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On nodal Enriques surfaces and quartic double solids

Colin Ingalls · Alexander Kuznetsov

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Abstract We consider the class of singular double coverings $X \rightarrow \mathbb{P}^3$ ramified in the degeneration locus D of a family of 2-dimensional quadrics. These are precisely the quartic double solids constructed by Artin and Mumford as examples of unirational but nonrational conic bundles. With such a quartic surface D , one can associate an Enriques surface S which is the factor of the blowup of D by a natural involution acting without fixed points (such Enriques surfaces are known as nodal Enriques surfaces or Reye congruences). We show that the nontrivial part of the derived category of coherent sheaves on this Enriques surface S is equivalent to the nontrivial part of the derived category of a minimal resolution of singularities of X .

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1 Introduction

We work over the field of complex numbers. Recall that an Enriques surface S is a smooth projective surface which can be represented as the quotient of a K3 surface by an involution acting without fixed points. Consequently the canonical class K_S of S is 2-torsion, $2K_S = 0$. More than 10 years ago, Zube [17] made the following nice observation for general Enriques surfaces.

Theorem 1.1 ([17]) *A choice of an ample polarization of degree 10 determines 10 elliptic pencils on S each with two double fibers. Let F_i^+, F_i^- , $i = 1, \dots, 10$ be the double fibres of these pencils. Then the line bundles $\{\mathcal{O}_S(-F_i^+)\}_{i=1}^{10}$ form a completely orthogonal exceptional collection in $\mathcal{D}^b(S)$.*

Since then, it has been an intriguing question to describe the orthogonal subcategory

$$\mathcal{A}_S = {}^\perp \langle \{\mathcal{O}_S(-F_i^+)\}_{i=1}^{10} \rangle. \quad (1)$$

The answer to this question is still not known. Our goal is to give a partial answer for nodal Enriques surfaces. Recall that an Enriques surface S is called **nodal** (another name is **Reye congruence**), if the image of S with respect to the morphism given by the linear system $\frac{1}{3} \sum F_i^+$ (called **Fano model** of S) is contained in a quadric. The main result of the present paper is a description of the category $\mathcal{A}_S$ for nodal Enriques surfaces in other terms. Actually, we show that $\mathcal{A}_S$ can be represented as a semiorthogonal component of the derived category of a very special quartic double solid.

Recall that a quartic double solid is a double covering of $\mathbb{P}^3$ ramified in a quartic surface $D \subset \mathbb{P}^3$. The bridge between quartic double solids and nodal Enriques surfaces is provided by special quartic surfaces, the so-called quartic symmetroids.

A **quartic symmetroid** is a quartic surface in $\mathbb{P}^3 = \mathbb{P}(W)$ which is a discriminant locus of a linear family (a web) of quadrics in another $\mathbb{P}^3 = \mathbb{P}(V)$ parameterized by $\mathbb{P}(W)$, such that the quadrics in the family have no common points in $\mathbb{P}(V)$ (see Sect. 4 for more details). The quartic symmetroid $D \subset \mathbb{P}(W)$ parameterizes degenerate quadrics in the family. The generic quartic symmetroid has 10 singular points corresponding to the quadrics of corank 2. It turns out that the resolution of singularities D' of D has a natural involution ι without fixed points and the quotient $S := D'/\iota$ is a nodal Enriques surface.

On the other hand, one can consider the double covering $X \rightarrow \mathbb{P}(W)$ ramified in the symmetroid D . As it was shown by Cossec [8] such double coverings are exactly the Artin–Mumford conic bundles [1]. The main result of the present paper is a representation of the subcategory $\mathcal{A}_S \subset \mathcal{D}^b(S)$ as a semiorthogonal component in $\mathcal{D}^b(X^+)$, the derived category of a small resolution X^+ of singularities of X . More precisely we show that there is a semiorthogonal decomposition

$$\mathcal{D}^b(X^+) = \langle \mathcal{A}_S, \mathcal{O}_{X^+}(-h), \mathcal{O}_{X^+} \rangle,$$

where h stands for the pullback to X^+ , of the ample generator of the Picard group of $\mathbb{P}(W)$.

There is a subtle point in the above result. Namely, in general a minimal resolution of singularities $X^+ \rightarrow X$ is not algebraic, in fact it is a Moishezon manifold. Recall that a manifold is Moishezon if its complex dimension is equal to the transcendence degree of its field of meromorphic functions over the complex numbers, as defined in [15]. However, since each Moishezon manifold is an algebraic space, its derived category of coherent sheaves (which by definition is the subcategory of the unbounded derived category of $\mathcal{O}$ -modules with bounded and coherent cohomology) is well-defined. Moreover, it satisfies the same properties as usual derived categories of coherent sheaves.

On the other hand, if one wants to avoid nonalgebraic varieties, one can consider the blowup X' of X which is again a resolution of singularities, but not small. We conjecture that one can realize the whole derived category $\mathcal{D}^b(S)$ of the Enriques surface S inside $\mathcal{D}^b(X')$. More precisely, we conjecture that there is a semiorthogonal decomposition

$$\mathcal{D}^b(X') = \langle \mathcal{D}^b(S), \mathcal{O}_{X'}(-h), \{\mathcal{O}_{X'}(-e_i)\}_{i=1}^{10}, \mathcal{O}_{X'} \rangle,$$

where the e_i are the classes of the exceptional divisors of the blowup $X' \rightarrow X$. We are going to return to this conjecture in the future.

The paper is organized as follows. In Sect. 2, we recall some notions and constructions, such as semiorthogonal decompositions, mutation functors and Serre functors. In Sect. 3, we discuss the derived categories of quartic double solids. In Sect. 4, we recall properties of quartic symmetroids and state the main result of the paper. In Sect. 5, we give the proof of the main result.

2 Preliminaries

The base field is the field $\mathbb{C}$ of complex numbers.

If X is a Moishezon manifold we denote by $\mathcal{D}^b(X)$ the full subcategory of the unbounded derived category of $\mathcal{O}_X$ -modules with bounded and coherent cohomology. This category is triangulated. We say that a morphism $f : X \rightarrow Y$ has finite Tor-dimension if the derived pullback of a sheaf has bounded cohomology. A flat morphism, or a closed embedding into a smooth variety has finite Tor-dimension. For any morphism $f : X \rightarrow Y$ of Moishezon manifolds we denote by $f_* : \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$ and by $f^* : \mathcal{D}^b(Y) \rightarrow \mathcal{D}^b(X)$ the *derived* push-forward and pull-back functors. In the first case we need f to be proper, and in the second, to have finite Tor-dimension. These assumptions ensure the functors preserve both boundedness and coherence. Similarly, $\otimes$ stands for the *derived* tensor product. If $F \in \mathcal{D}^b(X)$ the derived tensor product functor $F \otimes - : \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(X)$ will be denoted by $\mathbb{T}_F$.

For a proper morphism of finite Tor-dimension $f : X \rightarrow Y$ we will also use the right adjoint $f^!$ of the push-forward functor, which is given by the formula

$$f^!(F) \cong f^*(F) \otimes \omega_{X/Y}[\dim X - \dim Y],$$

where $\omega_{X/Y}$ is the relative canonical line bundle.

2.1 Semiorthogonal decompositions

Let $\mathcal{T}$ be a triangulated category.

Definition 2.1 ([5,6]) A semiorthogonal decomposition of a triangulated category $\mathcal{T}$ is a sequence of full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ in $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$ and for every object $T \in \mathcal{T}$ there exists a chain of morphisms $0 = T_n \rightarrow T_{n-1} \rightarrow \dots \rightarrow T_1 \rightarrow T_0 = T$ such that the cone of the morphism $T_k \rightarrow T_{k-1}$ is contained in $\mathcal{A}_k$ for each $k = 1, 2, \dots, n$.

We will write $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$ for a semiorthogonal decomposition of a triangulated category $\mathcal{T}$ with components $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$.

Admissibility is an important property of a triangulated subcategory $\mathcal{A} \subset \mathcal{T}$ which ensures that it can be extended to a semiorthogonal decomposition.

Definition 2.2 ([4,5]) A full triangulated subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$ is right admissible if the inclusion functor $i : \mathcal{A} \rightarrow \mathcal{T}$ has a right adjoint functor $i^! : \mathcal{T} \rightarrow \mathcal{A}$ and left admissible if it has a left adjoint functor $i^* : \mathcal{T} \rightarrow \mathcal{A}$. If i has both adjoint functors then $\mathcal{A}$ is called admissible.

Lemma 2.3 ([4,5]) (i) If $\mathcal{A}_1, \dots, \mathcal{A}_n$ is a semiorthogonal sequence of admissible subcategories in a triangulated category $\mathcal{T}$ (i.e. $\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$) then

$$\langle \mathcal{A}_1, \dots, \mathcal{A}_k, {}^\perp \langle \mathcal{A}_1, \dots, \mathcal{A}_k \rangle \cap \langle \mathcal{A}_{k+1}, \dots, \mathcal{A}_n \rangle^\perp, \mathcal{A}_{k+1}, \dots, \mathcal{A}_n \rangle$$

is a semiorthogonal decomposition.

(ii) If $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$ is a semiorthogonal decomposition of the derived category of a smooth projective variety X then each subcategory $\mathcal{A}_i \subset \mathcal{D}^b(X)$ is admissible.

Actually, the second part of the Lemma holds for any saturated (see [5]) triangulated category in place of $\mathcal{D}^b(X)$.

Definition 2.4 ([4]) An object $E \in \mathcal{T}$ is called exceptional if $\mathrm{Hom}(E, E) = \mathbb{C}$ and $\mathrm{Ext}^p(E, E) = 0$ for all $p \neq 0$. A collection of exceptional objects $(E_1, \dots, E_m)$ is called exceptional if $\mathrm{Ext}^p(E_l, E_k) = 0$ for all $l > k$ and all $p \in \mathbb{Z}$.

Assume that $\mathcal{T}$ is Ext-finite (which means that for any objects $G, G' \in \mathcal{T}$ the graded vector space $\mathrm{Ext}^\bullet(G, G') := \bigoplus_{t \in \mathbb{Z}} \mathrm{Hom}(G, G'[t])$ is finite-dimensional).

Lemma 2.5 ([4]) The subcategory $\langle E \rangle$ of $\mathcal{D}^b(X)$ generated by an exceptional object E is admissible and is equivalent to the derived category of vector spaces $\mathcal{D}^b(\mathbb{C})$.

We will often abuse notation by writing E in place of $\langle E \rangle$. As a consequence of Lemma 2.3 and of Lemma 2.5 one obtains the following

Corollary 2.6 ([6]) If $\mathcal{T}$ is an Ext-finite triangulated category then any exceptional collection $E_1, \dots, E_m$ in $\mathcal{T}$ induces a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}, E_1, \dots, E_m \rangle$$

where $\mathcal{A} = \langle E_1, \dots, E_m \rangle^\perp = \{F \in \mathcal{T} \mid \mathrm{Ext}^\bullet(E_k, F) = 0 \text{ for all } 1 \leq k \leq m\}$.

2.2 Serre functors

Definition 2.7 ([5, 7]) Let $\mathcal{T}$ be a triangulated category. A covariant additive functor $\mathbb{S} : \mathcal{T} \rightarrow \mathcal{T}$ is a **Serre functor** if it induces an equivalence of categories, and for all objects $F, G \in \mathcal{T}$ there are given bi-functorial isomorphisms $\mathrm{Hom}(F, G) \rightarrow \mathrm{Hom}(G, \mathbb{S}(F))^\vee$.

If a Serre functor exists then it is unique up to a canonical functorial isomorphism. If X is a smooth projective variety then $\mathbb{S}(F) := F \otimes \omega_X[\dim X]$ is a Serre functor in $\mathcal{D}^b(X)$.

