$. Similar to the case of smooth manifolds, every point has a neighborhood diffeomorphic to a standard chart. But now we have local charts of two types: 1) the usual ones, domains in $\mathbb{R}^n$; 2) a domain in the half-space $\mathbb{R}^n_+$ given by the inequality $x_1 \geq 0$. The points of the second type form the *boundary* ∂M. It is a smooth manifold (without boundary) of dimension $n-1$. By smooth functions, differential forms, vector fields etc on $(M, \partial M)$ we mean the corresponding objects on $M \setminus \partial M$ that can be extended smoothly outside the boundary in a neighborhood of any point (but such an extension is not fixed).

If M is oriented, then its boundary ∂M is also endowed with an orientation. The *natural orientation of the boundary* is given by the following rule: if $e_1, \dots, e_n$ is a positive frame of tangent vectors at a point of the boundary, such that $e_2, \dots, e_n$ are tangent to the boundary and e_1 is transversal to the boundary and directed outside the manifold, then the frame $e_2, \dots, e_n$ orients positively the boundary. Shortly, this rule says: **'exterior normal vector to the first place'**. In the dual terms, let $(M, \partial M)$ be given in the local coordinates by the inequality $x_1 \geq 0$. Then the form $dx_2 \wedge \dots \wedge dx_n$ orients positively the boundary $x_1 = 0$ iff the form $dx_1 \wedge \dots \wedge dx_n$ orients positively the manifold itself.

Theorem 12.1. (Stokes formula). *If $(M, \partial M)$ is a compact oriented n-dimensional manifold with boundary and ω is an $(n-1)$-form on it, then*

$$\int_{\partial M} \omega = \int_M d\omega.$$

[1]This traditional terminology is ambiguous: a manifold with boundary is *not* a manifold. Furthermore, a usual smooth manifold can be thought as a manifold with (empty) boundary.

First let us verity this formula for the cube I^n (the cube is not a manifold with boundary but it is irrelevant). Without loss of generality we assume that ω has the form $\omega = f(x)\,dx_2 \wedge \dots \wedge dx_n$. Then $d\omega = \frac{\partial f}{\partial x_1} dx_1 \wedge \dots \wedge dx_n$. By definition, we have

$$\int_{I^n} d\omega = \int_{I^n} \frac{\partial f}{\partial x_1} dx_1 \dots dx_n.$$

Integrating along x_1-coordinate we get by the Newton-Leibnitz formula

$$\int_{I^{n-1}} \left(f \Big|_{x_1=0}^{x_1=1} \right) dx_2 \dots dx_n = \int_{I^{n-1}} f \Big|_{x_1=1} dx_2 \dots dx_n - \int_{I^{n-1}} f \Big|_{x_1=0} dx_2 \dots dx_n.$$

By our agreement on the orientation of the boundary, these two terms correspond to the integral of ω over the top ($x_1 = 1$) and the bottom ($x_1 = 0$) faces of the cube I^n respectively. This proves the Stokes formula in this case since ω vanishes on other faces.

The general case can be reduced to the case of the cube using a partition of unity. Indeed, if $\omega = \sum \omega_i$ such that each ω_i is supported on some coordinate chart, then it is sufficient to prove the Stokes formula for each ω_i. For ω_i we have one of the two options: 1) $\operatorname{supp}\omega \cap \partial M = \emptyset$. In this case the support of ω embeds into the cube I^n and does not intersect the boundary of that cube. Therefore, we have $\int_M d\omega_i = \int_{\partial M} \omega = 0$. 2) $\operatorname{supp}\omega \cap \partial M \neq \emptyset$. In this case the support of ω embeds into the cube I^n and the only face on which ω is non-trivial is the face $x_1 = 0$ corresponding to a part of the boundary ∂M. Therefore, by the Stokes formula for a cube, we have

$$\int_M d\omega = \int_{I^n} d\omega = \int_{\partial I^n} \omega = \int_{I^{n-1}} \omega = \int_{\partial M} \omega.$$

Theorem is proved completely.

12.1. Verify the Stokes formula for

a) $\omega = x^2y^3\,dx + dy + z\,dz$ on the top half-sphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$;

b) $\omega = y\,dx + z\,dy + x\,dz$ on the disc of the intersection of the plane $x + z = a$ with the ball $x^2 + y^2 + z^2 \leq a^2$;

c) $\omega = (z^2 - x^2)dx + (x^2 - y^2)dy + (y^2 - z^2)dz$ on the helix surface $x = u\cos v$, $y = u\sin v$, $z = cv$ ($a \leq u \leq b$, $0 \leq v \leq 2\pi$).

