

A triangulated category  $\mathcal{J}$  is an additive category ( $k$ -linear). This comes with

- Shift functors:  $[t]: \mathcal{J} \rightarrow \mathcal{J}$  for each  $t \in \mathbb{Z}$  (Note  $[t_1] \cdot [t_2] = [t_1 + t_2]$ ).
- Class of distinguished triangles

$$T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} T_3 \xrightarrow{f_3} T_1[1].$$

↑ better to have equality and not  $\simeq$ . Can always pass to an equiv. category to make this equality.

The reader is referred to Gelfand-Manin for all axioms of triangulated categories. The most important axiom is:

For any  $f: T_1 \rightarrow T_2$ ,  $\exists$  a distinguished triangle  $T_1 \xrightarrow{f} T_2 \rightarrow T_3 \rightarrow T_1[1]$

$$T_3 =: \text{Cone}(f).$$

The isomorphism class of  $\text{Cone}(f)$  is well defined but is not functorial.

Some remarks about

$$\begin{array}{ccccccc} T_1' & \rightarrow & T_2' & \rightarrow & T_3' & \rightarrow & T_1'[1] \\ \downarrow & & \downarrow & & \downarrow & & \\ T_1 & \xrightarrow{f} & T_2 & \rightarrow & T_3 & \rightarrow & T_1[1] \end{array}$$

Some notation:

- $D(\mathcal{C}) :=$  the derived category of the abelian category  $\mathcal{C}$ .
- $D^b(\mathcal{C}) =$  Bounded derived category.
- $\text{Ext}^i(T, T') = \text{Hom}^i(T, T') = \text{Hom}(T, T'[i])$
- $\text{Hom}^i(T, T') = \bigoplus_{j \in \mathbb{Z}} \text{Hom}^i(T, T'[j]) \in \text{grmod}(k) = D(k)$ .
- $\forall v \in \text{mod}(k); T \in \mathcal{J}$  then  $V \otimes_k T = T^{\oplus \dim V}$

This is not a good definition (non-functorial)

So you define it by the functor it represents:

$$\left. \begin{array}{l} \text{Hom}(V \otimes T, T') = \text{Hom}(V^*, \text{Hom}(T, T')) \\ \text{Hom}(T', V \otimes T) = V \otimes \text{Hom}(T', T) \end{array} \right\} \textcircled{*}$$

- $V^* \in \text{grmod}(k); V^* = \bigoplus_{i \in \mathbb{Z}} V_i[-i]$  (Replace  $\text{Hom}$  with  $\text{Hom}^*$ )
- $V^* \otimes T = \bigoplus_{i \in \mathbb{Z}} V_i \otimes T[-i]$  and  $V$  with  $V^*$  in the above conditions  $\textcircled{**}$



The most important notion for these lectures is that of:

### Semiorthogonal Decomposition

Let  $\mathcal{J}$  be a triangulated category,  $\mathcal{A}, \mathcal{B} \subset \mathcal{J}$  be strictly full triangulated subcategories

is closed under shifts and cones.

Def:  $\mathcal{J} = \langle \mathcal{A}, \mathcal{B} \rangle$  is a semiorthogonal decomposition if:

Can also take  $\text{Hom}^*$

1)  $\text{Hom}(\mathcal{B}, \mathcal{A}) = 0$  (ie  $\forall A \in \mathcal{A}, B \in \mathcal{B}$  we have  $\text{Hom}(B, A) = 0$ )

2) For any  $T \in \mathcal{J}$  there is a distinguished triangle  $T_{\mathcal{B}} \rightarrow T \rightarrow T_{\mathcal{A}} \rightarrow T_{\mathcal{B}}[1]$ .

with  $T_{\mathcal{A}} \in \mathcal{A}$  and  $T_{\mathcal{B}} \in \mathcal{B}$ .

### Properties:

1) Functoriality: For any  $f: T \rightarrow T'$  there exist unique  $f_{\mathcal{A}}$  and  $f_{\mathcal{B}}$  such that the following

diagram commutes:

$$\begin{array}{ccccccc}
 & & & \text{in proof} & & & \\
 & & & \dots \rightarrow \text{map below is just composition with this arrow} & & & \\
 T_{\mathcal{B}} & \rightarrow & T & \rightarrow & T_{\mathcal{A}} & \rightarrow & T_{\mathcal{B}}[1] \\
 f_{\mathcal{B}} \downarrow & & \downarrow f & & \downarrow f_{\mathcal{A}} & & \downarrow f_{\mathcal{B}}[1] \\
 T'_{\mathcal{B}} & \rightarrow & T' & \rightarrow & T'_{\mathcal{A}} & \rightarrow & T'_{\mathcal{B}}[1] \\
 & & & & \text{comments in proof} & & 
 \end{array}$$

Proof: Apply  $\text{Hom}(-, T'_{\mathcal{A}})$ . Get long exact sequence

$$\dots \rightarrow \text{Hom}(T_{\mathcal{B}}[1], T'_{\mathcal{A}}) \rightarrow \text{Hom}(T_{\mathcal{A}}, T'_{\mathcal{A}}) \xrightarrow{\cong} \text{Hom}(T, T'_{\mathcal{A}}) \rightarrow \text{Hom}(T_{\mathcal{B}}, T'_{\mathcal{A}}) \rightarrow \dots$$

$\begin{matrix} \text{"} & & \text{"} \\ 0 & & f_{\mathcal{A}} & & 0 \end{matrix}$

$\Rightarrow \exists! f_{\mathcal{A}}$  s.t. the middle square commutes.

The exercise shows that the

Exercise:  $\exists! f_{\mathcal{B}}$  s.t. the left square commutes.  $\square$

left square commutes.

### Corollary:

$$\left. \begin{array}{l} T \mapsto T_{\mathcal{A}} \\ f \mapsto f_{\mathcal{A}} \end{array} \right\} \text{ is a functor and so is } \left. \begin{array}{l} T \mapsto T_{\mathcal{B}} \\ f \mapsto f_{\mathcal{B}} \end{array} \right\}$$

These are both functors from  $\mathcal{J} \rightarrow \mathcal{A}$  and  $\mathcal{J} \rightarrow \mathcal{B}$ .

Let the adjoint functors be:

$$\mathcal{A} \xrightarrow{\alpha} \mathcal{J} \text{ and } \mathcal{B} \xrightarrow{\beta} \mathcal{J}$$

Then  $T \mapsto T_{\mathcal{A}}$  is left adjoint to  $\alpha$  (denoted  $\alpha^*$ ) and  $T \mapsto T_{\mathcal{B}}$  is right adjoint to  $\beta$  (denoted  $\beta^!$ )