Lemma 2.8 ([4]) If $\mathcal{T}$ admits a Serre functor $\mathbb{S}$ and $\mathcal{A} \subset \mathcal{T}$ is right admissible then $\mathcal{A}$ admits a Serre functor $\mathbb{S}_{\mathcal{A}} = i^! \circ \mathbb{S} \circ i$, where $i : \mathcal{A} \rightarrow \mathcal{T}$ is the inclusion functor.

Proof If $A, A' \in \mathcal{A}$ then $\mathrm{Hom}_{\mathcal{A}}(A, i^! \mathbb{S} i A') \cong \mathrm{Hom}_{\mathcal{T}}(i A, \mathbb{S} i A') \cong \mathrm{Hom}_{\mathcal{T}}(i A', i A)^\vee \cong \mathrm{Hom}_{\mathcal{A}}(A', A)^\vee$. $\square$

2.3 Mutations

If a triangulated category $\mathcal{T}$ has a semiorthogonal decomposition then usually it has quite a lot of them. More precisely, there are two groups acting on the set of semiorthogonal decompositions — the group of autoequivalences of $\mathcal{T}$, and a braid group. The action of the braid group is given by the mutations.

Roughly speaking, the mutated decomposition is obtained by dropping one of the components of the decomposition and then extending the obtained semiorthogonal collection by inserting new component at some other place as in Lemma 2.3. More precisely, the basic two operations are defined as follows.

Lemma 2.9 ([4]) Assume that $\mathcal{A} \subset \mathcal{T}$ is an admissible subcategory, so that we have two semiorthogonal decompositions $\mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$ and $\mathcal{T} = \langle \mathcal{A}, {}^\perp \mathcal{A} \rangle$. Then there are functors $\mathbb{L}_{\mathcal{A}}, \mathbb{R}_{\mathcal{A}} : \mathcal{T} \rightarrow \mathcal{T}$ vanishing on $\mathcal{A}$ and inducing mutually inverse equivalences ${}^\perp \mathcal{A} \rightarrow \mathcal{A}^\perp$ and $\mathcal{A}^\perp \rightarrow {}^\perp \mathcal{A}$ respectively.

Proof Let $i : \mathcal{A} \rightarrow \mathcal{T}$ be the embedding functor. For any $F \in \mathcal{T}$ we define

$$\mathbb{L}_{\mathcal{A}}(F) = \mathrm{Cone}(ii^! F \rightarrow F), \quad \mathbb{R}_{\mathcal{A}}(F) = \mathrm{Cone}(F \rightarrow ii^* F)[-1].$$

Note that the cones in this triangles are functorial due to the semiorthogonality. All the properties are verified directly. $\square$

The functors $\mathbb{L}_{\mathcal{A}}$ and $\mathbb{R}_{\mathcal{A}}$ are known as the **left** and the **right mutation functors**.

Remark 2.10 If $\mathcal{A}$ is generated by an exceptional object E we can use the following explicit formulas for the adjoint functors $i^!, i^*$ of the embedding functor $i : \mathcal{A} \rightarrow \mathcal{T}$

$$ii^!(F) = \mathrm{Ext}^\bullet(E, F) \otimes E \quad ii^*(F) = \mathrm{Ext}^\bullet(F, E)^\vee \otimes E.$$

Thus we obtain the following distinguished triangles

$$\mathrm{Ext}^\bullet(E, F) \otimes E \rightarrow F \rightarrow \mathbb{L}_E(F), \quad \mathbb{R}_E(F) \rightarrow F \rightarrow \mathrm{Ext}^\bullet(F, E)^\vee \otimes E. \quad (2)$$

It is easy to deduce from Lemma 2.9 the following

Corollary 2.11 ([4]) *Assume that $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$ is a semiorthogonal decomposition with all components being admissible. Then for each $1 \leq k \leq n-1$ there is a semiorthogonal decomposition*

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-1}, \mathbb{L}_{\mathcal{A}_k}(\mathcal{A}_{k+1}), \mathcal{A}_k, \mathcal{A}_{k+2}, \dots, \mathcal{A}_n \rangle$$

and for each $2 \leq k \leq n$ there is a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-2}, \mathcal{A}_k, \mathbb{R}_{\mathcal{A}_k}(\mathcal{A}_{k-1}), \mathcal{A}_{k+1}, \dots, \mathcal{A}_n \rangle$$

There are two cases when the action of the mutation functors is particularly simple.

Lemma 2.12 *Assume that $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$ is a semiorthogonal decomposition with all components being admissible. Assume also that the components $\mathcal{A}_k$ and $\mathcal{A}_{k+1}$ are completely orthogonal, i.e. $\mathrm{Hom}(\mathcal{A}_k, \mathcal{A}_{k+1}) = 0$ as well as $\mathrm{Hom}(\mathcal{A}_{k+1}, \mathcal{A}_k) = 0$. Then*

$$\mathbb{L}_{\mathcal{A}_k}(\mathcal{A}_{k+1}) = \mathcal{A}_{k+1} \quad \text{and} \quad \mathbb{R}_{\mathcal{A}_{k+1}}(\mathcal{A}_k) = \mathcal{A}_k,$$

so that both the left mutation of $\mathcal{A}_{k+1}$ through $\mathcal{A}_k$ and the right mutation of $\mathcal{A}_k$ through $\mathcal{A}_{k+1}$ simply result in a permutation and

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-1}, \mathcal{A}_{k+1}, \mathcal{A}_k, \mathcal{A}_{k+2}, \dots, \mathcal{A}_n \rangle$$

is the resulting semiorthogonal decomposition of $\mathcal{T}$.

Lemma 2.13 *Let X be a smooth projective algebraic variety and $\mathcal{A} \subset \mathcal{D}^b(X)$ an admissible subcategory. Then*

$$\mathbb{L}_{\mathcal{A}^\perp}(\mathcal{A}) = \mathcal{A} \otimes \omega_X \quad \text{and} \quad \mathbb{R}_{\mathcal{A}^\perp}(\mathcal{A}) = \mathcal{A} \otimes \omega_X^{-1}.$$

An analogue of this Lemma holds for any triangulated category which has a Serre functor (see [5]). In this case tensoring by the canonical class should be replaced by the action of the Serre functor. Whenever such mutation is performed we will say that the component $\mathcal{A}$ of a semiorthogonal decomposition $\mathcal{D}^b(X) = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$ is translated to the left (respectively the component $\mathcal{A}$ of a semiorthogonal decomposition $\mathcal{D}^b(X) = \langle \mathcal{A}, {}^\perp \mathcal{A} \rangle$ is translated to the right).

We will also need the following evident observation.

Lemma 2.14 *Let Φ be an autoequivalence of $\mathcal{T}$. Then*

$$\Phi \circ \mathbb{R}_{\mathcal{A}} \cong \mathbb{R}_{\Phi(\mathcal{A})} \circ \Phi, \quad \Phi \circ \mathbb{L}_{\mathcal{A}} \cong \mathbb{L}_{\Phi(\mathcal{A})} \circ \Phi.$$

In particular, if L is a line bundle on X and E is an exceptional object in $\mathcal{D}^b(X)$ then

$$\mathbb{T}_L \circ \mathbb{R}_E \cong \mathbb{R}_{E \otimes L} \circ \mathbb{T}_L, \quad \mathbb{T}_L \circ \mathbb{L}_E \cong \mathbb{L}_{E \otimes L} \circ \mathbb{T}_L,$$

where $\mathbb{T}_L : \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(X)$ is the functor $\mathbb{T}_L(F) = F \otimes L$.

3 Derived category of a quartic double solid

In this section we discuss the structure of the derived categories of quartic double solids (smooth and nodal) and their resolutions of singularities.

3.1 Smooth double covering

We start with the following general result.

Lemma 3.1 *Let $f : X \rightarrow Y$ be a 2-fold covering ramified in a divisor $D \subset Y$ such that $D \sim -K_Y$ and assume that $D \sim 2H$ for some $H \in \text{Pic } Y$. Assume that $\mathcal{E}_1, \dots, \mathcal{E}_n$ is an exceptional collection in $\mathcal{D}^b(Y)$ such that the collection*

$$\mathcal{E}_1(-H), \dots, \mathcal{E}_n(-H), \mathcal{E}_1, \dots, \mathcal{E}_n \quad (3)$$

is also exceptional. Then the collection

$$f^* \mathcal{E}_1, \dots, f^* \mathcal{E}_n \in \mathcal{D}^b(X)$$

is exceptional in $\mathcal{D}^b(X)$.

Proof Note that $X = \text{Spec}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(-H))$, hence $f_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-H)$. Now we have

$$\begin{aligned} \text{Ext}^\bullet(f^* \mathcal{E}_i, f^* \mathcal{E}_j) &= \text{Ext}^\bullet(\mathcal{E}_i, f_* f^* \mathcal{E}_j) = \text{Ext}^\bullet(\mathcal{E}_i, \mathcal{E}_j \otimes (\mathcal{O}_Y \oplus \mathcal{O}_Y(-H))) \\ &= \text{Ext}^\bullet(\mathcal{E}_i, \mathcal{E}_j) \oplus \text{Ext}^\bullet(\mathcal{E}_i, \mathcal{E}_j(-H)). \end{aligned}$$

Note that the second summand vanishes for all i, j and the first summand vanishes for $i > j$ since the collection (3) is exceptional. Moreover, for $i = j$ we obtain exceptionality of $f^* \mathcal{E}_i$. $\square$

Now once we have constructed an exceptional collection in $\mathcal{D}^b(X)$, we can extend it to a semiorthogonal decomposition. It turns out that the additional component has a very interesting property.

Proposition 3.2 *Assume that the conditions of Lemma 3.1 are satisfied. Denote the orthogonal to the exceptional collection of the Lemma by $\mathcal{A}_X = \langle f^*\mathcal{E}_1, \dots, f^*\mathcal{E}_n \rangle^\perp$. Then*

$$\mathcal{D}^b(X) = \langle \mathcal{A}_X, f^*\mathcal{E}_1, \dots, f^*\mathcal{E}_n \rangle$$

is a semiorthogonal decomposition. Moreover, if collection (3) is full in $\mathcal{D}^b(Y)$ and X is smooth and projective, then the Serre functor $\mathbb{S}_{\mathcal{A}_X}$ of the category $\mathcal{A}_X$ is given by

$$\mathbb{S}_{\mathcal{A}_X} \cong \tau[\dim X - 1],$$

where τ is the involution of the double covering $f : X \rightarrow Y$.

Proof The first part follows from Corollary 2.6 and Lemma 3.1. So the only thing we have to compute is the Serre functor. For this we have to check that for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{A}_X$ we have

$$\mathrm{Hom}(\mathcal{F}_1, \tau\mathcal{F}_2[\dim X - 1])^\vee \cong \mathrm{Hom}(\mathcal{F}_2, \mathcal{F}_1),$$

a bifunctorial isomorphism. For this we start with the Serre duality on X . Note that $K_X = -H$, hence

$$\mathrm{Hom}(\mathcal{F}_1, \tau\mathcal{F}_2[\dim X - 1])^\vee \cong \mathrm{Hom}(\tau\mathcal{F}_2, \mathcal{F}_1(-H)[1]) \cong \mathrm{Hom}(\mathcal{F}_2, \tau\mathcal{F}_1(-H)[1]).$$

By Lemma 3.4 below for each $\mathcal{F}_1$ we have a canonical distinguished triangle

$$f^*f_*\mathcal{F}_1 \rightarrow \mathcal{F}_1 \rightarrow \tau\mathcal{F}_1(-H)[1].$$

So, to prove the claim it suffices to check that $\mathrm{Hom}(\mathcal{F}_2, f^*f_*\mathcal{F}_1) = 0$ for all objects $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{A}_X$. But since $K_{X/Y} = \mathcal{O}(H)$ we have $f^! \cong f^* \circ \mathbb{T}_{\mathcal{O}(H)}$, so

$$\mathrm{Hom}(\mathcal{F}_2, f^*f_*\mathcal{F}_1) \cong \mathrm{Hom}(\mathcal{F}_2(H), f^!f_*\mathcal{F}_1) \cong \mathrm{Hom}(f_*\mathcal{F}_2(H), f_*\mathcal{F}_1).$$

By the definition of $\mathcal{A}_X$ we have $\mathrm{Hom}(\mathcal{E}_i, f_*\mathcal{F}_j) = 0$ for all i, j , hence

$$f_*\mathcal{F}_1 \in \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle^\perp = \langle \mathcal{E}_1(-H), \dots, \mathcal{E}_n(-H) \rangle$$

because (3) is full. By the same reasoning, $f_*\mathcal{F}_2(H) \in \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$. The latter two subcategories are semiorthogonal, hence the Hom in question vanishes. $\square$

Remark 3.3 This behavior of the Serre functor of $\mathcal{A}_X$ has a nice generalization, see [13]. One can take X to be a double covering of (or a hypersurface in) arbitrary smooth projective variety Y which has a rectangular Lefschetz decomposition. Then analogously defined subcategory $\mathcal{A}_X$ of $\mathcal{D}^b(X)$ is a fractional Calabi–Yau category.