13 Cartan Identity $L = id + di$

Let v be a vector field, and ω a k-form. The *Lie derivative* $L_v\omega$ is defined as follows (see page 11). The field v generates the flow g_v^t. Consider the one-parameter family of forms $\omega_t = (g_v^t)^*\omega$. Then we set

$$L_v\omega = \frac{d}{dt}\omega_t \Big|_{t=0}.$$

The exterior derivative d increases by 1 the degree of a form. There is another operation on forms, the 'substitution' i_v, that decreases the degree by 1. By definition, if ω is a k-form, then

$$(i_v\omega)(\xi_1, \dots, \xi_{k-1}) = \omega(v, \xi_1, \dots, \xi_{k-1}).$$

The three operations are related by the following identity.

Theorem 13.1. (Cartan identity). $L_v\omega = i_v d\omega + d\,i_v\omega$.

13.1. Prove this identity in the case when ω is a) a function; b) an exact 1-form, $\omega = df$, $f \in \mathcal{F}(M)$.

For forms of greater degree the identity can be proved by induction.

13.2. Prove that the three operations are subject to the following generalized Leibnitz rules:

$$L_v(\alpha \wedge \beta) = (L_v\alpha) \wedge \beta + \alpha \wedge (L_v\beta);$$
$$i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge (i_v\beta);$$
$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge (d\beta).$$

13.3. Prove the Cartan identity.

One of the applications of the Cartan identity is the following independent 'geometric' proof of the Stokes formula. Let $(M, \partial M)$ be an oriented n-manifold with boundary, ω be an $(n-1)$-form. Fix any non-vanishing n-form Ω on M (the 'volume' form). It provides the isomorphism between the spaces of vector fields on M and $(n-1)$-forms: $\xi \mapsto i_\xi\Omega$. In particular, there is a unique vector field v such that $\omega = i_v\Omega$. The Cartan identity implies

$$L_v\Omega = d\, i_v\Omega = d\omega.$$

Let us compute the integral of the left hand side over M. By definition of the Lie derivative,

$$\int_M d\omega = \int_M L_v\Omega \approx \frac{1}{t}\int_M ((g_v^t)^*\Omega - \Omega).$$

The invariance of the integral implies that $\int_M (g_v^t)^*\Omega = \int_{g_v^t M}\Omega$. Thus we must compute the difference of the integrals of the same form Ω over the domains $g_v^t M$ and M. But these two domains overlap on the most part and the only part that contributes to the integral is the narrow strip swept by the boundary ∂M under the flow of v. It is clear that the contribution to this integral of the parallelogram spanned by the vectors $\xi_1, \ldots, \xi_{n-1}$ tangent to the boundary is (asymptotically) equal to $t\,\Omega(v, \xi_1, \ldots, \xi_{n-1}) = t\, i_v\Omega(\xi_1, \ldots, \xi_{n-1})$. Therefore,

$$\int_{g_v^t M - M} \Omega \approx \int_{\partial M} i_v\Omega = \int_{\partial M} \omega.$$

The Stokes formula is proved.

If M is a bounded domain in $\mathbb{R}^n$, and $\Omega = dx_1 \wedge \cdots \wedge dx_n$, then $i_v\Omega\Big|_{\partial M}$ is equal to $(v, \vec{n})\,\sigma$, where $\vec{n}$ is the normal unit vector to the boundary ∂M, and σ is the area form on the boundary (see page 15). Besides, $d\, i_v\Omega$ is an n-form, so it can be represented as $f\,\Omega$ for some function f. This function is called the *divergence* of the field v and denoted by $\operatorname{div} v$. In this notation the Stokes formula is known as the *Gauss-Ostrogradskij formula*

$$\int_M \operatorname{div} v\, \Omega = \int_{\partial M} (v, \vec{n})\, \sigma.$$

If v is the field of velocities of some particles, then the right hand side of the integral describes the flow of these particles through the boundary of M, while the left hand side describes the productivity of the local sources in M. Thus, the formula above represents the conservation law of materia.

13.4. Find the flow of the Newton field $F = \frac{\vec{r}}{r^3}$, $\vec{r} = x\partial_x + y\partial_y + z\partial_z$, though the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

14 Poincaré Lemma. De Rham cohomology

If a differential form ω is exact, then it is closed since $d^2 = 0$. The inverse is not true. For example, the arc length 1-form $d\varphi$ on the circle S^1 is not exact, since the argument function φ is not globally defined on the circle.