Lemma 3.4 *Let $f : X \rightarrow Y$ be a double covering with involution $\tau : X \rightarrow X$ and ramification divisor R . Then for any $\mathcal{F} \in \mathcal{D}^b(X)$ there is a distinguished triangle*

$$f^* f_* \mathcal{F} \rightarrow \mathcal{F} \rightarrow \tau \mathcal{F}(-R)[1].$$

Proof Consider the decomposition of the fiber product $X \times_Y X = \Delta(X) \cup \Gamma_\tau(X)$ into the union of the diagonal and the graph of τ which intersect along the ramification divisor R . It gives the following exact sequence

$$0 \rightarrow \mathcal{O}_{\Gamma_\tau(X)}(-R) \rightarrow \mathcal{O}_{X \times_Y X} \rightarrow \mathcal{O}_{\Delta(X)} \rightarrow 0$$

of sheaves on $X \times X$. We consider the sheaves as kernels of Fourier–Mukai transforms $\mathcal{D}^b(X) \rightarrow \mathcal{D}^b(X)$ and note that the structure sheaf of the diagonal gives the identity functor, the structure sheaf of the fiber product gives by base change the functor $f^* f_*$, while the sheaf $\mathcal{O}_{\Gamma_\tau(X)}(-R)$ gives the composition of the twist by $\mathcal{O}_X(-R)$ with the pullback by τ .

So, applying these Fourier–Mukai functors to an object $\mathcal{F} \in \mathcal{D}^b(X)$ (i.e. pulling back $\mathcal{F}$ to $X \times X$, tensoring with the above exact sequence and then pushing forward the obtained distinguished triangle to another factor X) we obtain a distinguished triangle which after rotation gives the claim. $\square$

Now we apply this result to a quartic double solid. Recall that a quartic double solid is a double covering of $\mathbb{P}^3$ ramified in a quartic. Denote by h the ample generator of $\text{Pic } \mathbb{P}^3$ as well as its pullback to the double covering. The standard exceptional collection $\mathcal{O}(-3h), \mathcal{O}(-2h), \mathcal{O}(-h), \mathcal{O}$ on $\mathbb{P}^3$ has the form (3) for $H = 2h$, hence we obtain the following

Corollary 3.5 *Let $D \subset \mathbb{P}^3$ be a smooth quartic surface and let $f : X \rightarrow \mathbb{P}^3$ be the double cover ramified on D . Then $(\mathcal{O}_X(-h), \mathcal{O}_X)$ is an exceptional pair and we have a semiorthogonal decomposition*

$$\mathcal{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X(-h), \mathcal{O}_X \rangle,$$

where $\mathcal{A}_X = \langle \mathcal{O}_X(-h), \mathcal{O}_X \rangle^\perp$. Moreover, $\mathbb{S}_{\mathcal{A}_X} \cong \tau[2]$.

Note that the behavior of the Serre functor of $\mathcal{A}_X$ is analogous to that of an Enriques surface. We are going to show that this is not just a coincidence. But first of all, we are going to extend the construction of the subcategory $\mathcal{A}_X$ to the case of nodal quartic double solids.

3.2 Nodal case, big resolution

Let $D \subset \mathbb{P}^3$ be a quartic surface with N ordinary double points $y_1, \dots, y_N \in D$. Then the corresponding double covering $f : X \rightarrow \mathbb{P}^3$ also has N ordinary double points, $x_1 = f^{-1}(y_1), \dots, x_N = f^{-1}(y_N)$. Therefore, the category $\mathcal{A}_X$ in this case is singular, in the sense that there are objects with infinitely many non-trivial extensions.

If we want to replace it by a smooth category, we have two possibilities. The first, is to replace the singular threefold X by its blowup in points $x_1, \dots, x_N$,

$$X' = \mathrm{Bl}_{x_1, \dots, x_N} X.$$

It is easy to see that X' can also be represented as a double covering.

Lemma 3.6 *Let*

$$Y = \mathrm{Bl}_{y_1, \dots, y_N} \mathbb{P}^3$$

be the blowup of $\mathbb{P}^3$ in points $y_1, \dots, y_N$ and $D' \subset Y$ the proper preimage of D . Then X' is the double covering of Y ramified in D' .

Proof Evident. $\square$

Denote the morphism $X' \rightarrow Y$ also by f . Denote the exceptional divisors of $Y \rightarrow \mathbb{P}^3$ by $E_1, \dots, E_N$, and the exceptional divisors of $X' \rightarrow X$ by $Q_1, \dots, Q_N$. Each E_i is isomorphic to $\mathbb{P}^2$ and each Q_i is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (since x_i is an ordinary double point). Moreover, f maps Q_i to E_i and is a double covering ramified in a conic.

$$\begin{array}{ccccccc} \bigsqcup_{i=1}^N \mathbb{P}^1 \times \mathbb{P}^1 & \xlongequal{\quad} & \bigsqcup_{i=1}^N Q_i & \longrightarrow & X' & \xrightarrow{\mathrm{Bl}_{x_1, \dots, x_N}} & X \\ \downarrow 2:1 & & \downarrow 2:1 & & \downarrow f & & \downarrow f \\ \bigsqcup_{i=1}^N \mathbb{P}^2 & \xlongequal{\quad} & \bigsqcup_{i=1}^N E_i & \longrightarrow & Y & \xrightarrow{\mathrm{Bl}_{y_1, \dots, y_N}} & \mathbb{P}^3 \end{array}$$

The naive replacement of the category $\mathcal{A}_X$ would be the orthogonal to $\langle \mathcal{O}_{X'}(-h), \mathcal{O}_{X'} \rangle$ in $\mathcal{D}^b(X')$. However, there is a better replacement. Note that the ramification divisor D' lies in the linear system $4h - 2 \sum e_i$, where e_i is the class of the exceptional divisor of Y over the point $y_i \in \mathbb{P}^3$. Denote one half of it by H :

$$H = 2h - \sum e_i. \quad (4)$$

Note that

$$K_Y = -2H = 2 \sum e_i - 4h. \quad (5)$$

Proposition 3.7 *Consider the collection of line bundles*

$$(\mathcal{E}_1, \dots, \mathcal{E}_{N+2}) = (\mathcal{O}_Y(-h), \{\mathcal{O}_Y(-e_i)\}_{i=1}^N, \mathcal{O}_Y).$$

Then the collection $(\mathcal{E}_1(-H), \dots, \mathcal{E}_{N+2}(-H), \mathcal{E}_1, \dots, \mathcal{E}_{N+2})$ is a full exceptional collection in $\mathcal{D}^b(Y)$.

Proof We start with a standard exceptional collection of a blowup (see [16])

$$\mathcal{D}^b(Y) = \langle \mathcal{O}_Y(-3h), \mathcal{O}_Y(-2h), \mathcal{O}_Y(-h), \mathcal{O}_Y, \{\mathcal{O}_{E_i}\}_{i=1}^N, \{\mathcal{O}_{E_i}(1)\}_{i=1}^N \rangle$$

and perform a sequence of mutations.

Step 1. Translate the block $\{\mathcal{O}_{E_i}(1)\}_{i=1}^N$ to the left. Clearly, we have an isomorphism $(\omega_Y)_{|E_i} \cong \mathcal{O}_Y(2e_i - 4h)_{|E_i} = \mathcal{O}_{E_i}(-2)$, so it follows from Lemma 2.13 that we will obtain

$$\mathcal{D}^b(Y) = \langle \{\mathcal{O}_{E_i}(-1)\}_{i=1}^N, \mathcal{O}_Y(-3h), \mathcal{O}_Y(-2h), \mathcal{O}_Y(-h), \mathcal{O}_Y, \{\mathcal{O}_{E_i}\}_{i=1}^N \rangle.$$

Step 2. Apply the left mutation through the block $\{\mathcal{O}_{E_i}(-1)\}_{i=1}^N$ to $\mathcal{O}_Y(-3h)$ and $\mathcal{O}_Y(-2h)$. Note that

$$\begin{aligned} \mathrm{Ext}^t(\mathcal{O}_{E_i}(-1), \mathcal{O}_Y(-kh)) &= \mathrm{Ext}^{3-t}(\mathcal{O}_Y(-kh), \mathcal{O}_{E_i}(-3))^\vee \\ &= H^{3-t}(E_i, \mathcal{O}_{E_i}(-3))^\vee = H^{t-1}(E_i, \mathcal{O}_{E_i}) \end{aligned}$$

(the first equality is Serre duality on Y , in the second we use $\mathcal{O}_Y(h)_{|E_i} \cong \mathcal{O}_{E_i}$, and the third uses Serre duality on E_i). Thus we have a one dimensional Ext^1 for each i . So we see that the mutation is given by the following extension

$$0 \rightarrow \mathcal{O}_Y(-kh) \rightarrow \mathcal{O}_Y\left(-kh + \sum_{i=1}^N e_i\right) \rightarrow \bigoplus_{i=1}^N \mathcal{O}_{E_i}(-1) \rightarrow 0,$$

for both $k = 2$ and $k = 3$, that is $\mathbb{L}_{(\{\mathcal{O}_{E_i}(-1)\}_{i=1}^N)} \mathcal{O}_Y(-kh) = \mathcal{O}_Y(-kh + \sum_{i=1}^N e_i) = \mathcal{O}_Y(-H - (k-2)h)$. Thus, we obtain

$$\mathcal{D}^b(Y) = \left\langle \mathcal{O}_Y(-H-h), \mathcal{O}_Y(-H), \{\mathcal{O}_{E_i}(-1)\}_{i=1}^N, \mathcal{O}_Y(-h), \mathcal{O}_Y, \{\mathcal{O}_{E_i}\}_{i=1}^N \right\rangle.$$

Step 3. Apply the left mutation through $\mathcal{O}_Y$ to the block $\{\mathcal{O}_{E_i}\}_{i=1}^N$, and simultaneously the left mutation through $\mathcal{O}_Y(-H)$ to the block $\{\mathcal{O}_{E_i}(-1)\}_{i=1}^N$. Note that

$$\mathrm{Ext}^\bullet(\mathcal{O}_Y, \mathcal{O}_{E_i}) = H^\bullet(E_i, \mathcal{O}_{E_i}),$$

so we have a one dimensional Hom for each i . It follows that the mutation of $\mathcal{O}_{E_i}$ is given by the following exact sequence

$$0 \rightarrow \mathcal{O}_Y(-e_i) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{E_i} \rightarrow 0,$$

that is $\mathbb{L}_{\mathcal{O}_Y} \mathcal{O}_{E_i} = \mathcal{O}_Y(-e_i)[1]$. Twisting by $\mathcal{O}_Y(-H)$ we conclude that

$$\mathbb{L}_{\mathcal{O}_Y(-H)} \mathcal{O}_{E_i}(-1) = \mathcal{O}_Y(-H - e_i)[1]$$

by Lemma 2.14. So, finally we obtain

$$\mathcal{D}^b(Y) = \langle \mathcal{O}_Y(-h-H), \{\mathcal{O}_Y(-e_i-H)\}_{i=1}^N, \mathcal{O}_Y(-H), \mathcal{O}_Y(-h), \{\mathcal{O}_Y(-e_i)\}_{i=1}^N, \mathcal{O}_Y \rangle \quad (6)$$

which proves the Proposition. $\square$

Applying Lemma 3.1 and Proposition 3.2 we obtain the following

Corollary 3.8 *There is a semiorthogonal decomposition*

$$\mathcal{D}^b(X') = \langle \mathcal{A}_{X'}, \mathcal{O}_{X'}(-h), \{\mathcal{O}_{X'}(-e_i)\}_{i=1}^N, \mathcal{O}_{X'} \rangle, \quad (7)$$

where $\mathcal{A}_{X'} = \langle \mathcal{O}_{X'}(-h), \{\mathcal{O}_{X'}(-e_i)\}_{i=1}^N, \mathcal{O}_{X'} \rangle^\perp$. Moreover, $\mathbb{S}_{\mathcal{A}_{X'}} \cong \tau[2]$.