Another example is the form $\sigma_2 = \frac{1}{r^3}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)$ in $\mathbb{R}^3 \setminus \{0\}$, or, more general, the form $\sigma_{n-1} = \frac{1}{r^n}\sum(-1)^i x_i\,dx_1 \wedge \ldots \hat{dx_i} \wedge \ldots dx_n$ in $\mathbb{R}^n \setminus \{0\}$ of Problem 10.5. To see that it is not exact, we remark that the restriction of this form to the unit sphere is the area form, therefore, the integral

$$c_n = \int_{S^{n-1}} \sigma_{n-1}$$

is equal to the area of the unit sphere, in particular, it is non-zero. On the other hand, if σ_{r-1} were exact, we would have by the Stokes formula, $\int_{S^{n-1}} \sigma_{n-1} = 0$. Therefore, σ_{n-1} is not exact.

14.1. Compute the volume of the unit ball in $\mathbb{R}^n$ and the area of its boundary.

In all examples the existence of closed non-exact forms reflects global topological invariants of the manifold. If M is topologically trivial then all these invariants vanish and there is no more difference between closed forms and exact ones.

Theorem 14.1. (Poincaré lemma). *Any closed differential k-form, $k \geq 1$, in the open ball $B^n \subset \mathbb{R}^n$ is exact, $d\omega = 0 \Rightarrow \omega = d\eta$.*

Let ω be a closed k-form. We look for a form η such that $d\eta = \omega$. Instead of defining η itself we define its *integral* over $(k-1)$-dimensional manifolds (with boundary, and even corners etc.). Namely, we set

$$\int_U \eta \stackrel{\text{def}}{=} \int_{CU} \omega,$$

where CU is the *cone* over U, the union of all segments connecting the points of U with the origin. Making U infinitesimally small we may recover η itself.

14.2. Prove that the form η is given explicitly by

$$\eta_x(v_1, \ldots, v_{k-1}) = \int_0^1 \omega_{tx}(\textstyle\sum x_i \frac{\partial}{\partial x_i}, tv_1, \ldots, tv_{k-1})\,dt\,.$$

Denote by $h : \Omega^k(B^n) \to \Omega^{k-1}(B^n)$ the correspondence $\omega \mapsto \eta$ given by the formula above. Remark that this operation is defined on any form, not necessary closed.

Lemma 14.2. *The operation h satisfies the identity $h\,d + d\,h = \mathrm{id}$, i.e.*

$$h\,d\omega + d\,h\omega = \omega, \qquad \omega \in \Omega^k(B^n). \tag{$*$}$$

Poincaré Lemma is the corollary of this identity. Indeed, if ω is closed then $h\,d\omega = 0$, therefore, $\omega = d(h\omega)$.

It is possible to prove the lemma directly from the definition of the operation h. We shell use a different argument. Namely, we will verify that the integral of the both sides of the formula gives the same values for any k-dimensional manifold (with boundary) U. To see it, we remark the following set-theoretical decomposition

$$\partial CU = U \cup C\partial U.$$

Every term in this relation corresponds to a certain term of the identity (∗): by the Stokes formula,

$$\int_{\partial CU} \omega = \int_{CU} d\omega = \int_U hd\omega.$$

Similarly,

$$\int_{C\partial U} \omega = \int_{\partial U} h\omega = \int_U dh\omega.$$

This proves the identity (∗), at least up to some signs in the terms.

14.3. Complete the details of the proof.

14.4. Compute $h(\omega)$ for $\omega = A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$, where A, B, C are smooth functions ($n = 3$).

14.5. Compute $hd + dh$ on
a) 0-forms, i.e. functions;
b) exact 1-forms;
c) arbitrary 1-forms.

14.6. Let $\omega = c(x)\,dx_1 \wedge \cdots \wedge dx_k$. Prove that $h(\omega) = a(x) \sum (-1)^{i-1} x_i\, dx_1 \wedge \ldots \hat{dx_i} \wedge \cdots \wedge dx_k$. Find $a(x)$.

Definition 14.1. The kth *De Rham cohomology* $H^k_{\mathrm{DR}}(M)$ is the quotient space of the closed k-forms over exact k-forms.

The Poincaré lemma can be reformulated by saying $H^k_{\mathrm{DR}}(\mathbb{R}^n) = 0$ for all $k > 0$. The De Rham cohomology is a topological (in fact, homotopy) invariant of the manifold. If M is compact, then the space $H^k_{\mathrm{DR}}(M)$ is finite-dimensional for each k.