3.3 Nodal case, small resolution

Another alternative is to replace X by its *small resolution of singularities* as in Sect. 3 of [2]. Note that an analytic neighborhood of an ordinary double point $x_i \in X$ is analytically isomorphic to an analytic neighborhood of the vertex of the 3-dimensional quadratic cone. In particular, it has a resolution of singularities (actually two resolutions!) with the exceptional locus being $\mathbb{P}^1$. Choosing such resolution for each x_i , and gluing all of them together, we obtain a smooth analytic manifold X^+ with a map $\sigma : X^+ \rightarrow X$ such that the fiber over each point x_i is $\mathbb{P}^1$. This manifold does not admit an algebraic structure in general, however it follows immediately from the definition that it is always a Moishezon manifold, and in particular an algebraic space. Therefore one can speak about its derived category of coherent sheaves.

Note also that since the construction of the minimal resolution X^+ involves a choice of one of two local resolutions for each of the singular points $x_1, \dots, x_N$, there are 2^N different minimal resolutions. Each of them has a semiorthogonal decomposition analogous to that of X .

Proposition 3.9 *There is a semiorthogonal decomposition*

$$\mathcal{D}^b(X^+) = \langle \mathcal{A}_{X^+}, \mathcal{O}_{X^+}(-h), \mathcal{O}_{X^+} \rangle,$$

where $\mathcal{A}_{X^+} = \langle \mathcal{O}_{X^+}(-h), \mathcal{O}_{X^+} \rangle^\perp$.

Proof Note that $\sigma_* \mathcal{O}_{X^+} \cong \mathcal{O}_X$ since an ordinary double point is a rational singularity, so for locally free $\mathcal{E}, \mathcal{E}'$ on $\mathcal{O}_X$ we have

$$\mathrm{Ext}^\bullet(\sigma^* \mathcal{E}, \sigma^* \mathcal{E}') \cong \mathrm{Ext}^\bullet(\mathcal{E}, \sigma_* \sigma^* \mathcal{E}') \cong \mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{E}' \otimes \sigma_* \mathcal{O}_{X^+}) \cong \mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{E}').$$

This shows that $(\mathcal{O}_{X^+}(-h), \mathcal{O}_{X^+})$ is an exceptional pair in $\mathcal{D}^b(X^+)$. This gives the required decomposition. $\square$

Remark 3.10 It is natural to ask whether the Serre functor of $\mathcal{A}_{X^+}$ can be written as an involution composed with the shift by 2. It turns out that this is not the case. Indeed, since $K_{X^+} = -2h$, we have

$$\mathbb{S}_{\mathcal{A}_{X^+}}^{-1} \cong \mathbb{L}_{(\mathcal{O}_{X^+}(-h), \mathcal{O}_{X^+})} \circ \mathbb{S}_{\mathcal{D}^b(X^+)}^{-1} \cong \mathbb{L}_{(\mathcal{O}_{X^+}(-h), \mathcal{O}_{X^+})} \circ \mathbb{T}_{\mathcal{O}_{X^+}(2h)}[-3].$$

It is easy to see that applied to $\sigma^*\mathcal{F}$ this functor gives $\sigma^*\tau\mathcal{F}[-2]$. On the other hand, if C is an exceptional curve of X^+ then

$$\mathbb{S}_{\mathcal{A}_{X^+}}^{-1}(\mathcal{O}_C(-1)) \cong \mathcal{O}_C(-1)[-3].$$

The shifts are different, hence the behavior of the Serre functor on $\mathcal{A}_{X^+}$ is more complicated in this case.

Remark 3.11 If we have a family of double solids X_t with smooth generic member and X_0 being a nodal double solid, then one can consider $\mathcal{A}_{X_0^+}$ as a *smooth degeneration* of $\mathcal{A}_{X_t}$. And the category $\mathcal{A}_{X_0'}$ can be thought of as a (categorical) blowup of the category $\mathcal{A}_{X_0^+}$.

Lemma 3.12 *There is a fully faithful embedding $\mathcal{A}_{X^+} \rightarrow \mathcal{A}_{X'}$ which extends to a semiorthogonal decomposition $\mathcal{A}_{X'} = \langle \mathcal{A}_{X^+}, \{\mathcal{O}_{Q_i}(-1, 0)\}_{i=1}^N \rangle$.*

Proof Note that the blow up of the minimal resolution X^+ in the union of the exceptional curves $C_1 = \sigma^{-1}(x_1), \dots, C_N = \sigma^{-1}(x_N)$ coincides with the blowup X' of X . Denote the exceptional divisors by $Q_1, \dots, Q_N$. Note that $Q_i \cong C_i \times \mathbb{P}^1$ since the normal bundles of the C_i are isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

The standard semiorthogonal decomposition of the blowup in this case gives

$$\mathcal{D}^b(X') = \langle \mathcal{D}^b(X^+), \{\mathcal{O}_{Q_i}(-1, 0), \mathcal{O}_{Q_i}\}_{i=1}^N \rangle.$$

Replacing $\mathcal{D}^b(X^+)$ by decomposition of Proposition 3.9 we obtain

$$\mathcal{D}^b(X') = \langle \mathcal{A}_{X^+}, \mathcal{O}_{X'}(-h), \mathcal{O}_{X'}, \{\mathcal{O}_{Q_i}(-1, 0), \mathcal{O}_{Q_i}\}_{i=1}^N \rangle.$$

Now we note that the sheaves $\mathcal{O}_{Q_i}(-1, 0)$ are completely orthogonal to $\mathcal{O}_{X'}(-h)$ and $\mathcal{O}_{X'}$. Indeed, since the latter are pullbacks via $\sigma' : X' \rightarrow X$, it suffices to observe that $\sigma'_*\mathcal{O}_{Q_i}(-1, 0) = 0$, which is clear since σ' contracts each of Q_i to a point and the sheaf $\mathcal{O}_{Q_i}(-1, 0)$ on Q_i is acyclic. Therefore by Lemma 2.12 we can move all $\mathcal{O}_{Q_i}(-1, 0)$ to the left of $\mathcal{O}_{X'}(-h)$ and $\mathcal{O}_{X'}$. We obtain a semiorthogonal decomposition

$$\mathcal{D}^b(X') = \langle \mathcal{A}_{X^+}, \{\mathcal{O}_{Q_i}(-1, 0)\}_{i=1}^N, \mathcal{O}_{X'}(-h), \mathcal{O}_{X'}, \{\mathcal{O}_{Q_i}\}_{i=1}^N \rangle.$$

Finally, mutating $\mathcal{O}_{Q_i}$ through $\mathcal{O}_{X'}$ we obtain

$$\mathcal{D}^b(X') = \langle \mathcal{A}_{X^+}, \{\mathcal{O}_{Q_i}(-1, 0)\}_{i=1}^N, \mathcal{O}_{X'}(-h), \{\mathcal{O}_{X'}(-e_i)\}_{i=1}^N, \mathcal{O}_{X'} \rangle.$$

Comparing this with (7) we deduce the claim. $\square$

4 Quartic symmetroids and special double solids

In this section we discuss the quartic double solids associated with quartic symmetroids and state the main result of the paper. The following construction is very classical. For more details see e.g. [8].

Assume V and W are vector spaces, both of dimension 4, and choose a generic linear embedding

$$s : W \subset S^2 V^\vee.$$

Alternatively, s can be considered as a tensor in $W^\vee \otimes S^2 V^\vee$. The latter space can be considered as the space of global sections of various natural vector bundles. Considering zero loci of these sections we obtain some natural varieties associated with s . For example, considering s as a global section of the line bundle $\mathcal{O}(1, 2)$ on $\mathbb{P}(W) \times \mathbb{P}(V)$ we obtain a divisor of bidegree $(1, 2)$

$$Q \subset \mathbb{P}(W) \times \mathbb{P}(V)$$

which can be thought of as a family of quadrics in $\mathbb{P}(V)$ parameterized by $\mathbb{P}(W)$. The degeneration locus of this family $D \subset \mathbb{P}(W)$ is the zero locus of $\det s \in S^4 W^\vee \otimes (\det V^\vee)^2$, hence is a quartic surface. Quartic surfaces obtained in this way are called **quartic symmetroids**. If s is generic then D has 10 ordinary double points. The blowup D' of D can be realized as the zero locus of s considered as a global section of the vector bundle $W^\vee \otimes \mathcal{O}(1, 1)$ on $\mathbb{P}(V) \times \mathbb{P}(V)$ (the map $D' \rightarrow D$ takes a point $(v, v') \in D'$ to the unique $w \in \mathbb{P}(W)$ such that quadric Q_w is singular at v). On the other hand, the transposition of factors $\iota : \mathbb{P}(V) \rightarrow \mathbb{P}(V)$ acts on the K3-surface D' without fixed points, hence

$$S = D'/(\iota)$$

is an Enriques surface.

Remark 4.1 Enriques surfaces associated in this way with quartic symmetroids are Reye congruences of lines. They are **nodal Enriques surfaces** (the name is slightly misleading, since they are smooth), as in the sense of [9]. It is a non-trivial result that a nodal Enriques surface is a Reye congruence [10]. The family of nodal Enriques surfaces is known to be a divisor in the family of all Enriques surfaces.

Our goal is the following

Conjecture 4.2 *Let X' be the blowup of the nodal quartic double solid X associated with the quartic symmetroid $D \subset \mathbb{P}^3$ and $S = D'/(\iota)$ be the corresponding Enriques surface. Then*

$$\mathcal{A}_{X'} \cong \mathcal{D}^b(S),$$

where the subcategory $\mathcal{A}_{X'} \subset \mathcal{D}^b(X')$ is as defined in Corollary 3.8.

Instead of proving this conjecture we concentrate on proving the following slightly simpler result. Note that both categories $\mathcal{A}_{X'}$ and $\mathcal{D}^b(S)$ have a completely orthogonal exceptional collections of length 10: for $\mathcal{A}_{X'}$ it is given by the sheaves $\mathcal{O}_{Q_i}(-1, 0)$ (see Lemma 3.12), and for $\mathcal{D}^b(S)$ it is given by the line bundles $\mathcal{O}_S(-F_i^+)$ (the embedding of S into $\mathrm{Gr}(2, V)$ which we construct below gives a polarization of degree 10 on S , so Theorem 1.1 applies; in fact we will see the double fibers F_i^+ explicitly in Lemma 5.13). The orthogonal complements of these collections are the categories $\mathcal{A}_{X^+}$ and $\mathcal{A}_S$ respectively. The main result of the paper is the following

Theorem 4.3 *Let X^+ be a small resolution of singularities of the nodal quartic double solid X associated with the quartic symmetroid $D \subset \mathbb{P}^3$ and S the corresponding Enriques surface. Then*

$$\mathcal{A}_{X^+} \cong \mathcal{A}_S,$$

where the subcategories $\mathcal{A}_{X^+} \subset \mathcal{D}^b(X^+)$ and $\mathcal{A}_S \subset \mathcal{D}^b(S)$ are as defined in Proposition 3.9 and (1) respectively.

Remark 4.4 Note that there are 2^{10} ways to choose one multiple fibre in each of 10 elliptic pencils on S . This is precisely the same as the number of small resolutions of singularities of X . We will show below that there is a bijection between these sets and the equivalence of Theorem 4.3 is proved for compatible choices.

Remark 4.5 Note that different small resolutions of X are related by flops. These flops induce equivalences of the corresponding subcategories $\mathcal{A}_{X^+}$. On the other hand, different choices of multiple fibres in elliptic pencils on S lead to subcategories $\mathcal{A}_S$ related by appropriate mutation functors.

The proof of the Theorem is given in the next section.

5 The relative scheme of lines

The main idea of the proof is to analyze carefully the derived category of the relative scheme of lines M on the family $Q \rightarrow \mathbb{P}(W)$ of quadrics associated with s . On one hand, we show that M is the blowup of $\mathrm{Gr}(2, V)$ in the Enriques surface S , whereof one realizes $\mathcal{D}^b(S)$ as a component of $\mathcal{D}^b(M)$. On the other hand, there is a general description of the relative scheme of lines [14], one component of which is the derived category of X^+ . Finally, a sequence of mutations transforms one of these decompositions of $\mathcal{D}^b(M)$ to the other and establishes the required equivalence.

5.1 Relation to the Enriques surface

To start with, we will need another description of the Enriques surface associated with the quartic symmetroid. For this, consider the product of the Grassmannians $\mathrm{Gr}(2, V) \times \mathrm{Gr}(2, W)$. Denote the tautological subbundles on those as $\mathcal{U}$ and $\mathcal{U}_W$ respectively. Then s can be thought of as a global section of the vector bundle $S^2\mathcal{U}^\vee \boxtimes \mathcal{U}_W^\vee$. Consider its zero locus $\tilde{S} \subset \mathrm{Gr}(2, V) \times \mathrm{Gr}(2, W)$. Set-theoretically, $\tilde{S}$

is the set of all pairs (L, K) of two dimensional subspaces $L \subset V$, $K \subset W$ such that the line $\mathbb{P}(L)$ is contained in all the quadrics in the pencil $\mathbb{P}(K) \subset \mathbb{P}(W)$. It is known (see e.g. [3], Chapter VIII) that $\tilde{S}$ is isomorphic to the Enriques surface associated with s (the map $D' \rightarrow \tilde{S}$ takes a point $(v, v') \in D' \subset \mathbb{P}(V) \times \mathbb{P}(V)$ to the point $(L, K) \in \text{Gr}(2, V) \times \text{Gr}(2, W)$, where L is the linear span of v and v' and K consists of all $w \in \mathbb{P}(W)$ such that Q_w contains both v and v' and v, v' are polar with respect to Q_w .) Furthermore, the projection onto $\text{Gr}(2, V)$ gives an embedding of $S = \tilde{S}$ into $\text{Gr}(2, V)$.

Denote the ample generator of $\text{Pic}(\text{Gr}(2, V))$ by $\mathcal{O}(g)$. The following resolution of the structure sheaf of S on $\text{Gr}(2, V)$ is well known and can be also deduced by classical geometric arguments. However, we include here a proof based on cohomological computations for completeness.

Lemma 5.1 *The structure sheaf of S in $\text{Gr}(2, V)$ has the following resolution*

$$0 \rightarrow S^2\mathcal{U}(-3g) \rightarrow W^\vee \otimes \mathcal{O}_{\text{Gr}(2, V)}(-3g) \rightarrow \mathcal{O}_{\text{Gr}(2, V)} \rightarrow \mathcal{O}_S \rightarrow 0. \quad (8)$$

Proof By the definition of $\tilde{S}$ we have the following Koszul resolution for $\mathcal{O}_{\tilde{S}}$:

$$0 \rightarrow \Lambda^6(S^2\mathcal{U} \boxtimes \mathcal{U}_W) \rightarrow \cdots \rightarrow \Lambda^2(S^2\mathcal{U} \boxtimes \mathcal{U}_W) \rightarrow S^2\mathcal{U} \boxtimes \mathcal{U}_W \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\tilde{S}} \rightarrow 0.$$

The wedge powers of the tensor product can be rewritten (see e.g. [11, Ex. 6.11]) as

$$\Lambda^k(A \otimes B) = \bigoplus_{|\alpha|=k} \Sigma^\alpha A \otimes \Sigma^{\alpha^T} B,$$

where the sum is taken over the set of all Young diagrams inscribed in the rectangle of the size $\dim A$ by $\dim B$ with k boxes, α^T stands for the transposed diagram, and Σ^α is the corresponding Schur functor. Further, by the Pieri formula (see e.g. [11, Ex. 6.16]) we have

$$\Sigma^{a_1, a_2} \mathcal{U}_W = S^{a_1 - a_2} \mathcal{U}_W \otimes (\det \mathcal{U}_W)^{\otimes a_2} = S^{a_1 - a_2} \mathcal{U}_W(-a_2).$$

Finally, using the definition of the Schur functors and the Littlewood–Richardson rule one can rewrite

$$\begin{aligned} \Sigma^{(2)} S^2\mathcal{U} &= S^4\mathcal{U} \oplus (\det \mathcal{U})^{\otimes 2} = S^4\mathcal{U} \oplus \mathcal{O}(-2), \\ \Sigma^{(1,1)} S^2\mathcal{U} &= S^2\mathcal{U} \otimes \det \mathcal{U} = S^2\mathcal{U}(-1), \\ \Sigma^{(2,1)} S^2\mathcal{U} &= S^4\mathcal{U} \otimes \det \mathcal{U} \oplus S^2\mathcal{U} \otimes (\det \mathcal{U})^{\otimes 2} = S^4\mathcal{U}(-1) \oplus S^2\mathcal{U}(-2), \\ \Sigma^{(1,1,1)} S^2\mathcal{U} &= (\det \mathcal{U})^{\otimes 3} = \mathcal{O}(-3), \\ \Sigma^{(2,2)} S^2\mathcal{U} &= S^4\mathcal{U} \otimes (\det \mathcal{U})^{\otimes 2} \oplus (\det \mathcal{U})^{\otimes 4} = S^4\mathcal{U}(-2) \oplus \mathcal{O}(-4), \\ \Sigma^{(2,1,1)} S^2\mathcal{U} &= S^2\mathcal{U} \otimes (\det \mathcal{U})^{\otimes 3} = S^2\mathcal{U}(-3), \\ \Sigma^{(2,2,1)} S^2\mathcal{U} &= S^2\mathcal{U} \otimes (\det \mathcal{U})^{\otimes 4} = S^2\mathcal{U}(-4), \\ \Sigma^{(1,1,1,1)} S^2\mathcal{U} &= (\det \mathcal{U})^{\otimes 6} = \mathcal{O}(-6). \end{aligned}$$

Combining all these equalities together we rewrite the Koszul complex explicitly as

$$\begin{aligned} 0 \rightarrow \mathcal{O}(-6, -3) &\rightarrow S^2\mathcal{U}(-4) \boxtimes \mathcal{U}_W(-2) \\ &\rightarrow S^2\mathcal{U}(-3) \boxtimes S^2\mathcal{U}_W(-1) \oplus (S^4\mathcal{U}(-2) \oplus \mathcal{O}(-4)) \boxtimes \mathcal{O}(-2) \\ &\rightarrow \mathcal{O}(-3) \boxtimes S^3\mathcal{U}_W \oplus (S^4\mathcal{U}(-1) \oplus S^2\mathcal{U}(-2)) \boxtimes \mathcal{U}_W(-1) \\ &\rightarrow S^2\mathcal{U}(-1) \boxtimes S^2\mathcal{U}_W \oplus (S^4\mathcal{U} \oplus \mathcal{O}(-2)) \boxtimes \mathcal{O}(-1) \\ &\rightarrow S^2\mathcal{U} \boxtimes \mathcal{U}_W \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\tilde{S}} \rightarrow 0. \end{aligned}$$

Since the projection to $\mathrm{Gr}(2, V)$ identifies $\tilde{S}$ with S , we can take the pushforward of the above resolution to get a resolution of $\mathcal{O}_S$. Note that the bundles $\mathcal{U}_W, \mathcal{O}(-1), S^2\mathcal{U}_W, \mathcal{U}_W(-1), \mathcal{O}(-2), \mathcal{U}_W(-2)$ and $\mathcal{O}(-3)$ on $\mathrm{Gr}(2, W)$ are acyclic. So the only nontrivial pushforward will come from the terms $S^2\mathcal{U}(-3) \boxtimes S^2\mathcal{U}_W(-1), \mathcal{O}(-3) \boxtimes S^3\mathcal{U}_W$, and $\mathcal{O}$. Since

$$\begin{aligned} H^2(\mathrm{Gr}(2, W), S^2\mathcal{U}_W(-1)) &= \mathbb{C}, \quad H^2(\mathrm{Gr}(2, W), S^3\mathcal{U}_W) = W^\vee, \\ H^0(\mathrm{Gr}(2, W), \mathcal{O}) &= \mathbb{C}, \end{aligned}$$

and all other cohomology vanishes, we obtain the desired resolution. $\square$

Now consider the relative scheme of lines M for the family of quadrics $Q \rightarrow \mathbb{P}(W)$. By definition M is a subscheme in $\mathrm{Gr}(2, V) \times \mathbb{P}(W)$, consisting of all pairs (L, w) , where L is a 2-dimensional subspace of V and $w \in \mathbb{P}(W)$, such that the line $\mathbb{P}(L)$ lies on the quadric $Q_w \subset \mathbb{P}(V)$. Thus, it is the zero locus of a section of the vector bundle $S^2\mathcal{U}^\vee \otimes \mathcal{O}_{\mathbb{P}(W)}(1)$ on $\mathrm{Gr}(2, V) \times \mathbb{P}(W)$, given by s . Since s is generic the section is regular. We obtain the following.

Lemma 5.2 *The structure sheaf of M in $\mathrm{Gr}(2, V) \times \mathbb{P}(W)$ has the following resolution*

$$0 \rightarrow \mathcal{O}(-3g - 3h) \rightarrow S^2\mathcal{U}(-g - 2h) \rightarrow S^2\mathcal{U}(-h) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_M \rightarrow 0. \quad (9)$$

Proof This is just the Koszul resolution. $\square$

Now we are ready to show that map $\pi : M \rightarrow \mathrm{Gr}(2, V)$ is the blowup of $S \subset \mathrm{Gr}(2, V)$.