14.7. Compute the following De Rham cohomology spaces:
a) $H^0(M)$ for any manifold;
b) $H^k_{\mathrm{DR}}(M)$ for any $k > \dim M$;
b) $H^1_{\mathrm{DR}}(S^1)$;
c) $H^1_{\mathrm{DR}}(S^2)$;
d) $H^1_{\mathrm{DR}}(M)$ for any simply-connected manifold M;
e) $H^2_{\mathrm{DR}}(S^1 \times S^1)$;
f) $H^2_{\mathrm{DR}}(S^2)$.

14.8. Let ω be an n-form on a compact oriented n-manifold M such that $\int_M \omega = 0$. Is it true that $\omega = d\eta$ for some $(k-1)$-form η?
[Answer: yes. In fact, the integral provides the isomorphism

$$\int_M : H^n_{\mathrm{DR}}(M) \xrightarrow{\sim} \mathbb{R}. \,]$$

15 Short review of principle formulas

1. *Vector fields.* A vector field $v = \sum v_i(x)\dfrac{\partial}{\partial x_i}$ is given in coordinates by n functions v_i where n is the dimension of the manifold. The *derivative* of a function along the field, $f \mapsto vf = \sum v_i\, \partial f/\partial x_i$ is a *derivation* of the ring of functions that is it is $\mathbb{R}$-linear and satisfies the Leibnitz rule, $v(fg) = f\,vg + g\,vf$. Moreover, there is one-to-one correspondence between the spaces of vector fields and derivations.

2. *Lie derivative* L_v along the field v acts on various objects of tensor nature as follows: the phase flow g^t of the field v translates the given tensor field w and we get a family of tensor fields w_t. By definition, $L_v w = \frac{d}{dt} w_t \Big|_{t=0}$.

3. *Commutator* $[u, v]$ of vector fields $u = \sum u_i \dfrac{\partial}{\partial x_i}$ and $v = \sum v_j \dfrac{\partial}{\partial x_j}$ is defined as the derivation $f \mapsto uvf - vuf$. Equivalently, it can be defined as the Lie derivative, $[u, v] = L_u v = \frac{d}{dt} v_t \Big|_{t=0}$, where $v_t(x) = \left(g^t\right)_*^{-1} v(g^t x)$, and the family of diffeomorphisms g^t is the flow of u. In coordinates the commutator is written as

$$[u, v] = \sum (uv_j)\frac{\partial}{\partial x_j} - \sum (vu_i)\frac{\partial}{\partial x_i} = \sum \left(u_j \frac{\partial v_i}{\partial x_j} - v_j \frac{\partial u_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

The commutator is not linear with respect to the multiplication of the fields by functions but satisfies the identities

$$[u, gv] = g[u, v] + ug\, v, \qquad [fu, v] = f[u, v] - vf\, u.$$

4. *Differential forms* of degree k are multilinear skew-symmetric functions of a collection of k vector fields. The simplest example is the 1-form df, where f is a function. By definition, $df(\xi) = \xi f$.

5. *Exterior product* of two forms α and β of degrees k and l respectively is the $(k + l)$-form $\alpha \wedge \beta$ given by

$$\alpha^k \wedge \beta^l(\xi_1, \ldots, \xi_{k+l}) = \sum_{\substack{\sigma_1 < \cdots < \sigma_k \\ \sigma_{k+1} < \cdots < \sigma_{k+l}}} (-1)^\sigma \alpha^k(\xi_{\sigma_1}, \ldots, \xi_{\sigma_k})\, \beta^l(\xi_{\sigma_{k+1}}, \ldots, \xi_{\sigma_{k+l}}) \,,$$

where $(-1)^\sigma$ is the sign of the permutation $(1, \ldots, k + l) \mapsto (\sigma_1, \ldots, \sigma_{k+l})$. This operation is *associative and (super)commutative*, $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$. Using the exterior product any k-form can uniquely be written locally in the form

$$\alpha = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where x_i are the coordinate functions of the chosen coordinate system.

6. *The Lie derivative* $L_v \alpha$ satisfies the identity

$$v\alpha(\xi_1, \ldots, \xi_k) = (L_v\alpha)(\xi_1, \ldots, \xi_k) + \alpha([v, \xi_1], \xi_2, \ldots, \xi_k) + \cdots + \alpha(\xi_1, \ldots, \xi_{k-1}, [v, \xi_k]).$$

7. The Lie derivative is defined also by the following properties

1) it is $\mathbb{R}$-linear;
2) for a 0-form f, that is for a function, $L_v f = vf$;
3) for 1-forms of the form df we have $L_v df = d(vf)$;
4) $L_v(\alpha^k \wedge \beta^l) = (L_v \alpha) \wedge \beta + \alpha \wedge (L_v \beta)$.