Lemma 5.3 *The map $\pi : M \rightarrow \mathrm{Gr}(2, V)$ is the blowup of $S \subset \mathrm{Gr}(2, V)$. Moreover, if e is the class of the exceptional divisor then $h = 3g - e$.*

Proof First, let us describe the fibres of π . It is clear that the fibre over a point of $\mathrm{Gr}(2, V)$ corresponding to a subspace $L \subset V$ is the linear subspace of $\mathbb{P}(W)$ consisting of quadrics, containing the line $\mathbb{P}(L)$. This is a condition of codimension 3, hence the generic fibre is a point. Moreover, the fibre is nontrivial if there is a pencil of quadrics containing $\mathbb{P}(L)$. This occurs when L is in the image of the map $S = \tilde{S} \rightarrow \mathrm{Gr}(2, V)$. This shows that π is the blowup of an ideal supported at S . It remains to check that this is the ideal of S . For this consider the sheaf $\pi_*(\mathcal{O}_M(h))$. Since h is very ample

over $\mathrm{Gr}(2, V)$ this sheaf is a twist of the ideal in question. To compute $\pi_* \mathcal{O}_M(h)$ we use (9). It gives

$$0 \rightarrow S^2 \mathcal{U} \rightarrow W^\vee \otimes \mathcal{O} \rightarrow \pi_* \mathcal{O}_M(h) \rightarrow 0.$$

Comparing with (8) we see that $\pi_* \mathcal{O}_M(h) \cong I_S(3g)$. Thus M is the blowup of S . Moreover, it follows also that $h = 3g - e$. $\square$

Denote the exceptional divisor of π by E , and the restriction of π to E by p . Let also $i : E \rightarrow M$ be the embedding. Using [16] we obtain the following

Corollary 5.4 *We have the following semiorthogonal decomposition*

$$\mathcal{D}^b(M) = \langle \pi^*(\mathcal{D}^b(\mathrm{Gr}(2, V))), i_* p^*(\mathcal{D}^b(S)) \rangle. \quad (10)$$

Since $\mathcal{D}^b(\mathrm{Gr}(2, V))$ is generated by an exceptional collection, we see that $\mathcal{D}^b(M)$ consists of several exceptional objects and the category $\mathcal{A}_S$. We will continue using this description in Subsect. 5.3, but now we will look at M from another point of view.

5.2 Relation to the double solid

So far we considered M as a blowup of the Grassmannian $\mathrm{Gr}(2, V)$. Now consider the projection $\rho : M \rightarrow \mathbb{P}(W)$. By definition, M is the relative scheme of lines for the family of quadrics $\mathcal{Q} \rightarrow \mathbb{P}(W)$. Therefore the generic fibre of M is a disjoint union of two conics. The fibre is a single conic over the divisor D of degenerate quadrics, and over the 10 points corresponding to quadrics of corank 2, the fibre is a union of two planes intersecting in a point. Note that the Stein factorization for the map $\rho : M \rightarrow \mathbb{P}(W)$ is

$$\begin{array}{ccc} M & \xrightarrow{\mu} & X, \\ & \searrow \rho \quad \swarrow f & \\ & \mathbb{P}(W) & \end{array}$$

where $f : X \rightarrow \mathbb{P}(W)$ is the double covering ramified in D . This X , is the special double solid of interest.

The derived category of a relative scheme of lines for a family of 2-dimensional quadrics was described in [14]. To state the answer obtained therein we need to introduce some notation. Firstly, for each point $y_i \in \mathbb{P}(W)$ corresponding to a quadric of corank 2, choose one of two planes in the fibre of M over y_i . Denote it by Σ_i^+ , and the other by Σ_i^- . Further, consider the sheaf $\mathcal{B}_0$ of even Clifford algebras on $\mathbb{P}(W)$ associated with the family of quadrics $\mathcal{Q} \rightarrow \mathbb{P}(W)$ (see [12] for details). Let $\mathcal{D}^b(\mathbb{P}(W), \mathcal{B}_0)$ denote the derived category of sheaves of $\mathcal{B}_0$ -modules on $\mathbb{P}(W)$.

Theorem 5.5 ([14]) *For each choice of 10 planes Σ_i^+ there exists a minimal resolution X^+ of X and a semiorthogonal decomposition*

$$\mathcal{D}^b(M) = \langle \mathcal{D}^b(X^+), \mathcal{D}^b(\mathbb{P}(W)), \mathcal{B}_0, \{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^{10} \rangle.$$

We will also need a description of the embedding functors for the first two components of this decomposition. Firstly, the functor $\mathcal{D}^b(X^+) \rightarrow \mathcal{D}^b(M)$ can be constructed as follows. One can check (see [14]) that the normal bundle to any of the planes Σ_i^+ is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Hence one can perform a flip $M \dashrightarrow M^+$ in all these planes. The obtained Moishezon manifold M^+ has a structure of a $\mathbb{P}^1$ -fibration $\mu_+ : M^+ \rightarrow X^+$ over X^+ . We remark without proof that M^+ is the Severi-Brauer Scheme of $\mathcal{B}_+$ which is an Azumaya algebra on X^+ that is a maximal order containing $f_+^* \mathcal{B}_0$. In particular, the pullback functor μ_+^* gives an embedding $\mathcal{D}^b(X^+) \rightarrow \mathcal{D}^b(M^+)$. Further, the flip $M \dashrightarrow M^+$ can be represented as a composition of a blowup $\xi : \tilde{M} \rightarrow M$ and a blowdown $\xi_+ : \tilde{M} \rightarrow M^+$. All these maps can be summarized in the following commutative diagram

$$\begin{array}{ccccc} & & \tilde{M} & & \\ & \swarrow \xi & & \searrow \xi_+ & \\ M & & & & M_+ \\ \downarrow \mu & \searrow \rho & & \swarrow \rho_+ & \downarrow \mu_+ \\ X & \xleftarrow{\sigma_+} & & & X_+ \\ & \searrow f & & \swarrow f_+ & \\ & & \mathbb{P}(W) & & \end{array}$$

The composition of functors $\xi_* \xi_+^* : \mathcal{D}^b(M^+) \rightarrow \mathcal{D}^b(M)$ is also fully faithful by a result of Bondal and Orlov (see [6]). Composing we obtain a fully faithful functor $\Phi_0 = \xi_* \xi_+^* \mu_+^* : \mathcal{D}^b(X^+) \rightarrow \mathcal{D}^b(M)$. We also need the following observation

Lemma 5.6 *Let $f_+ : X_+ \rightarrow \mathbb{P}(W)$ be the composition of the contraction $X^+ \rightarrow X$ and the double covering $X \rightarrow \mathbb{P}(W)$.*

For any object $\mathcal{F} \in \mathcal{D}^b(\mathbb{P}(W))$ one has

$$\Phi_0(f_+^* \mathcal{F}) \cong \rho^* \mathcal{F}.$$

Moreover, the functor Φ_0 is $\mathbb{P}(W)$ -linear, that is

$$\Phi_0(\mathcal{G} \otimes f_+^* \mathcal{F}) \cong \Phi_0(\mathcal{G}) \otimes \rho^* \mathcal{F}.$$

Proof This follows immediately from the projection formula. $\square$

Now consider the second component. Note that in addition to the sheaf of even Clifford algebras $\mathcal{B}_0$, one has also the sheaf $\mathcal{B}_1$ of the odd components of Clifford

algebras on $\mathbb{P}(W)$. As sheaves on $\mathbb{P}(W)$ they have the following structure

$$\mathcal{B}_0 = \mathcal{O} \oplus \Lambda^2 V(-h) \oplus \Lambda^4 V(-2h), \quad \mathcal{B}_1 = V \otimes \mathcal{O} \oplus \Lambda^3 V(-h).$$

This pair of sheaves naturally extends to a sequence of sheaves $\mathcal{B}_k$ of $\mathcal{B}_0$ -modules defined by

$$\mathcal{B}_{2k} = \mathcal{B}_0(kh), \quad \mathcal{B}_{2k+1} = \mathcal{B}_1(kh).$$

For each $k \in \mathbb{Z}$ consider the morphism $\mathcal{U} \boxtimes \mathcal{B}_{k-1} \rightarrow \mathcal{B}_k$ of sheaves of $\mathcal{B}_0$ -modules on $\mathrm{Gr}(2, V) \times \mathbb{P}(W)$ induced by the embedding $\mathcal{U} \subset V \otimes \mathcal{O}$ and the Clifford action $V \otimes \mathcal{B}_k \rightarrow \mathcal{B}_{k+1}$. Let $\alpha : M \rightarrow \mathrm{Gr}(2, V) \times \mathbb{P}(W)$ be the embedding. Another result of [14] is the following

Theorem 5.7 ([14]) *There are isomorphisms $\mathrm{Coker}(\mathcal{U} \boxtimes \mathcal{B}_{k-1} \rightarrow \mathcal{B}_k) \cong \alpha_* \mathcal{S}_k$, where the sheaves $\mathcal{S}_k$ are defined as*

$$0 \rightarrow \mathcal{O}(kh) \rightarrow \mathcal{S}_{2k} \rightarrow \mathcal{O}(g + (k-1)h) \rightarrow 0, \quad \mathcal{S}_{2k+1} = (V/\mathcal{U})(kh), \quad (11)$$

and the extension is nontrivial. Moreover, we have

$$\rho_*(\mathcal{S}_k(-g)) = 0 \quad (12)$$

for all $k \in \mathbb{Z}$. Finally, the functor $\Phi_1 : \mathcal{D}^b(\mathbb{P}(W), \mathcal{B}_0) \rightarrow \mathcal{D}^b(M)$ defined by the formula, $\mathcal{F} \mapsto \mathcal{S}_0 \otimes_{\mathcal{B}_0} \rho^* \mathcal{F}$, is fully faithful and

$$\Phi_1(\mathcal{B}_k) \cong \mathcal{S}_k.$$

On the other hand, it follows from [12], Theorem 5.5 that $\mathcal{D}^b(\mathbb{P}(W), \mathcal{B}_0)$ is generated by an exceptional collection $\mathcal{B}_{k-3}, \mathcal{B}_{k-2}, \mathcal{B}_{k-1}, \mathcal{B}_k$, where k can be taken arbitrarily. Indeed there might be an additional component, the derived category of the intersection of all quadrics in $\mathbb{P}(V)$ parameterized by W . But as s is generic this is empty since it is the intersection of 4 general quadrics in $\mathbb{P}^3$. Combining this result with the Theorem above we obtain the following

Corollary 5.8 *The subcategory $\Phi_1(\mathcal{D}^b(\mathbb{P}(W), \mathcal{B}_0)) \subset \mathcal{D}^b(M)$ is generated by an exceptional collection*

$$\langle \mathcal{S}_{k-3}, \mathcal{S}_{k-2}, \mathcal{S}_{k-1}, \mathcal{S}_k \rangle,$$

where k can be taken arbitrarily.

Combining the above results we can write down the following semiorthogonal decomposition

$$\mathcal{D}^b(M) = \langle \Phi_0(\mathcal{D}^b(X^+)), \mathcal{S}_{-1}, \mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^{10} \rangle. \quad (13)$$

5.3 The mutations

Now we adopt the following strategy. We start with decomposition (10) of $\mathcal{D}^b(M)$ and then apply a sequence of mutations to get decomposition (13). As a result we will obtain a relation between $\mathcal{A}_{X^+}$ and $\mathcal{D}^b(S)$. More precisely, instead of the first component of (10) we substitute the following exceptional collection on $\mathrm{Gr}(2, V)$:

$$\mathcal{D}^b(\mathrm{Gr}(2, V)) = \langle \mathcal{O}(-2g), \mathcal{O}(-g), V/\mathcal{U}(-g), \mathcal{O}, V/\mathcal{U}, \mathcal{O}(g) \rangle.$$

Plugging it into (10) and denoting

$$\Psi_0 = i_* p^* : \mathcal{D}^b(S) \rightarrow \mathcal{D}^b(M), \quad (14)$$

we obtain a semiorthogonal decomposition

$$\mathcal{D}^b(M) = \langle \mathcal{O}(-2g), \mathcal{O}(-g), V/\mathcal{U}(-g), \mathcal{O}, V/\mathcal{U}, \mathcal{O}(g), \Psi_0(\mathcal{D}^b(S)) \rangle. \quad (15)$$

Now we perform a sequence of mutations.