8. ***Differential*** of a k-form ω is the $(k+1)$-form defined by the relation

$$d\omega(v_0, ..., v_k) = \sum_{i=0}^{k} (-1)^i v_i \omega(v_0, ..., \hat{v}_i, ..., v_k) + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([v_i, v_j], v_0, ..., \hat{v}_i, ..., \hat{v}_j, ..., v_k \,.)$$

In particular,

$$\begin{aligned} k=1: \qquad & d\alpha(\xi, \eta) = \quad & \xi\alpha(\eta) - \eta\alpha(\xi) - \alpha([\xi, \eta]) \\ k=2: \qquad & d\alpha(\xi_1, \xi_2, \xi_3) = \quad & \xi_1 \alpha(\xi_2, \xi_3) - \alpha([\xi_1, \xi_2], \xi_3) + \circlearrowleft, \end{aligned}$$

where the terms $\circlearrowleft$ are given by cyclic permutations of the indices $1, 2, 3$.

9. In an equivalent way the differential can be defined by the following axioms
1) it is $\mathbb{R}$-linear;
2) for 0-forms that is for a function, $df(\xi) = \xi f$;
3) $d(\alpha^k \wedge \beta^l) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$;
4) $d^2 = 0$.

10. The second definition implies the *coordinate representation* of the differential,

$$d\Bigg(\sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \Bigg) = \sum_{i_1 < \cdots < i_k, j} \frac{\partial a_{i_1, \ldots, i_k}}{\partial x_j} \, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

11. ***Cartan identity*** $L = id + di$. Denote by i_v the operation of the substitution to the first place of the field v to the form (this operation decreases by one the degree of the form). Then the following identity holds

$$L_v \alpha = i_v d\alpha + d i_v \alpha.$$

It can be proven either directly from the definition or by induction decomposing the form into the forms of smaller degrees.

12. ***Integral*** of a k-form over a k-dimensional domain $U \subset \mathbb{R}^n$ is defined as the multiple integral

$$\int_U a(x) \, dx_1 \wedge \cdots \wedge dx_k = \iint_U a(x) \, dx_1 \ldots dx_k.$$

This definition does not depend on the choice of the coordinates whenever the coordinate change *preserves the orientation*. The integral of a k-form over a compact *oriented* k-dimensional manifold M is determined by catting the manifold into small pieces or by a partition of unity.

13. ***Stokes formula***. If an oriented manifold M has a boundary ∂M then for any form ω of degree $k-1$ one has

$$\int_M d\omega = \int_{\partial M} \omega.$$

The boundary ∂M is oriented by the rule: '**the exterior normal to the first place**', namely, the frame $\xi_2, \ldots, \xi_k$ orients positively the boundary ∂M iff the frame $\xi_1, \xi_2, \ldots, \xi_k$ orients positively M where the vector ξ_1 is directed outwards.

Different system of notations

Some more algebraically educated authors use different agreements about the definition of the wedge product. Namely, they consider exterior forms as skew-symmetric tensors and they set

$$\alpha \wedge \beta = \mathrm{Alt}(\alpha \otimes \beta),$$

where the *alternation operation* Alt acts on monomials of degree k by

$$\mathrm{Alt}(\alpha_1 \otimes \cdots \otimes \alpha_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\operatorname{sign} \sigma} \alpha_{\sigma_1} \otimes \cdots \otimes \alpha_{\sigma_k}.$$

This definition of the wedge product differs from our by the factor $\binom{k+l}{k} = \frac{(k+l)!}{k!\, l!}$, where $k = \deg \alpha$, $l = \deg \beta$.

Similar factor appear in all other relations. For example, the value of the volume form $dx_1 \wedge \cdots \wedge dx_n$ on the coordinate frame $(\partial_{x_1}, \ldots, \partial_{x_n})$ is equal, according to this system of notations, to $\frac{1}{n!}$ instead of 1, which is quite unnatural from the geometric point of view.

Another example is the Cartan identity. To keep it in the classical form $L = id + di$ one needs to define the substitution operator i_v in this system of notations by $i_v \omega = k\, \omega(v, \cdots)$, where $k = \deg \omega$.

Both systems of notations are quite popular and it is not clear immediately from a particular paper which agreement is used by the author.

Bibliography

There exists an incredible number of textbooks on Calculus but only few of them explain the language of differentiable manifolds. In the list below only those books are included that are available at the library of Independent Moscow University.

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