Step 1. Apply the left mutation through $V/\mathcal{U}$ and $\mathcal{O}(g)$ to $\Psi_0(\mathcal{D}^b(S))$. We obtain

$$\mathcal{D}^b(M) = \langle \mathcal{O}(-2g), \mathcal{O}(-g), V/\mathcal{U}(-g), \mathcal{O}, \Psi_1(\mathcal{D}^b(S)), V/\mathcal{U}, \mathcal{O}(g) \rangle, \quad (16)$$

where

$$\Psi_1 = \mathbb{L}_{(V/\mathcal{U}, \mathcal{O}(g))} \circ i_* p^* : \mathcal{D}^b(S) \rightarrow \mathcal{D}^b(M). \quad (17)$$

Step 2. Translate bundles $V/\mathcal{U}$ and $\mathcal{O}(g)$ to the left. By Lemma 2.13 we obtain

$$\mathcal{D}^b(M) = \langle V/\mathcal{U}(-g-h), \mathcal{O}(-h), \mathcal{O}(-2g), \mathcal{O}(-g), V/\mathcal{U}(-g), \mathcal{O}, \Psi_1(\mathcal{D}^b(S)) \rangle \quad (18)$$

since $K_M = -4g + e = -g - h$.

Step 3. Apply the left mutation through $V/\mathcal{U}(-g-h)$ and $\mathcal{O}(-h)$ to $\mathcal{O}(-2g)$.

Lemma 5.9 *We have $\mathrm{Ext}^\bullet(\mathcal{O}(-h), \mathcal{O}(-2g)) = \mathrm{Ext}^\bullet(V/\mathcal{U}(-g-h), \mathcal{O}(-2g)) = 0$.*

Proof We have to compute

$$\begin{aligned} H^\bullet(M, \mathcal{O}(h-2g)) &= H^\bullet(M, \mathcal{O}(g-e)), \quad \text{and} \\ H^\bullet(M, \mathcal{U}^\perp(h-g)) &= H^\bullet(M, \mathcal{U}^\perp(2g-e)), \end{aligned}$$

where $\mathcal{U}^\perp := (V/\mathcal{U})^\vee$. Since $\pi_*(\mathcal{O}(-e)) = I_S$, the projection formula implies that these spaces are isomorphic to $H^\bullet(\mathrm{Gr}(2, V), I_S(g))$ and $H^\bullet(\mathrm{Gr}(2, M), I_S \otimes \mathcal{U}^\perp(2g))$ respectively. Using (8) as a resolution of I_S and Borel–Bott–Weil on $\mathrm{Gr}(2, V)$ we obtain the required vanishing. $\square$

Because of the orthogonality proved above, and by Lemma 2.12 we obtain

$$\mathcal{D}^b(M) = \langle \mathcal{O}(-2g), V/\mathcal{U}(-g-h), \mathcal{O}(-h), \mathcal{O}(-g), V/\mathcal{U}(-g), \mathcal{O}, \Psi_1(\mathcal{D}^b(S)) \rangle. \quad (19)$$

Step 4. Apply the left mutation through $\mathcal{O}(-h)$ to $\mathcal{O}(-g)$.

Lemma 5.10 *We have $\mathrm{Ext}^\bullet(\mathcal{O}(-h), \mathcal{O}(-g)) = \mathbb{C}[-1]$.*

Proof We have to compute $H^\bullet(M, \mathcal{O}(h - g)) = H^\bullet(M, \mathcal{O}(2g - e))$. Since $\pi_*(\mathcal{O}(-e)) = I_S$, the projection formula implies that this is the same as $H^\bullet(\mathrm{Gr}(2, V), I_S(2g))$. By Borel–Bott–Weil we have $H^\bullet(\mathrm{Gr}(2, V), \mathcal{O}(-g)) = 0$ and $H^\bullet(\mathrm{Gr}(2, V), S^2\mathcal{U}(-g)) = \mathbb{C}[-2]$. Using (8) as a resolution of I_S we deduce the claim. $\square$

It follows that $\mathbb{L}_{\mathcal{O}(-h)}(\mathcal{O}(-g))$ is the unique extension of $\mathcal{O}(-h)$ by $\mathcal{O}(-g)$. By Theorem 5.7 it is isomorphic to $S_0(-g)$. Thus we obtain

$$\mathcal{D}^b(M) = \langle \mathcal{O}(-2g), V/\mathcal{U}(-g - h), S_0(-g), \mathcal{O}(-h), V/\mathcal{U}(-g), \mathcal{O}, \Psi_1(\mathcal{D}^b(S)) \rangle. \quad (20)$$

Step 5. Apply the right mutation through $V/\mathcal{U}(-g)$ to $\mathcal{O}(-h)$.

Lemma 5.11 *We have $\mathrm{Ext}^\bullet(\mathcal{O}(-h), V/\mathcal{U}(-g)) = 0$.*

Proof We have to compute $H^\bullet(M, V/\mathcal{U}(h - g)) = H^\bullet(M, V/\mathcal{U}(2g - e))$. Since $\pi_*(\mathcal{O}(-e)) = I_S$, the projection formula implies that this is the same as $H^\bullet(\mathrm{Gr}(2, M), I_S \otimes V/\mathcal{U}(2g))$. Using (8) as a resolution of I_S and Borel–Bott–Weil on $\mathrm{Gr}(2, V)$ we obtain the required vanishing. $\square$

Because of the orthogonality proved above and by Lemma 2.12 we obtain

$$\mathcal{D}^b(M) = \langle \mathcal{O}(-2g), V/\mathcal{U}(-g - h), S_0(-g), V/\mathcal{U}(-g), \mathcal{O}(-h), \mathcal{O}, \Psi_1(\mathcal{D}^b(S)) \rangle. \quad (21)$$

Step 6. Translate bundle $\mathcal{O}(-2g)$ to the right. By Lemma 2.13 we obtain

$$\mathcal{D}^b(M) = \langle V/\mathcal{U}(-g - h), S_0(-g), V/\mathcal{U}(-g), \mathcal{O}(-h), \mathcal{O}, \Psi_1(\mathcal{D}^b(S)), \mathcal{O}(h - g) \rangle. \quad (22)$$

Step 7. Apply the right mutation through $\mathcal{O}(h - g)$ to $\Psi_1(\mathcal{D}^b(S))$. We obtain

$$\mathcal{D}^b(M) = \langle V/\mathcal{U}(-g - h), S_0(-g), V/\mathcal{U}(-g), \mathcal{O}(-h), \mathcal{O}, \mathcal{O}(h - g), \Psi_2(\mathcal{D}^b(S)) \rangle, \quad (23)$$

where

$$\Psi_2 = \mathbb{R}\mathcal{O}_{(h-g)} \circ \mathbb{L}_{(V/\mathcal{U}, \mathcal{O}(g))} \circ i_* p^* : \mathcal{D}^b(S) \rightarrow \mathcal{D}^b(M). \quad (24)$$

Step 8. Apply the left mutation through $\mathcal{O}$ to $\mathcal{O}(h - g)$. Analogously to step 4 we have $\mathbb{L}_{\mathcal{O}}(\mathcal{O}(h - g)) = S_0(h - g) = S_2(-g)$. Thus we obtain

$$\mathcal{D}^b(M) = \langle V/\mathcal{U}(-g - h), S_0(-g), V/\mathcal{U}(-g), \mathcal{O}(-h), S_2(-g), \mathcal{O}, \Psi_2(\mathcal{D}^b(S)) \rangle. \quad (25)$$

Step 9. Finally, apply the left mutation through $\mathcal{O}(-h)$ to $S_2(-g)$.

Lemma 5.12 *We have $\mathrm{Ext}^\bullet(\mathcal{O}(-h), S_2(-g)) = 0$.*

Proof This follows immediately from (12). $\square$

Because of the orthogonality and by Lemma 2.12 we obtain

$$\mathcal{D}^b(M) = \langle V/\mathcal{U}(-g-h), \mathcal{S}_0(-g), V/\mathcal{U}(-g), \mathcal{S}_2(-g), \mathcal{O}(-h), \mathcal{O}, \Psi_2(\mathcal{D}^b(S)) \rangle.$$

Finally, rewriting $V/\mathcal{U} = \mathcal{S}_1$, $V/\mathcal{U}(-h) = \mathcal{S}_{-1}$ (see Theorem 5.7) we see that we have obtained

$$\mathcal{D}^b(M) = \langle \mathcal{S}_{-1}(-g), \mathcal{S}_0(-g), \mathcal{S}_1(-g), \mathcal{S}_2(-g), \mathcal{O}(-h), \mathcal{O}, \Psi_2(\mathcal{D}^b(S)) \rangle. \quad (26)$$

5.4 Multiple elliptic fibres

Now we have to take into account the structure of $\mathcal{D}^b(S)$, namely the semiorthogonal decomposition

$$\mathcal{D}^b(S) = \langle \{\mathcal{O}_S(-F_i^+)\}_{i=1}^{10}, \mathcal{A}_S \rangle, \quad (27)$$

where $F_i^\pm$ are the multiple fibres of the 10 elliptic pencils on S and the category $\mathcal{A}_S$ is defined by $\mathcal{A}_S = {}^\perp \langle \{\mathcal{O}_S(-F_i^+)\}_{i=1}^{10} \rangle$. It turns out that the functor Ψ_2 relates the multiple elliptic fibres $F_i^\pm$ to the planes $\Sigma_i^\pm$ of M .

Let $y_i \in \mathbb{P}(W)$ be one of the 10 nodes of the symmetroid D . Then as we know the quadric $\mathcal{Q}_{y_i}$ is a union of two planes, and $\rho^{-1}(y_i) = \Sigma_i^+ \cup \Sigma_i^-$ is also a union of two planes.

Lemma 5.13 *The exceptional divisor E of the blowup $\pi : M \rightarrow \mathrm{Gr}(2, V)$ intersects $\Sigma_i^\pm$ along an elliptic curve. Its image $F_i^\pm = \pi(\Sigma_i^\pm \cap E) \subset S$ is a multiple elliptic fibre of S .*

Proof Note that $e = 3g - h$, hence $\mathcal{O}(e)|_{\Sigma_i^\pm} = \mathcal{O}(3)$. On the other hand, $\Sigma_i^\pm$ is not contained in E , since otherwise we would have $\mathbb{P}^2 = \pi(\Sigma_i^\pm) \subset S$ which is impossible. Hence the intersection is a plane cubic. But the only plane cubic curves on S are the multiple elliptic fibres. $\square$

Proposition 5.14 *Let $F_i^\pm = \pi(E \cap \Sigma_i^\pm)$ be a multiple elliptic fibre of S . Then*

$$\Psi_2(\mathcal{O}_S(2g - F_i^\pm)) \cong \mathcal{O}_{\Sigma_i^\pm}(-1).$$

Proof To unburden the notation we will write Σ instead of $\Sigma_i^\pm$ and F instead of $F_i^\pm$. Now consider several exact sequences. First, note that $\pi(\Sigma) = \mathrm{Gr}(2, 3) \subset \mathrm{Gr}(2, V)$ is the zero locus of a regular section of a vector bundle $\mathcal{U}^\vee$; hence we have the following Koszul resolution on $\mathrm{Gr}(2, V)$

$$0 \rightarrow \mathcal{O}(-g) \rightarrow \mathcal{U} \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\pi(\Sigma)} \rightarrow 0.$$

Identifying $\mathcal{U} = \mathcal{U}^\vee(-g)$ and pulling back to M we obtain

$$0 \rightarrow \mathcal{O}(-g) \rightarrow \mathcal{U}^\vee(-g) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\pi^{-1}(\pi(\Sigma))} \rightarrow 0. \quad (28)$$

On the other hand, the preimage $\pi^{-1}(\pi(\Sigma))$ clearly has two components, one is the plane Σ itself, and the other is $p^{-1}(\pi(\Sigma) \cap S) = p^{-1}(F)$. The components intersect along the cubic curve $\Sigma \cap p^{-1}(F)$. So, the structure sheaf of $\pi^{-1}(\pi(\Sigma))$ fits into the following exact sequence

$$0 \rightarrow \mathcal{O}_{\Sigma}(-3) \rightarrow \mathcal{O}_{\pi^{-1}(\pi(\Sigma))} \rightarrow i_*p^*\mathcal{O}_F \rightarrow 0. \quad (29)$$

Also we have the following Koszul resolutions

$$0 \rightarrow i_*p^*\mathcal{O}_S(-F) \rightarrow i_*p^*\mathcal{O}_S \rightarrow i_*p^*\mathcal{O}_F \rightarrow 0 \quad (30)$$

and

$$0 \rightarrow \mathcal{O}(-e) \rightarrow \mathcal{O} \rightarrow i_*p^*\mathcal{O}_S \rightarrow 0. \quad (31)$$

Combining (28), (29), (30), and (31) and twisting everything by $\mathcal{O}(2g)$ we obtain a bicomplex

$$\begin{array}{ccccccc} & & & \mathcal{O}(2g-e) & \longrightarrow & \mathcal{O}_{\Sigma}(-1) & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{O}(g) & \longrightarrow & \mathcal{U}^{\vee}(g) & \longrightarrow & \mathcal{O}(2g) \longrightarrow \mathcal{O}_{\pi^{-1}(\pi(\Sigma))}(2g) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & i_*p^*\mathcal{O}_S(2g-F) & \longrightarrow & i_*p^*\mathcal{O}_S(2g) \longrightarrow i_*p^*\mathcal{O}_F(2g) \longrightarrow 0 \end{array}$$

Here the middle row is (28), the bottom row is (30), the third column is (31), and the fourth column is (29), where everything twisted by $\mathcal{O}(2g)$ as noted above.

Since the bottom two rows of the bicomplex are exact, the total complex is quasi-isomorphic to the top row, which can be rewritten as

$$0 \rightarrow \mathcal{O}(h-g) \rightarrow \mathcal{O}_{\Sigma}(-1) \rightarrow 0. \quad (32)$$

The map here is just the restriction $\mathcal{O}(h-g) \rightarrow \mathcal{O}(h-g)|_{\Sigma} \cong \mathcal{O}(2g-e)|_{\Sigma} \cong \mathcal{O}_{\Sigma}(-1)$. On the other hand, since the right two columns are exact, the total complex is quasi-isomorphic to

$$0 \rightarrow \mathcal{O}(g) \rightarrow \mathcal{U}^{\vee}(g) \rightarrow i_*p^*\mathcal{O}_S(2g-F) \rightarrow 0. \quad (33)$$

Combining these observations, we conclude that complexes (32) and (33) are quasi-isomorphic.

Now we claim that complex (32) is quasi-isomorphic to $\mathbb{L}_{\mathcal{O}(h-g)}(\mathcal{O}_{\Sigma}(-1))$ and complex (33) is quasi-isomorphic to $\mathbb{L}_{(\mathcal{O}(g), \mathcal{U}^{\vee}(g))}(i_*p^*\mathcal{O}_S(2g-F))$. Indeed, the first is clear since

$$\mathrm{Ext}^{\bullet}(\mathcal{O}(h-g), \mathcal{O}_{\Sigma}(-1)) \cong H^{\bullet}(\mathcal{O}(g-h) \otimes \mathcal{O}_{\Sigma}(-1)) \cong H^{\bullet}(\Sigma, \mathcal{O}_{\Sigma}) \cong \mathbb{C}.$$

For the second it suffices to check that (33), or equivalently (32), is orthogonal to $\mathcal{O}(g)$ and $\mathcal{U}^\vee(g)$. For this we check that both terms of (32) are orthogonal to $\mathcal{O}(g)$ and $\mathcal{U}^\vee(g)$. Indeed, for the second term, $\mathcal{O}_\Sigma(-1)$, we have

$$\begin{aligned}\mathrm{Ext}^\bullet(\mathcal{O}(g), \mathcal{O}_\Sigma(-1)) &\cong H^\bullet(\Sigma, \mathcal{O}_\Sigma(-2)) = 0, \\ \mathrm{Ext}^\bullet(\mathcal{U}^\vee(g), \mathcal{O}_\Sigma(-1)) &\cong H^\bullet(\Sigma, \mathcal{U}_\Sigma(-2)) = 0,\end{aligned}$$

Since $\Sigma \cong \mathrm{Gr}(2, 3)$ and the restriction $\mathcal{U}_\Sigma$ is the tautological bundle on $\mathrm{Gr}(2, 3)$. Further, for the first term, $\mathcal{O}(h - g)$, we have

$$\begin{aligned}\mathrm{Ext}^\bullet(\mathcal{O}(g), \mathcal{O}(h - g)) &= H^\bullet(M, \mathcal{O}(h - 2g)) = H^\bullet(M, \mathcal{O}(g - e)) \\ &= H^\bullet(\mathrm{Gr}(2, V), I_S(g)), \\ \mathrm{Ext}^\bullet(\mathcal{U}^\vee(g), \mathcal{O}(h - g)) &= H^\bullet(M, \mathcal{U}(h - 2g)) = H^\bullet(M, \mathcal{U}(g - e)) \\ &= H^\bullet(\mathrm{Gr}(2, V), I_S \otimes \mathcal{U}(g)).\end{aligned}$$

Using the resolution (8) and the Borel–Bott–Weil Theorem we conclude that these cohomologies vanish.

Thus we have checked that

$$\mathbb{L}_{\mathcal{O}(h-g)}(\mathcal{O}_\Sigma(-1)) \cong \mathbb{L}_{(\mathcal{O}(g), \mathcal{U}^\vee(g))}(i_* p^* \mathcal{O}_S(2g - F)). \quad (34)$$

On the other hand, recall that by Lemma 2.9 the mutation functors $\mathbb{L}_{\mathcal{O}(h-g)}$ and $\mathbb{R}_{\mathcal{O}(h-g)}$ are mutually inverse equivalences between ${}^\perp \mathcal{O}(h - g)$ and $\mathcal{O}(h - g)^\perp$. So, let us check that $\mathcal{O}_\Sigma(-1) \in {}^\perp \mathcal{O}(h - g)$, i.e. that $\mathrm{Ext}^\bullet(\mathcal{O}_\Sigma(-1), \mathcal{O}(h - g)) = 0$. Indeed, by Serre duality we have

$$\mathrm{Ext}^t(\mathcal{O}_\Sigma(-1), \mathcal{O}(h - g))^\vee = \mathrm{Ext}^{4-t}(\mathcal{O}(2h), \mathcal{O}_\Sigma(-1)) = H^{4-t}(\Sigma, \mathcal{O}_\Sigma(-1)) = 0.$$

Thus (34) implies that

$$\mathbb{R}_{\mathcal{O}(h-g)} \mathbb{L}_{(\mathcal{O}(g), \mathcal{U}^\vee(g))}(i_* p^* \mathcal{O}_S(2g - F)) \cong \mathcal{O}_\Sigma(-1).$$

Finally note that $\langle V/\mathcal{U}, \mathcal{O}(g) \rangle = \langle \mathcal{U}^\perp(g), \mathcal{O}(g) \rangle = \langle \mathcal{O}(g), \mathcal{U}^\vee(g) \rangle$, hence $\mathbb{L}_{(V/\mathcal{U}, \mathcal{O}(g))} = \mathbb{L}_{(\mathcal{O}(g), \mathcal{U}^\vee(g))}$. In view of this, the above isomorphism proves the Proposition. $\square$

5.5 The proof of the Main Theorem

Now we are in a position to prove Theorem 4.3.

Indeed, let us start with the semiorthogonal decomposition (26). Replacing $\mathcal{D}^b(S)$ by its semiorthogonal decomposition (27), twisting by $\mathcal{O}_S(2g)$, and taking into account Proposition 5.14 we obtain the following semiorthogonal decomposition

$$\begin{aligned}\mathcal{D}^b(M) &= \langle \mathcal{S}_{-1}(-g), \mathcal{S}_0(-g), \mathcal{S}_1(-g), \mathcal{S}_2(-g), \mathcal{O}(-h), \\ &\quad \mathcal{O}, \{\mathcal{O}_{\Sigma_i^+}(-1)\}_{i=1}^{10}, \Psi_2(\mathcal{A}_S \otimes \mathcal{O}_S(2g)) \rangle.\end{aligned}$$

Let us twist it by $\mathcal{O}_M(g)$. As a result we obtain

$$\mathcal{D}^b(M) = \langle \mathcal{S}_{-1}, \mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{O}(g-h), \mathcal{O}(g), \{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^{10}, \Psi_3(\mathcal{A}_S) \rangle, \quad (35)$$

where $\Psi_3 = \mathbb{T}_{\mathcal{O}(g)} \circ \Psi_2 \circ \mathbb{T}_{\mathcal{O}(2g)}$. It remains to make several additional mutations.

Step 9. Apply left mutation through $\mathcal{O}(g-h)$ and $\mathcal{O}(g)$ to the block $\{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^{10}$. The sheaves $\mathcal{O}_{\Sigma_i^+}$ are completely orthogonal to $\mathcal{O}(g)$ and $\mathcal{O}(g-h)$, since

$$\mathrm{Ext}^\bullet(\mathcal{O}(g-th), \mathcal{O}_{\Sigma_i^+}) = H^\bullet(\Sigma_i^+, \mathcal{O}_{\Sigma_i^+}(-1)) = 0.$$

Therefore, by Lemma 2.12 we obtain

$$\mathcal{D}^b(M) = \langle \mathcal{S}_{-1}, \mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^{10}, \mathcal{O}(g-h), \mathcal{O}(g), \Psi_3(\mathcal{A}_S) \rangle. \quad (36)$$

Step 10. Translate $\mathcal{O}(g-h)$, $\mathcal{O}(g)$ and $\Psi_3(\mathcal{A}_S)$ to the left. By Lemma 2.13 we obtain

$$\mathcal{D}^b(M) = \langle \mathcal{O}(-2h), \mathcal{O}(-h), \Psi_4(\mathcal{A}_S), \mathcal{S}_{-1}, \mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^{10} \rangle, \quad (37)$$

where $\Psi_4 = \mathbb{T}(-g-h) \circ \Psi_3 = \mathbb{T}_{\mathcal{O}(-h)} \circ \Psi_2 \circ \mathbb{T}_{\mathcal{O}(2g)}$.

Comparing this decomposition with (13) we conclude that

$$\Phi_0(\mathcal{D}^b(X_+)) = \langle \mathcal{O}_M(-2h), \mathcal{O}_M(-h), \Psi_4(\mathcal{A}_S) \rangle.$$

By Lemma 5.6 the functor Φ_0 commutes with the twist by $\mathcal{O}(-h)$, hence we can rewrite

$$\Phi_0(\mathcal{D}^b(X_+)) = \langle \mathcal{O}_M(-h), \mathcal{O}_M, \Psi_5(\mathcal{A}_S) \rangle,$$

where $\Psi_5 = \mathbb{T}_{\mathcal{O}(h)} \circ \Psi_4 = \Psi_2 \circ \mathbb{T}_{\mathcal{O}(2g)}$. Since again by Lemma 5.6 we have $\mathcal{O}_M(-th) = \Phi_0(\mathcal{O}_{X^+}(-th))$, and Φ_0 is fully faithful, we conclude that

$$\Psi_5(\mathcal{A}_S) = \Phi_0 \circ \mathbb{T}_{\mathcal{O}(2h)}(\mathcal{A}_{X^+}). \quad (38)$$

Thus the functor $\Psi_5^{-1} \circ \Phi_0 \circ \mathbb{T}_{\mathcal{O}(2h)} : \mathcal{A}_{X^+} \rightarrow \mathcal{A}_S$ gives the required equivalence.

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